Sharp criteria for the waiting time phenomenon in solutions to the thin-film equation

Nicola De Nitti

(joint work with Julian Fischer)

Università degli Studi di Bari Aldo Moro

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CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

Thin-film models

The *thin-film equation* (**degenerate fourth order** parabolic equation)

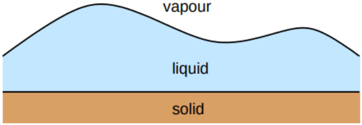
$$\partial_t u + \operatorname{div}(u^n \nabla \Delta u) = 0 \tag{TFE}$$

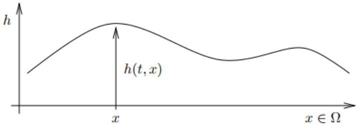
(with positive real parameter $n > 0$) describes the **surface-tension-driven evolution of the height of a viscous thin-liquid film on a flat surface**.

Like the *porous medium equation*

$$\partial_t u = \Delta u^m = m \nabla \cdot (u^{m-1} \nabla u) \tag{PME}$$

(with $m > 1$), the thin-film equation gives rise to a *free boundary problem*, the free boundary being the boundary of the liquid film $\partial\{u(\cdot, t) > 0\}$.





The thin-film equation is mostly of interest in the regime $n \in (1, 3)$, as for $n \geq 3$ it is conjectured that the support of solutions remains constant in time.

Derivation: From the incompressible *Navier-Stokes equations*,

$$\begin{cases} \partial_t v + (v \cdot \nabla)v - \nu\Delta v + \nabla q = 0, \\ \mathrm{div} v = 0, \end{cases} \tag{INS}$$

where $\nu > 0$ is the *viscosity* – assuming a scaling of height and length

$$\boxed{\varepsilon = \frac{\text{height}}{\text{length}} \ll 1.}$$

Applications of thin liquid films:

- industrial coating processes for decorative, insulating, or protective purposes;
- cooling of microelectronic devices;
- microfluidics to model and replicate biological systems (e.g. blood circulation systems) or biological processes (e.g. in-vivo protein crystallisation and bone formation).

Key estimates for the thin-film equation

For solutions to (TFE), maximum or comparison principles cannot be valid.

Existence of *non-negativity preserving* weak solutions and their qualitative properties are obtained thanks to two types of integral estimates:

(1) **energy estimate:**

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla u_0|^2 \geq \frac{1}{2}\int_\Omega |\nabla u(T,x)|^2 + \int_0^T \int_\Omega u^n |\nabla \Delta u|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\geq \frac{1}{2}\int_\Omega |\nabla u(T,x)|^2 \\
&+ C \int_0^T \int_\Omega \left|\nabla \Delta u^{\frac{n+2}{2}}\right|^2 + u^{n-2}|\nabla u|^2 |D^2 u|^2 + |\nabla u^{\frac{n+2}{6}}|^6 \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

N.B. The second inequality plays a key role (makes the dissipation term more amenable to interpolation arguments):

- for $d = 1$ and $n \in \left(\frac{1}{2}, 3\right)$ in [Bernis, Proc. SIAM 1996];
- for $d \in \{2,3\}$ and $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$ in [Grün, Comm. P.D.E. 2004].

(2) **entropy estimate:**

$$\frac{1}{\alpha(\alpha+1)}\int_\Omega u_0^{\alpha+1}\,\mathrm{d}x \geq \frac{1}{\alpha(\alpha+1)}\int_\Omega u^{\alpha+1}\,\mathrm{d}x + C\int_0^T\int_\Omega \left|\nabla u^{\frac{n+\alpha+1}{4}}\right|^4 + \left|D^2 u^{\frac{n+\alpha+1}{2}}\right|^2 \mathrm{d}x\,\mathrm{d}t,$$

for $\alpha \in \left(\frac{1}{2}-n, 2-n\right) \setminus \{1,0\}$.

Proved in [Bernis-Friedman, JDE 1990], [Beretta-Bertsch-Dal Passo, ARMA 1995].

N.B. For many purposed we need to restrict ourselves to $n \in (1,2)$ because for $n \in (2,3)$ only "backward" entropy estimates hold.

Conclusions. For $d \in \{1,2,3\}$, we can prove existence (but uniqueness is an open problem!) for two different notions of solution depending on the value of the parameter $n \in (1,3)$:

- for $1 < n < 2$ (*strong slippage* regime): *weak solutions*;
- for $2 \leq n < 3$ (*weak slippage* regime): *energy dissipating weak solutions*.

In both cases, the integral estimates enforce a *zero contact angle condition* $|\nabla u| = 0$ at the free boundary.

Weak solutions to the thin-film equation with zero contact angle $(1 < n < 2)$

Let $d \in \{1,2,3\}$ and $n \in (1,2)$. Let $T > 0$ and let $u_0 \in H^1(\mathbb{R}^d)$ have compact support. We say that a nonnegative function $u \in L^\infty([0,T); H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$ is a *weak solution of* (TFE) *with zero contact angle and initial data* u_0 if the following conditions are satisfied:

a) $u \in H^1_{loc}\left([0,T); (W^{1,p}(\mathbb{R}^d))'\right)$ for all $p > \frac{4d}{2d+n(2-d)}$;

b) For any $\alpha \in \left(\max\left\{-1, \frac{1}{2} - n\right\}, 2-n\right) \setminus \{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2(\mathbb{R}^d \times [0,T))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4(\mathbb{R}^d \times [0,T))$.

c) for any $\psi \in L^\infty([0,T); C^3_c(\mathbb{R}^d))$ we have for any $T > 0$

$$\int_0^T \langle \partial_t u, \psi \rangle_{(W^{1,p}(\Omega))' \times W^{1,p}(\Omega)} \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^n \nabla u \cdot \nabla \Delta \psi \, \mathrm{d}x \, \mathrm{d}t$$
$$+ n \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^{n-1} \nabla u \cdot D^2 \psi \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \frac{n}{2} \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^{n-1} |\nabla u|^2 \Delta \psi \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \frac{n(n-1)}{2} \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^{n-2} |\nabla u|^2 \nabla u \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t.$$

d) u attains its initial data u_0 in the sense $\lim_{t \to 0} u(\cdot, t) = u_0(\cdot)$ in $L^1(\mathbb{R}^d)$.

Weak energy dissipating solutions to the thin-film equation with zero contact angle ($2 \leq n < 3$)

Let $d \in \{1,2,3\}$ and $n \in [2,3)$. Let $T > 0$ and let $u_0 \in H^1(\mathbb{R}^d)$ have compact support. We call a nonnegative function $u \in L^\infty([0,T); H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$, $u \geq 0$, an *energy-dissipating weak solution of the thin-film equation with zero contact angle and initial data* u_0 if the following conditions are satisfied:

a) We have $\nabla u^{\frac{n+2}{6}} \in L^6(\mathbb{R}^d \times [0,T))$, $u^{\frac{n-2}{2}} \nabla u \otimes D^2 u \in L^2(\mathbb{R}^d \times [0,T))$, and $\chi_{\{u>0\}} u^{\frac{n}{2}} \nabla \Delta u \in L^2(\mathbb{R}^d \times [0,T))$.

b) For all $\alpha \in \left(\max\left\{-1, \frac{1}{2} - n\right\}, 2-n\right) \setminus \{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2(\mathbb{R}^d \times [0,T))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4(\mathbb{R}^d \times [0,T))$.

c) It holds that $u \in H^1_{loc}([0,T); (W^{1,p}(\mathbb{R}^d))')$ for all $p > \frac{4d}{2d+n(2-d)}$.

d) For any $\psi \in L^2([0,T), W^{1,\infty}(\mathbb{R}^d))$ and any $T > 0$, we have

$$\int_0^T \langle \partial_t u, \psi \rangle_{(W^{1,p}(\mathbb{R}^d))' \times W^{1,p}(\mathbb{R}^d)} \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^n \nabla \Delta u \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t.$$

e) u attains its initial data u_0 in the sense $\lim_{t \to 0} u(\cdot, t) = u_0(\cdot)$ in $L^1(\mathbb{R}^d)$.

Finite speed of propagation

Localized versions of the entropy and energy inequalities are the base of most studies of the *qualitative properties of the thin-film equation*.

Finite speed of propagation: For each ball $\overline{B_{R_0}(x_0)}$, with $x_0 \in \mathbb{R}^d$ and $R_0 > 0$, that contains $\operatorname{supp} u_0$, a continuous, monotonically increasing function $R : [0, T) \to \mathbb{R}_0^+$, with $R(0) = 0$, exists such that, for all $t \in (0, T)$, we have

$$\operatorname{supp}(u(\cdot, t)) \subset \overline{B(x_0, R_0 + R(t))}.$$

Optimal upper bound on interface propagation rates [Grün, Interfaces Free Bound. 2002]. For $n \in (1, 3)$, $t \in (0, T)$:

$$\operatorname{supp}(u(\cdot, t)) \subset B\left(0, R_0 + C(n, d)\|u_0\|_{L^1(\Omega)}^{\frac{n}{4+d\cdot n}} \cdot t^{\frac{4}{4+nd}}\right).$$

Optimal lower bound on interface propagation rates [Fischer, JDE 2013]. For $n \in \left(\frac{3}{2}, 3\right)$, $t \in (0, T)$:

$$B\left(0, C(n, d)\|u_0\|_{L^1(\Omega)}^{\frac{n}{4+d\cdot n}} \cdot t^{\frac{4}{4+nd}}\right) - \operatorname{diam}(\operatorname{supp}(u_0)) \subset \operatorname{supp}(u(\cdot, t)).$$

Waiting time phenomenon

If the **initial data** u_0 are **flat enough** near some point x_0 of the initial free boundary, the interface will **locally remain stationary** (or at most move backward) for some time before it finally starts moving forward.

Waiting time: the amount of time that passes before the free boundary moves beyond its initial condition.

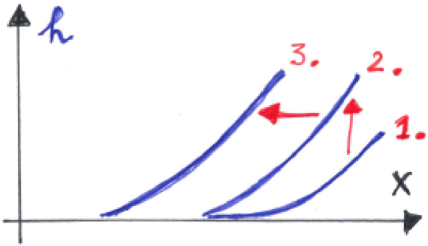


Figure: Illustration of the waiting time phenomenon (by Giacomelli-Knüpfer-Otto).

Consider the one-dimensional thin-film equation $\partial_t u = -\partial_x(u^n \partial^3_{xxx} u)$ in the regime $n \in (2,3)$ and with compactly supported nonnegative initial data $u_0 \in H^1(\mathbb{R})$; denote by x_0 the leftmost point in the support of u_0.

- **Instantaneous forward motion of the free boundary at** x_0 occurs *if and only if* u_0 grows faster than $(x-x_0)_+^{4/n}$ near the free boundary x_0 in the sense of "averages of the mass"

$$\limsup_{r\to 0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \, \mathrm{d}x = \infty. \tag{1}$$

- Hence a **waiting time phenomenon** occurs *if and only if*

$$\limsup_{r\to 0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \, \mathrm{d}x < \infty. \tag{2}$$

- The **optimal upper and lower bounds for waiting times** are both formulated in terms of the quantity

$$\sup_{r>0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \, \mathrm{d}x \tag{3}$$

and differ from each other only by a constant factor.

Literature review: previous almost optimal results

- [Dal Passo-Giacomelli-Grün, Ann. SNS 2001].

  Previously known condition for the **occurrence of a waiting time phenomenon** for the thin-film equation for $n \in [2, 3)$:

$$\limsup_{r \to 0} r^{-4/n+1} \left(\fint_{(x_0, x_0+r)} |\nabla u_0|^2 \, \mathrm{d}x \right)^{1/2} < \infty, \tag{4}$$

- [Fischer, ARMA 2014 & AHP 2016].

  Previously known condition for **instantaneous forward motion of the free boundary** in solutions to the thin-film equation:

$$\limsup_{r \to 0} r^{-4/n} \left(\fint_{(x_0, x_0+r)} u_0^p \, \mathrm{d}x \right)^{1/p} = \infty \tag{5}$$

  for a certain $p \in (0, 1)$, with typically $0 < p \leq \frac{1}{2}$.

Occurrence of the waiting time phenomenon: the case of highly oscillatory initial data

$$u_0(x) := \left(2 + \sin \frac{1}{x - x_0}\right)(x - x_0)_+^{4/n}.$$

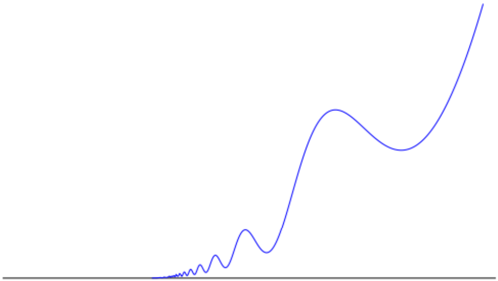


Figure: While the initial data u_0 are clearly bounded from above and from below by a multiple of $(x - x_0)^{4/n}$, due to the rapid oscillations near the free boundary the limit (4) is infinite. As a result, the previous sufficient criterion for waiting times from is not applicable. In contrast, **our sufficient condition shows that for this initial data indeed a waiting time phenomenon occurs**.

Instantaneous forward motion of the free boundary: the case of highly concentrated initial data

$$u_0(x) := (x - x_0)_+^{4/n} + (x - x_0)_+^{4/n-\delta} \cdot \sum_{k=2}^{\infty} k^2 \varphi\Big(k^2\Big(x - x_0 - \frac{1}{k}\Big)\Big),$$

where $\varphi : \mathbb{R} \to \mathbb{R}_0^+$ is a bump function and $\delta > 0$ is fixed.

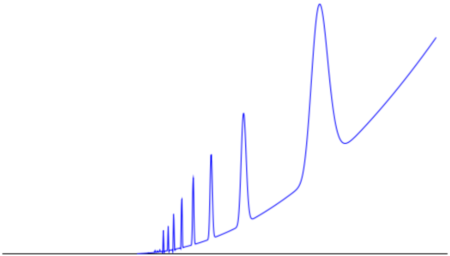


Figure: The initial data features infinitely many bumps accumulating at x_0. The bumps near a point $x > x_0$ have mass of order $(x - x_0)^{4/n-\delta}$ but width of order $|x - x_0|^2$. As a consequence of the mass estimate for the bumps, **our sufficient condition for instantaneous forward motion of the free boundary is applicable**. In contrast, the previous sufficient conditions for instantaneous forward motion from are not applicable for $\delta > 0$ small enough, as the increasingly strong concentration of the bumps cause the limit in (5) to be finite.

Rigorous definition of the waiting time phenomenon

- Let $u_0 \in L^1(\mathbb{R}^d)$ and $u \in L^\infty([0,T); L^1(\mathbb{R}^d))$. For any point $x_0 \in \mathbb{R}^d \setminus \operatorname{supp} u_0$ in the complement of the support of u_0, we define the *waiting time* T^* of u at x_0 as

$$T^* := \operatorname{essinf}\{t > 0 : x_0 \in \operatorname{supp} u(\cdot,t)\},$$

  where $\operatorname{supp} u(\cdot,t)$ is understood in the sense of support of a distribution.

  - In other words, for a point x_0 which lies outside of the support of the initial data, we define the waiting time T^* to be the first time at which the support of the solution u reaches x_0.

- For any point $x_0 \in \partial \operatorname{supp} u_0$ on the boundary of the initial support, we define the waiting time T^* of u at x_0 as

$$T^* := \operatorname{essinf}\{t > 0 : x_0 \notin \overline{\mathbb{R}^d \setminus \operatorname{supp} u(\cdot,t)}\}.$$

  - In other words, for a point x_0 on the initial free boundary $\partial \operatorname{supp} u_0$, we define the waiting time to be the first time at which x_0 is contained in the interior of the support of the solution u.

Occurrence of the waiting time phenomenon and estimate from below for the waiting time

Theorem 1

Let $u : \mathbb{R}^d \times [0, T) \to \mathbb{R}$ be an energy-dissipating weak solution to (TFE) *with zero contact angle and initial data $u_0 \in L^1(\mathbb{R}^d)$.*

Let $x_0 \in \partial \operatorname{supp} u_0 \bigcup (\mathbb{R}^d \setminus \operatorname{supp} u_0)$ be a point on the boundary or outside of the support of the initial data.

Suppose that there exists a constant $\kappa > 0$ such that for all $r > 0$ the estimate

$$\boxed{\fint_{B_r(x_0)} u_0 \, \mathrm{d}x \leq \kappa r^{\frac{4}{n}}} \tag{6}$$

holds.

If $x_0 \in \partial \operatorname{supp} u_0$, suppose furthermore that $\operatorname{supp} u_0$ satisfies an exterior cone condition at x_0 with some positive opening angle $\lambda > 0$, i.e. either $(x_0, x_0 + \delta) \cap \operatorname{supp} u_0$ or $(x_0 - \delta, x_0) \cap \operatorname{supp} u_0$ is empty for some $\delta > 0$ small enough.

Then u has a positive waiting time T^* at x_0 and there exists a constant c depending only on d, n, and possibly λ such that the waiting time T^* is bounded from below by

$$\boxed{T^* \geq c\kappa^{-n}.}$$

Sketch of the proof: Down-propagation of degeneracy argument

In the regime $n \in [2,3)$, we say that the solution u of (TFE) is *degenerate* on a parabolic cylinder $B_r(x_0) \times [0,T]$ if it satisfies

$$\sup_{t\in(0,T)} \fint_{B_r(x_0)} u \,\mathrm{d}x \le \varepsilon T^{-1/n} r^{4/n}; \tag{7a}$$

$$\sup_{t\in(0,T)} \fint_{B_r(x_0)} \frac{t^\beta}{T^\beta} |\nabla u|^2 \,\mathrm{d}x + \int_0^T \fint_{B_r(x_0)} \frac{t^\beta}{T^\beta} \left|\nabla u^{\frac{n+2}{6}}\right|^6 \mathrm{d}x\,\mathrm{d}t \le \varepsilon T^{-\frac{2}{n}} r^{\frac{8}{n-1}}. \tag{7b}$$

for some appropriately chosen $\varepsilon = \varepsilon(d,n) > 0$ and $\beta \in (0,1)$.

Provided that the initial data also satisfy a degeneracy condition of the type $\limsup_{r\to 0} r^{-4/n} \fint_{(x_0,x_0+r)} u_0 \,\mathrm{d}x < \infty$, the degeneracy of u on a parabolic cylinder $B_r(x_0) \times [0,T]$ implies the degeneracy of u on the spatially smaller parabolic cylinder $B_{r/2}(x_0) \times [0,T]$ with the same time horizon T.

Propagating the degeneracy down to $r \to 0$, this essentially shows $u(x_0,t) = 0$ for $t \le T$.

To propagate the degeneracy, we need to iterate back and forth between a **localized mass estimate** and a **localized time-weighted energy estimate**.

- **Propagation of the first degeneracy.**

Starting with degenerate initial data u_0, after choosing T appropriately, the degeneracy properties (7a) and (7b) on a spatially larger parabolic cylinder ensure that the **influx of mass into the smaller ball** $B_{r/2}(x_0)$ remains sufficiently limited up to time T.

- **Propagation of the second degeneracy.**

To propagate the second degeneracy condition (7b) we need to control the **influx of energy into the smaller ball** $B_{r/2}(x_0)$ suitably. We rely on the regularization properties of the nonlinear fourth-order parabolic operator. Heuristically, our approach is close in spirit to the consideration

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \le -c\int_\Omega |\nabla u^{\frac{n+2}{6}}|^6 \,\mathrm{d}x \le -c(\Omega)\Big(\int_\Omega u\,\mathrm{d}x\Big)^{n-4}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^3.$$

This estimate implies by an elementary ODE argument a bound of the form

$$\int_\Omega |\nabla u(\cdot,t)|^2\,\mathrm{d}x \le C(\Omega)t^{-1/2}\Big(\sup_{s\in[0,t]}\int_\Omega u(\cdot,s)\,\mathrm{d}x\Big)^{2-n/2},$$

which is now independent of $\int_\Omega |\nabla u_0|^2\,\mathrm{d}x$, but blows up for $t\to 0$.
The blowup near initial time is the reason for the factor t^β in our condition (7b).

Instant forward motion of the interface and estimate from above for the waiting time

Theorem 2

Let $u : \mathbb{R}^d \times [0,T) \to \mathbb{R}$ *be an energy-dissipating weak solution to* (TFE) *with zero contact angle and initial data* $u_0 \in L^1(\mathbb{R}^d)$.

Let $x_0 \in \partial \operatorname{supp} u_0 \bigcup (\mathbb{R}^d \setminus \operatorname{supp} u_0)$ *be a point on the boundary or outside of the support of the initial data.*

Then there exists a constant C *depending only on* n *such that the waiting time* T^* *of* u *at* x_0 *is bounded from above by*

$$\boxed{T^* \leq C \left(\sup_{r>0} r^{-\frac{4}{n}} \fint_{(x_0-r,x_0+r)} u_0 \, \mathrm{d}x \right)^{-n}.}$$

In particular, if the initial data u_0 *satisfy*

$$\limsup_{r\to 0} r^{-\frac{4}{n}} \fint_{(x_0-r,x_0+r)} u_0 \, \mathrm{d}x = \infty$$

at a point on the initial free boundary $x_0 \in \partial \operatorname{supp} u_0$, *the free boundary starts moving forward immediately at* x_0, *without waiting time.*

Sketch of the proof: Monotonicity formula and differential inequality argument

Step 1. *Almost optimal estimate* [Fischer, ARMA 2014 & AHP 2016].

Monotonicity formula. Weighted entropy inequality:

$$\partial_t \int_{\mathbb{R}} u^{1+\alpha}|x-x_0|^\gamma \,\mathrm{d}x \geq c \int_{\mathbb{R}} u^{1+\alpha+n}|x-x_0|^{\gamma-4} + |\nabla u^{\frac{1+\alpha+n}{4}}|^4|x-x_0|^\gamma \,\mathrm{d}x$$

for suitable $-1 < \alpha < 0$ and suitable $\gamma < -1$, as long as the support of the solution $u(\cdot, t)$ does not touch the singularity of the weight at x_0.

Differential inequality argument [Chipot-Sideris, Trans. AMS 1985]. Using Hölder's inequality and assuming that the support of u remains to the right of x_0, one obtains from the monotonicity formula applied with $x_0 - \delta$ in place of x_0

$$\partial_t \int_{\mathbb{R}} u^{1+\alpha}|x-x_0+\delta|^\gamma \,\mathrm{d}x \geq c\delta^{-\frac{(\gamma+1)n}{(1+\alpha)}-4}\left(\int_{\mathbb{R}} u^{1+\alpha}|x-x_0+\delta|^\gamma \,\mathrm{d}x\right)^{\frac{1+\alpha+n}{1+\alpha}}.$$

This implies finite-time blowup of $\int_{\mathbb{R}} u^{1+\alpha}(\cdot,t)|x-x_0+\delta|^\gamma \,\mathrm{d}x$ and thereby a contradiction to the assumption that the support of $u(\cdot,T)$ remains to the right of x_0 as soon as

$$T \geq C\delta^{\frac{(1+\gamma)n}{(1+\alpha)}+4}\left(\int_{\mathbb{R}} u_0^{1+\alpha}|x-x_0|^\gamma \,\mathrm{d}x\right)^{-\frac{n}{(1+\alpha)}},$$

so, in particular, as soon as

$$T \geq C\left(\delta^{-4(1+\alpha)/n} \fint_{(x_0,x_0+\delta)} u_0^{1+\alpha} \,\mathrm{d}x\right)^{-n/(1+\alpha)}.$$

Step 2. *Improvement: estimates in terms of mass.*

Remark. For "concentrated" initial data, the integral on the right-hand side of the previous formula is much smaller than suggested by the relation

$$\fint_{(x_0,x_0+\delta)} u_0^{1+\alpha} \,\mathrm{d}x \sim \left(\fint_{(x_0,x_0+\delta)} u_0 \,\mathrm{d}x\right)^{1+\alpha}$$

which would be valid for initial data like $u_0(x) \sim (x-x_0)_+^\beta$.

Idea to prove the sharp lower bound in terms of mass. Combine the previous almost optimal estimates with a new estimate connecting motion of mass to entropy production.

Remarks on the cases $d \in \{2,3\}$ and $n \in (1,2)$

- Our sufficient condition for a waiting time (1) is not limited to the regime $n \in (2,3)$, but holds also for $n \in (1,2)$ – we need to use a *localized entropy estimate*.
- In higher dimension, the estimate from below for the waiting time is the same; the estimate from above is weaker, more subtle and technically involved.
- The stationary state $u(x,t) = (x-x_0)_+^2$ shows that in the regime $n < 2$ one cannot expect a condition like (2) to be sufficient for instantaneous forward motion of the free boundary, as $(x-x_0)_+^2$ grows steeper than $(x-x_0)_+^{4/n}$ in this regime.
- The constructions in [Fischer, AHP 2016] show that our condition (1) is in fact sharp among all conditions formulated in terms of the growth of the initial data at the free boundary: It is shown that there exist initial data with only slightly steeper growth than $(x-x_0)_+^{4/n}$ for which instantaneous forward motion occurs.

Some future perspectives

- *Qualitative properties* for the *nonlocal* thin-film equation (modelling hydraulic fractures):

$$\partial_t u + \partial_x \left(u^n \partial_x (-\Delta)^s u\right) = 0.$$

Existence of non-negative solutions: [Imbert-Mellet, Nonlinearity 2011], [Tarhini AHP 2015].

Asymptotic profile: [Imbert-Mellet, Comm. Math. Phys. 2015], [Seratti-Vázquez, ArXiv preprint 2019].

Finite vs. infinite speed of propagation: open problem!

Non-negativity preserving *numerical schemes*: [work in progress].

- *Qualitative properties* for the thin-film equation with *nonlinear surface tension term*:

$$\partial_t u = \operatorname{div}(\mathcal{M}(u)\nabla p), \quad \text{where} \quad p = -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

Existence of non-negative solutions: [Friederich-Grün, Dip. Thesis 2009].

Some future perspectives

- *Singular limit*: $\partial_t u + \partial_x u^n + \varepsilon \partial_x (u^n \partial^3_{xxx} u) = 0$ as $\varepsilon \to 0^+$.

  The $n \in (1,2)$ case: convergence to *entropy solution* of the scalar conservation law for [Otto-Westdickenberg, J. Hyp. Diff. Eq. 2005] via *compensated compactness* and *minimal entropy condition*.

  The $n \in (2,3)$ case: open problem!

- *Singular limit for the nonlocal problem*: $\partial_t u + \partial_x u^n + \varepsilon \partial_x \left(u^n \partial_x (-\Delta)^s u\right) = 0$ as $\varepsilon \to 0^+$.
  Well-posedness maybe is not too difficult. Compactness of the sequence seems to require general α-entropy estimates (currently only known for $\alpha = 1 - n$).

- *Control for free boundary problems*.

  Porous medium equation: [Geshkovski-Zuazua, work in progress].

Thank you for your attention.

Nicola De Nitti

Department of Mathematics, Università degli Studi di Bari Aldo Moro

Email address: n.denitti@studenti.uniba.it