Bilinear control problems on quantum graphs

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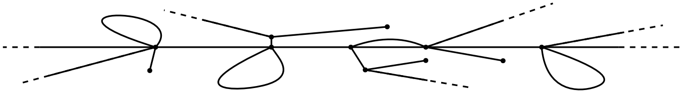
VIII Partial differential equations, optimal design and numerics,

Benasque, Aug 18 – Aug 30, 2019.

Quantum graph



- Set of points (vertices) connected by segments or half-lines (edges),
- equipped with a metric structure (equipped with a distance),
- equipped with a self-adjoint operator as a Schrödinger Hamiltonian.

Let the bilinear Schrödinger equation in $\mathscr{H} = L^2(\mathscr{G}, \mathbb{C})$ for $\mathscr{G}$ a graph

$$\begin{cases} i\partial_t \psi(t) = -\Delta\psi(t) + u(t)B\psi(t), \\ \psi(0) = \psi^0. \end{cases} \tag{BSE}$$

- The operator $-\Delta$ is a self-adjoint Laplacian.
- The operator B is bounded and symmetric in $\mathscr{H}$.
- The function $u \in L^2((0,T),\mathbb{R})$ is the control for $T > 0$.
- We call Γ_t^u the unitary propagator of the (BSE).
- Let S be the unit sphere in $\mathscr{H}$.

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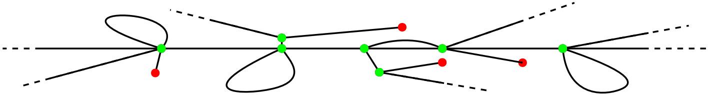
Aim: Study the controllability of the (BSE) in a suitable $X \subset \mathscr{H}$.

$$\forall \psi^1, \psi^2 \in X \cap S,\ \exists T > 0,\ u \in L^2((0, T), \mathbb{R}) \quad \Longrightarrow \quad \Gamma_T^u \psi^1 = \psi^2.$$

Boundary conditions of $-\Delta$



● Internal vertices ● External vertices .

Let $N(v)$ be the set of edges containing an internal vertex v.

Neumann-Kirchhoff:

$$\begin{cases} f \text{ is continuous in } v, \\ \sum_{e \in N(v)} \frac{\partial f}{\partial x_e}(v) = 0. \end{cases}$$

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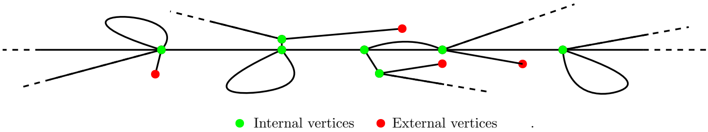

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Let v be an external vertex of the graph.

Dirichlet: $f(v) = 0$, **Neumann:** $\frac{\partial f}{\partial x}(v) = 0.$

Controllability on compacts graphs

Heuristically speaking: Let $(\lambda_k)_{k\in\mathbb{N}^*}$ be the (ordered) spectrum of $-\Delta$.

$$\text{If } \mathscr{G} \text{ is an interval} \quad \Longrightarrow \quad \inf_{k\in\mathbb{N}^*} |\lambda_{k+1} - \lambda_k| > 0, \tag{1}$$

$\Longrightarrow$ **well-posedness** and **local exact controllability**.

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$\Longrightarrow$ **well-posedness** and **local exact controllability**.

$$\text{If } \mathscr{G} \text{ is generic} \quad \Longrightarrow \quad \begin{cases} \text{(1) is not guaranteed but} \\ \text{there exists } \mathcal{M} \in \mathbb{N}^* \text{ so that} \\ \inf_{k\in\mathbb{N}^*} |\lambda_{k+\mathcal{M}} - \lambda_k| > 0. \end{cases}$$

Further assumptions are required.

Controllability on compacts graphs

Let $\{\phi_k\}_{k\in\mathbb{N}^*}$ be the eigenfunctions of $-\Delta$ on $\mathscr{G}$ composed by the edges $\{e_j\}_{j\leq N}$. For $s>0$, let $H^s_{\mathscr{G}} := D(|\Delta|^{\frac{s}{2}})$ and $H^s := \prod_{j=1}^N H^s(e_j,\mathbb{C})$.

Theorem (D.)

Let $\mathscr{G}$ be a compact graph and, for $d\in[0,1/2)$,

$$B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}, \qquad B : H^{3+d}_{\mathscr{G}} \to H^{3+d}\cap H^2_{\mathscr{G}},$$

$$\exists C>0 \quad : \quad |\langle \phi_k, B\phi_1\rangle_{L^2}| \geq \frac{C}{k^3} \qquad \forall k\in\mathbb{N}^*,$$

$$\exists C_1>0 \quad : \quad |\lambda_{k+1}-\lambda_k| \geq \frac{C_1}{k^{\frac{d}{\mathcal{M}-1}}} \qquad \forall k\in\mathbb{N}^*.$$

If, for $k,l,m,n\in\mathbb{N}^*$ distinct such that $\lambda_k-\lambda_l-\lambda_m-\lambda_n=0$, we have

$$\langle \phi_k, B\phi_k\rangle_{L^2} - \langle \phi_l, B\phi_l\rangle_{L^2} - \langle \phi_m, B\phi_m\rangle_{L^2} + \langle \phi_n, B\phi_n\rangle_{L^2} \neq 0,$$

then the (BSE) is well-posed and globally exactly controllable in $H^{3+d}_{\mathscr{G}}$:

$$\forall \psi^1,\psi^2\in H^{3+d}_{\mathscr{G}}\cap S,\ \exists T>0,\ u\in L^2((0,T),\mathbb{R}) \quad \Longrightarrow \quad \Gamma^u_T\psi^1=\psi^2.$$

Thank you for your attention!