Quadratic behaviors of the 1D linear Schrödinger equation, with bilinear control

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Idea in finite dimension.

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 + x_2^2. \end{cases} \qquad \begin{cases} x_1 = u_1, \\ x_2 = u_2, \\ x_3(T) = \int_0^T \left(u_1^3(t) + u_2^2(t)\right) dt. \end{cases}$$

- At **order one:**

$$x_3^L(t) \equiv 0,$$

- At **order two:**

$$x_3^Q(T) = \int_0^T u_2^2(t)dt \geqslant 0.$$

- For the **non-linear** system: cubic remainder ?
  When $\|u\|_{W^{1,\infty}} << 1$,

$$\int_0^T u_1^3(t)dt << \int_0^T u_2^2(t)dt.$$

Schrödinger equation.

$$\begin{cases} i\partial_t \psi(t,x) = -\partial_x^2 \psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T). \end{cases}$$

Bilinear control system

- the **state**: ψ, such that $\|\psi(t)\|_{L^2(0,1)} = 1$ for all time,
- $\mu : (0,1) \to \mathbb{R}$ **dipolar moment** of the quantum particle
- and $u : (0,T) \to \mathbb{R}$ denotes a **scalar control**.

Schrödinger equation.

$$\begin{cases} i\partial_t \psi(t,x) = -\partial_x^2 \psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T). \end{cases}$$

Notations:

- $D(A) := H^2(0,1) \cap H_0^1(0,1), \quad A\varphi := -\frac{d^2\varphi}{dx^2}$
- $\lambda_j := (j\pi)^2, \quad \varphi_j(x) := \sqrt{2}\sin(j\pi x), \quad \forall j \in \mathbb{N}^*.$
- $H^s_{(0)}(0,1) := D(A^{\frac{s}{2}}), \quad \|\varphi\|_{H^s_{(0)}(0,1)} := \left(\sum_{j=1}^{+\infty} |j^s \langle \varphi, \varphi_j \rangle|^2 \right)^{\frac{1}{2}}.$

Question.

$$\begin{cases} i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi, & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T). \end{cases} \tag{1}$$

Definition (Small-time controllability around the ground state.)

Let $(E_T, \|\cdot\|_{E_T})$ be a family of normed vector spaces of scalar functions defined on $[0,T]$, for $T>0$. The system (1) is said to be ***E-STLC around the ground state*** *if:*

$$\exists s \in \mathbb{N}, \quad \forall T > 0, \quad \forall \varepsilon > 0, \quad \exists \delta > 0,$$

$$\forall \psi_f \in \mathcal{S}, \|\psi_f - \psi_1(T)\|_{H^s_{(0)}(0,1)} < \delta, \quad \exists u \in L^2(0,T) \cap E_T,$$

$$\|u\|_{E_T} < \varepsilon, \quad \psi(0) = \varphi_1, \quad \psi(T) = \psi_f.$$

Previous results.

Theorem (Ball, Marsden and Slemrod, 1982)
The system is not controllable in $\mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$ *with controls in* $L^2_{loc}([0,+\infty),\mathbb{R})$.

Theorem (Beauchard and Laurent, 2010)
Let $T > 0$ *and* $\mu \in H^3((0,1),\mathbb{R})$ *be such that*

$$\exists c > 0 \text{ such that } \frac{c}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle|, \quad \forall k \in \mathbb{N}^*.$$

Then, the system is controllable in $\mathcal{S} \cap H^3_{(0)}$, *locally around the ground state in arbitrary time* $T > 0$ *with controls in* $L^2((0,T),\mathbb{R})$.

Moment method.

First-order: $i\partial_t \psi_L = -\partial_x^2 \psi_L - u(t)\mu(x)\psi_1$
Explicit solution:

$$\psi_L(t) = i \sum_{j=1}^{+\infty} \langle \mu\varphi_1, \varphi_j \rangle \int_0^t u(\tau) e^{i(\lambda_j - \lambda_1)\tau} d\tau \varphi_j e^{-i\lambda_j t}, \quad t \in (0, T).$$

- If $\langle \mu\varphi_1, \varphi_K \rangle = 0$, then

$$\langle \psi_L(t), \varphi_K \rangle \equiv 0.$$

- If for all $j \in \mathbb{N}^*$, $\langle \mu\varphi_1, \varphi_j \rangle \neq 0$, the equality $\psi_L(T) = \psi_f$ is equivalent to

$$\int_0^T u(t) e^{i(\lambda_j - \lambda_1)t} dt = -i \frac{\langle \psi_f, \varphi_j \rangle}{\langle \mu\varphi_1, \varphi_j \rangle} e^{i\lambda_j T}, \quad \forall j \in \mathbb{N}^*.$$

When the linear system is not controllable.

$$i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x), \quad \psi(t,0) = \psi(t,1) = 0.$$

Theorem (Beauchard and Morancey, 2014)

Let $K \in \mathbb{N}^*$, $\mu \in H^3((0,1),\mathbb{R})$ *be such that*

$$\langle \mu\varphi_1, \varphi_K\rangle = 0 \text{ and } A_K := \langle \mu'^2\varphi_1, \varphi_K\rangle \neq 0.$$

There exists $T_K^* > 0$ *such that, for every* $T < T_K^*$, *there exists* $\varepsilon > 0$ *such that, for every* $u \in L^2((0,T),\mathbb{R})$ *with*

$$\|u\|_{L^2(0,T)} < \varepsilon,$$

the solution with initial condition $\varphi_1 = \sqrt{2}\sin(\pi\cdot)$ *satisfies*

$$\psi(T) \neq [\sqrt{1-\delta^2}\varphi_1 + i\,\mathrm{sign}(A_K)\delta\varphi_K]e^{-i\lambda_1 T}, \quad \delta \in (0,1).$$

Goals of our work.

- **First drift** : already used to deny STLC with controls small in L^2. New : deny STLC with controls small in $W^{-1,\infty}$.
- Formulate assumptions to observe a drift quantified by the H^{-k}-norm of the control. Then, deny STLC with controls small in H^{2n-3}.

Theorem : First quadratic obstruction.

Theorem (First quadratic obstruction, B - 2019)

Let $\mu \in H^3((0,1),\mathbb{R})$ *satisfying that there exists* $K \in \mathbb{N}^*$ *such that*

$$\langle \mu\varphi_1, \varphi_K\rangle = 0,$$

and

$$A_K := \langle (\mu')^2\varphi_1, \varphi_K\rangle \neq 0.$$

Then the Schrodinger system is not $W^{-1,\infty}$*-STLC.*

Rk: Quadratic obstruction to STLC in finite dimension for $\frac{dx}{dt} = f_0(x) + uf_1(x)$

$$\mu'^2\varphi_1 = [f_1,[f_1,f_0]](\varphi_1), \quad f_0 = \partial_x^2 \text{ and } f_1 = \mu.$$

Theorem: First quadratic obstruction.

Theorem (First quadratic obstruction, B - 2019)

More precisely,

$$\forall 0 < A < |A_K|, \quad \forall R > 0, \quad \exists T^* > 0, \quad \forall T \in (0, T^*),$$

$\exists \eta > 0, \quad \forall u \in L^2(0,T)$ *with* $\|u\|_{L^2(0,T)} < R$ *and* $\|u_1\|_{L^\infty(0,T)} \leqslant \eta$,

if the solution ψ *of* (1), *with initial data* φ_1, *satisfies*

$$\langle \psi(T), \varphi_j \rangle = 0,$$

then

$$\operatorname{Im}\left(\langle \psi(T), \varphi_K e^{-i\lambda_1 T} \rangle\right) \begin{cases} \leqslant \frac{A - A_K}{4} \|u_1\|^2_{L^2(0,T)}, & \text{if } A_K > 0, \\ \geqslant -\frac{A + A_K}{4} \|u_1\|^2_{L^2(0,T)}, & \text{if } A_K < 0. \end{cases}$$

The n-th quadratic obstruction.

Theorem (The n-th quadratic obstruction, B - 2019)

Let $n \in \mathbb{N}$, $n \geqslant 2$. Let $\mu \in H^{2n+1}((0,1),\mathbb{R})$ be such that

- *its first $n-1$ odd derivatives are zero at $x=0$ and $x=1$,*
- *there exists $K \in \mathbb{N}^*$ such that $\langle \mu\varphi_1, \varphi_K\rangle = 0$, for $p = 1, \dots, n-1$, $A_K^p = 0$ and $A_K^n \neq 0$, where A_K^p is defined as*

$$A_K^p := (-1)^{p-1} \sum_{j=1}^{+\infty} \left(\lambda_j - \frac{\lambda_1 + \lambda_K}{2}\right) (\lambda_K - \lambda_j)^{p-1} (\lambda_j - \lambda_1)^{p-1} \times \langle \mu\varphi_1, \varphi_j\rangle \langle \mu\varphi_K, \varphi_j\rangle, \quad p \in \mathbb{N}^*,$$

- *there exists J a finite subset of $\mathbb{N}^* \setminus \{1\}$ of cardinal n such that for all $j \in J$, we have $\langle \mu\varphi_1, \varphi_j\rangle \neq 0$.*

Then the system (1) is not H^{2n-3}-STLC.

Theorem (The n-th quadratic obstruction, B - 2019)

More precisely,

$$\forall 0 < A < |A_K^n|, \quad \exists T^* > 0, \quad \forall T \in (0, T^*),$$

$$\exists \eta > 0, \quad \forall u \in H^{2n-3}(0,T) \text{ with } \|u\|_{H^{2n-3}(0,T)} \leqslant \eta,$$

if the solution ψ of (1), with initial data φ_1, satisfies

$$\langle \psi(T), \varphi_j \rangle = 0, \quad \forall j \in J,$$

then

$$\operatorname{Im}\left(\langle \psi(T), \varphi_K e^{-i\lambda_1 T} \rangle \right) \begin{cases} \leqslant \frac{A - A_K^n}{4} \|u_n\|^2_{L^2(0,T)}, & \text{if } A_K^n > 0, \\ \geqslant -\frac{A + A_K^n}{4} \|u_n\|^2_{L^2(0,T)}, & \text{if } A_K^n < 0. \end{cases}$$

Strategy of proof.

1. Study of the **quadratic** term.

   Goal: Reveal coercive drift, quantified by the H^{-n}-norm of the control.

$$\mathrm{Im}\left(\langle \psi_Q(T), \varphi_K e^{i\lambda_1 T} \rangle\right) = Q_{T,K}(u_n) \approx ||u_n||^2_{L^2(0,T)}.$$

2. Estimation of the **cubic remainder**?

   Goal: Find the functional setting allowing us to neglect it in front of the drift.

$$||\left(\psi - \psi_1 - \psi_L - \psi_Q\right)(T)|| = o(||u_n||^2_{L^2}).$$

Estimate on the cubic remainder:

Pb: Seek estimates involving u_1.

Idea: Introduction of an auxiliary system. For ψ a solution of the system,

$$\widetilde{\psi}(t,x) := \psi(t,x)e^{-iu_1(t)\mu(x)}, \quad (t,x) \in (0,T) \times \mathbb{R},$$

which is a weak solution of,

$$i\partial_t \widetilde{\psi} = -\partial_x^2 \widetilde{\psi} - i u_1(t) \left[2\mu'(x)\partial_x \widetilde{\psi} + \mu''(x)\widetilde{\psi}\right] + u_1(t)^2 \mu'(x)^2 \widetilde{\psi}.$$

Estimate on the cubic remainder.

Proposition

If $\|u\|_{L^2(0,T)} < R$ *for some* $R > 0$*, when* $||u_1||_{L^2} \to 0$*, we have,*

$$\begin{aligned} \|\widetilde{\psi} - \psi_1\|_{L^\infty((0,T);H_0^1(0,1))} &= O(\|u_1\|_{L^2}), \\ \|\widetilde{\psi} - \psi_1 - \widetilde{\psi_L}\|_{L^\infty((0,T);L^2(0,1))} &= O(\|u_1\|_{L^2}^2), \\ |\langle(\widetilde{\psi} - \psi_1 - \widetilde{\psi_L} - \widetilde{\psi_Q})(T), \varphi_K e^{-i\lambda_1 T}\rangle| &= O(\|u_1\|_{L^2(0,T)}^3). \end{aligned}$$

Estimate on the cubic remainder : linear remainder

$$(\widetilde{\psi} - \psi_1)(t) = -\int_0^t e^{-iA(t-\tau)} \Big[u_1(\tau) \left(2\mu' \partial_x \widetilde{\psi}(\tau) + \mu'' \widetilde{\psi}(\tau) \right) + i u_1(\tau)^2 (\mu')^2 \widetilde{\psi}(\tau) \Big] d\tau, \quad t \in (0, T).$$

- First,

$$\left\| \int_0^t e^{-iA(t-\tau)} \left[u_1(\tau)\mu'' + i u_1(\tau)^2 (\mu')^2 \right] \widetilde{\psi}(\tau) d\tau \right\|_{H_0^1} \leqslant C \left(\|\mu''\|_{H^1} \|u_1\|_{L^1(0,T)} + \|\mu'^2\|_{H^1} \|u_1\|_{L^2}^2 \right) \|\widetilde{\psi}\|_{L^\infty((0,T);H_0^1)}$$

- Then,

$$\left\| \int_0^t e^{-iA(t-\tau)} u_1(\tau) \mu' \partial_x \widetilde{\psi}(\tau) d\tau \right\|_{H_0^1} \leqslant C \|u_1\|_{L^2(0,T)} \|\widetilde{\psi}\|_{L^\infty((0,T);H^2)}.$$

And next?

Initial goal: Estimate the cubic remainder of the initial system.
Pb: Not enough to go back to the initial system.

$$\langle(\psi - \psi_1 - \psi_L - \psi_Q)(T), \varphi_K e^{-i\lambda_1 T}\rangle$$
$$= \langle e^{iu_1(T)\mu} \left(\widetilde{\psi} - \psi_1 - \widetilde{\psi_L} - \widetilde{\psi_Q}\right)(T)), \varphi_K e^{-i\lambda_1 T} > + \dots$$

Ideas:

- Seek L^2 estimates on the cubic remainder. Ok if $\mu'(0) = \mu'(1) = 0$ and ... $\rightarrow$ Second obstructions and the following + Gagliardo-Nirenberg inequalities
- State the quadratic drift on the auxiliary system and then try to go back to the initial system. $\rightarrow$ First obstruction
- Choosing a better projection : drift on $\langle e^{-i\alpha(T)\mu}\psi(T), \varphi_K e^{-i\lambda_1 T}\rangle$ instead $\rightarrow$ First obstruction

Study of the quadratic term.

Qu: Sign of $\text{Im}\langle \psi_Q(T), \varphi_K e^{-i\lambda_1 T}\rangle$?

- First, $\langle \psi_Q(T), \varphi_K e^{-i\lambda_1 T}\rangle = \int_0^T u(t) \int_0^t u(\tau) h(t,\tau) dt d\tau$.
- Then, integrations by parts :

$$\langle \psi_Q(T), \varphi_K e^{-i\lambda_1 T}\rangle = \tilde{Q}(u_1(T), \dots, u_n(T), \alpha_1, \dots, \alpha_n) \\ - i \sum_{p=1}^{n-1} A_K^p \int_0^T u_p(t)^2 e^{i(\lambda_K - \lambda_1)t} dt + Q(u_n),$$

And there exists $T_K^* > 0$ such that, for every $T < T_K^*$,

$$Q(u_n) \begin{cases} \leqslant -\frac{-A_K^n}{4} \int_0^T u_n(t)^2 dt, & \text{if } A_K^n > 0, \\ \geqslant -\frac{A_K^n}{4} \int_0^T u_n(t)^2 dt, & \text{if } A_K^n < 0. \end{cases}$$

"Proof" of the theorem.

Doing an expansion of the solution of Schrodinger around the ground state ($\psi_{eq} = \psi_1, u_{eq} = 0$),

$$\begin{aligned}\mathrm{Im}\langle \psi(T), \varphi_K e^{-i\lambda_1 T}\rangle = \mathrm{Im}\langle \psi_1(T), \varphi_K e^{-i\lambda_1 T}\rangle + \mathrm{Im}\langle \psi_L(T), \varphi_K e^{-i\lambda_1 T}\rangle \\ + \mathrm{Im}\langle \psi_Q(T), \varphi_K e^{-i\lambda_1 T}\rangle + O(restecubique)\end{aligned}$$

So,

$$\begin{aligned}\mathrm{Im}\langle \psi(T), \varphi_K e^{-i\lambda_1 T}\rangle = 0 + 0 + Q(u_n) + o(\|u_n\|_{L^2}^2), \\ \begin{cases} \leqslant -\frac{A_K^n}{8}\|u_n\|_{L^2}^2 \leqslant 0, & \text{if } A_K^n > 0, \\ \geqslant -\frac{A_K^n}{8}\|u_n\|_{L^2}^2 \geqslant 0, & \text{if } A_K^n < 0. \end{cases}\end{aligned}$$

Perspectives.

- On the contrary, using the cubic term to recover some missed directions, and prove some controllability as done by Beauchard and Marbach for parabolic equations.

- Applying this strategy to other equations: Kdv, Burgers, ...

Thank you !