CONTROLLABILITY OF THE KORTEWEG-DE VRIES EQUATION ON A STAR-SHAPED NETWORK

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Motivation

- The Korteweg-de Vries (KdV) equation:

$$\partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0,$$

Figure: Solitary-type waves.

Motivation

- In last years, a lot of works have studied controllability properties for the Korteweg-de Vries equation.
  The control of systems of partial differential equations is interesting because it appears in many physical models and there are very challenging problems.
- Question:

Can you control one system if you have N coupled PDE, what is the minimum number of controls?

Previous Works

The control of the KdV equation in a bounded domain has been studied for different authors:

- (Rosier, 1997)
- (Coron & Crépeau, 2004)
- (Cerpa, 2014).

In last years some authors have proposed models using the KdV equations on a finite star shaped network. For example in (Ammari & Crépeau, 2017) the exact controllability results are proven for N+1 controls.

Star-Shaped Network. N=3

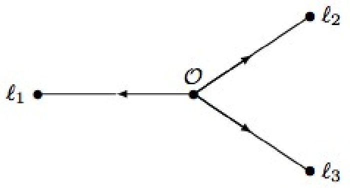

Figure: Star-Shaped Network

Korteweg-de Vries on a Star-Shaped Network

The system is the following:

$$\begin{cases} \partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j = 0, & x \in (0, l_j), t > 0, \\ u_j(t,0) = u_k(t,0), & t > 0, j,k = 1, \cdots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2 + g(t), & t > 0, j = 1, \cdots N, \\ u_j(t, l_j) = 0, & t > 0, j = 1, \cdots, N \\ \partial_x u_j(t, l_j) = g_j(t), & t > 0, j = 1, \cdots, N, \\ u_j(0,x) = u_j^0(x), & x \in (0, l_j), j = 1, \cdots, N. \end{cases}$$

Where $\alpha > N/2$.

Previous Result

(Ammari & Crépeau, 2017). The Korteweg-de Vries equation on a finite star-shaped network is exactly controllable.

Goal

We want to show the controllability without control g at the central node for the system of N Korteweg-de Vries equations.

Problem

For the system of N Korteweg-de Vries equations:

$$\begin{cases} \partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j = 0, & \forall x \in (0, l_j), \forall t > 0, \\ u_j(t,0) = u_k(t,0), & \forall t > 0, j,k = 1, \cdots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2 & \forall t > 0, j = 1, \cdots N, \\ u_j(t, l_j) = 0 & \forall t > 0, j = 1, \cdots, N, \\ \partial_x u_j(t, l_j) = g_j(t), & \forall t > 0, j = 1, \cdots, N, \\ u_j(0,x) = u_j^0(x), & \forall x \in (0, l_j), j = 1, \cdots, N. \end{cases}$$

Where $\alpha > N/2$.

We define

$$\mathbb{L}^2(T) = \prod_{j=1}^{N} L^2(0, l_j), \ \ \mathbb{C}_0^2([0,T]) = \prod_{j=1}^{N} C_0^2([0,T]),$$

$$H_r^s(0, l_j) = \{v \in H^s(0, l_j), \ \ \left(\frac{d}{dx}\right)^{i-1} v(l_j) = 0, 1 \leq i \leq s\},$$

$$\mathbb{H}_e^s(T) = \{\underline{u} = (u_1, \cdots, u_N) \in \prod_{j=1}^{N} H_r^s(0, l_j), u_j(0) = u_k(0), \forall j, k = 1, \cdots N\},$$

and the space

$$\mathbb{B} := C([0,T], L^2(T)) \cap L^2(0, T; \mathbb{H}_e^1(T)).$$

Controllability of the linear system

Observability Inequality

Proposition [H. Brezis. Analyse fonctionnelle. Théorie et applications]

Let E, F be two Banach spaces and $A : E \to F$ a closed operator with $D(A)$ dense in E. Then, we have:

- $A(E) = F$ if and only if there exists a constant $C > 0$ such that:

$$\| v \|_{F^*} \leq C \| A^*(v) \|_{E^*}$$

for every $v \in D(A^*)$.

Theorem 1. [E. Cerpa, E. Crépeau, C. Moreno.] IMA J. of Math. Control and Inf.

Let $(l_j)_{j=1,..N} \in (0,+\infty)^N$ such that $L = \underset{j=1,..,N}{max}\, l_j$ is sufficiently small. Then exist $a, C > 0$ such that for $a < T$, and $\varphi_T \in L^2(T)$ we have:

$$\|\varphi(T,\cdot)\|^2_{L^2(T)} \leq C \sum_{j=1}^{N} \|\partial_x \varphi_j(t,l_j)\|^2_{L^2(0,T)}. \qquad (1)$$

For the backward adjoint problem:

$$\begin{cases} (\partial_t\varphi_j + \partial_x\varphi_j + \partial_x^3\varphi_j)(t,x) = 0, & \forall x \in (0,l_j), \forall t > 0 \\ \varphi_j(t,0) = \varphi_k(t,0), & \forall t > 0, j,k = 1,\cdots N \\ \sum_{j=1}^N \partial_x^2\varphi_j(t,0) = (\alpha - N)\varphi_1(t,0), & \forall t > 0, j = 1,\cdots N \\ \varphi_j(t,l_j) = 0, & \forall t > 0, j = 1,..N \\ \partial_x\varphi_j(t,l_j) = 0, & \forall t > 0, j = 1,..N \\ \varphi_j(T,x) = \varphi_j^T(x), & \forall x \in (0,l_j), j = 1,\cdots N. \end{cases}$$

Proof steps.
Multiplying the linear adjoint system by $q_j\varphi_j$ and integrating by parts in $[s,T]\times[0,l_j]$.

1. $q_j(t,x)=t,\ \ s=0,$
2. $q_j(t,x)=1,$
3. $q_j(t,x)=\frac{(2l_j-x)(l_j-x)}{2l_j^2},\ \ s=0.$

For $L=\max l_j$ and $l=\min l_j$ and the Poincaire inequality we deduce:

$$M\|\varphi(T,x)\|^2_{L^2(T)}dx\leq T\sum_{j=1}^{N}\|\partial_x\varphi_j(t,l)\|^2_{L^2(0,T)},$$

where

$$M=\{T-C^2(\frac{2L}{3}+\frac{LT}{l})-\frac{T(1+\frac{3T}{2l})(2\alpha-N)}{(2\alpha-N)+\sum_{j=1}^{N}\frac{1}{l_j^2}}\}. \tag{2}$$

We want to study for $C = L/\pi$ the sign of discriminant given by

$$\triangle = (1 - \frac{L^3}{l\pi^2} - \frac{2\alpha - N}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}})^2 - 4\frac{\frac{3}{2l}(2\alpha - N)}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}(\frac{2L^3}{3\pi^2}).$$

The discriminant is positive if and only if:

$$\left(1 - \frac{L^3}{l\pi^2} - \frac{2\alpha - N}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\right)^2 > 4\frac{(2\alpha - N)}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\left(\frac{L^3}{l\pi^2}\right),$$

this is true for L sufficiently small. Therefore, for $a < T < b$ we have the controllability of the system.

Controllability Nonlinear System

We also study the local exact controllability for the nonlinear system:

$$\begin{cases} (\partial_t u_j + \partial_x u_j + u_j \partial_x u_j + \partial_x^3 u_j)(t,x) = 0, & \forall x \in (0,l_j), \forall t > 0, \\ u_j(t,0) = u_k(t,0), & \forall t > 0, j,k = 1,\cdots N \\ \sum_{j=1}^N \partial_x^2 u(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & \forall t > 0, j = 1,\cdots N \\ u(t,l_j) = 0, & \forall t > 0, j = 1,\cdots N \\ \partial_x u(t,l_j) = g_j(t), & \forall t > 0, j = 1,..N \\ u_j(0,x) = u_j^0(x), & \forall x \in (0,l_j), j = 1,\cdots N. \end{cases} \tag{3}$$

Theorem 2. [E. Cerpa, E. Crépeau, C. Moreno.] IMA J. of Math. Control and Inf. Accepted.

Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geq N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min}, \tag{4}$$

then the nonlinear control system (3) is locally exactly controllable. This means that there exists $\varepsilon > 0$ such that for any states $u^0 \in \mathbb{L}^2(\mathcal{T})$ and $u^T \in \mathbb{L}^2(\mathcal{T})$ with $\|u^0\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$ and $\|u^T\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$ there exists some controls $g = (g_1, \cdots, g_N) \in L^2(0,T)^N$ such that the solution $u = (u_1, \cdots, u_N) \in \mathbb{B}$ of (3) satisfies

$$u_1(T,\cdot) = u_1^T, \quad u_2(T,\cdot) = u_2^T, \ \cdots, u_N(T,\cdot) = u_N^T.$$

Proof

Let $u^0 \in \mathbb{L}^2(T)$ such that for $\varepsilon > 0$ sufficiently small, $||u^0|| < \varepsilon$ and $||u^T|| < \varepsilon$, for the Theorem 1 there exists a unique solution u. We can decompose this solution into $u^1 + u^2 + u^3$, where u^1 is solutions of

$$\begin{cases} (\partial_t u_j^1 + \partial_x u_j^1 + \partial_x^3 u_j^1)(t,x) = 0, & \forall x \in (0,l_j), \forall t > 0, \\ u_j^1(t,0) = u_k^1(t,0), & \forall t > 0, j,k = 1, \cdots N \\ \sum_{j=1}^N \partial_x^2 u_j^1(t,0) = -\alpha u_1^1(t,0), & \forall t > 0, j = 1, \cdots N \\ u_j^1(t,l_j) = 0, & \forall t > 0, j = 1, \cdots N \\ \partial_x u_j^1(t,l_j) = 0, & \forall t > 0, j = 1, \cdots N \\ u_j^1(0,x) = u_j^0(x), & \forall x \in (0,l_j), j = 1, \cdots, N. \end{cases}$$

u^2 is solution of

$$\begin{cases} (\partial_t u_j^2 + \partial_x u_j^2 + \partial_x^3 u_j^2)(t,x) = -u_j \partial_x u_j, & \forall x \in (0,l_j), \forall t > 0, \\ u_j^2(t,0) = u_k^2(t,0), & \forall t > 0, j,k = 1,\cdots N \\ \sum_{j=1}^N \partial_x^2 u_j^2(t,0) = -\alpha u_1^2(t,0) - \frac{N}{3}(u_1(t,0))^2, & \forall t > 0, j = 1,\cdots N \\ u_j^2(t,l_j) = 0, & \forall t > 0, j = 1,..N \\ \partial_x u_j^2(t,l_j) = 0, & \forall t > 0, j = 1,\cdots N \\ u_j^2(0,x) = 0, & \forall x \in (0,l_j), j = 1,\cdots N, \end{cases}$$

and u^3 is solution of system

$$
\begin{cases}
(\partial_t u_j^3 + \partial_x u_j^3 + \partial_x^3 u_j^3)(t,x) = 0, & \forall x \in (0,l_j), \forall t > 0, \\
u_j^3(t,0) = u_k^3(t,0), & \forall t > 0, j,k = 1, \cdots N \\
\sum_{j=1}^N \partial_x^2 u_j^3(t,0) = -\alpha u_1^3(t,0), & \forall t > 0, j = 1 \cdots N \\
u_j^3(t,l_j) = 0, & \forall t > 0, j = 1, ..N \\
\partial_x u_j^3(t,l_j) = g_j(t), & \forall t > 0, j = 1, \cdots N \\
u_j^3(0,x) = 0, & \forall x \in (0,l_j), j = 1, \cdots N.
\end{cases}
$$

Where $g = (g_1, , g_N) \in L^2(0,T)^N$ is a control such that

$$u^3(T,\cdot) = u^T - u^1(T,\cdot) - u^2(T,\cdot).$$

We consider the map:

$$\Pi : u \in \mathbb{B} \to u^1 + u^2 + u^3 \in \mathbb{B}.$$

We want to find a fixed point $u \in \mathbb{B}$ of the operator Π.

Let $R > 0$ we define

$$B(0,R) = \{u \in L^2(0,T,\mathbb{H}^1(T)) : \|u\|_{L^2(0,T,\mathbb{H}^1(T))} \leq R\}.$$

From previous computation we get

$$\|\Pi(u)\|_{\mathbb{B}} \leq C_1\|u_0\|_{\mathbb{L}^2(T)} + C_2\|u(T)\|_{\mathbb{L}^2(0,L)} + C_3\|u\|_{L^2(0,T;\mathbb{L}^2(T))},$$

with R and ϵ small enough so that $(C_1 + C_2)\varepsilon + C^3 R^2 < R$, we have that $\Pi(B(0,R)) \subset B(0,R)$. Furthermore, $\forall u, v \in B(0,R)$:

$$\| \Pi(u) - \Pi(v) \|_{\mathbb{B}} \leq 2C_4 R \| u - v \|_{L^2(0,T,\mathbb{H}^1(T))},$$

and then for R, ϵ small enough, $C_4 R \in (0,1)$, thus, we obtain that Π is a contraction in $B(0,R) \subset \mathbb{B}$, which ends the proof of Theorem 2.

Other control poblems for system

We study the following Hirota-Sutsuma system, with $T > 0$ and $Q = (0,T) \times (0,L)$:

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + 3w_x, & (t,x) \in Q, \\ v_t + \frac{1}{2}v_{xxx} = -3vv_x, & (t,x) \in Q, \\ w_t + \frac{1}{2}w_{xxx} = -3uw_x & (t,x) \in Q. \end{cases} \tag{5}$$

with the initial and boundary conditions:

$$\begin{cases} u(t,0) = 0,\ u(t,L) = 0,\ u_x(t,0) = h_1(t), & t \in (0,T), \\ v(t,0) = 0,\ v(t,L) = 0,\ v_x(t,L) = h_2(t), & t \in (0,T), \\ w(t,0) = 0,\ w(t,L) = 0,\ w_x(t,L) = h_3(t), & t \in (0,T), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x), & x \in (0,L). \end{cases} \tag{6}$$

Problem 1.
Given $T > 0, (u_0, v_0, w_0) \in (L^2(0,L))^3$ and $(u^1, v^1, w^1) \in (L^2(0,L))^3$ does there exist h_i for $i = 1, 2, 3$ in certain spaces such that the corresponding solution (u, v, w) of (5) - (6) satisfies:

$$u(T,x) = u^1(x), \;\; v(T,x) = v^1(x), \;\; w(T,x) = w^1(x)?$$

Finally, we study the Kuramoto - Sivashinsky (KS) system with $T > 0$, a bounded domain Ω of $\mathbb{R}$ and ω a nonempty open subset Ω. In $(0,T)\times\Omega$ the system is described by the following equation:

$$\left\{ \begin{array}{c} u_t + u_{xxxx} + a(x)u_{xxx} + b(x)u_{xx} = v_x + g_1(x)v + f_1(x)u_x + g_2(x)u + 1_\omega h, \\ v_t + v_{xxxx} + c(x)v_{xxx} + d(x)v_{xx} = u_x + g_3(x)u + f_2(x)v_x + g_4(x)v. \end{array} \right. \tag{7}$$

with the initial and boundary conditions:

$$\left\{ \begin{array}{lll} u(t,0) = u_x(t,0) = 0, & u(t,L) = u_x(t,L) = 0, & t \in (0,T), \\ v(t,0) = v_x(t,0) = 0, & v(t,L) = v_x(t,L) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & v(0,x) = v_0(x). & x \in \Omega, \end{array} \right. \tag{8}$$

Problem 2. Let $T > 0, (u_0, v_0) \in (L^2(0,L))^2$ and $(u^1, v^1) \in (L^2(0,L))^2$ does there exist h in certain spaces such that the corresponding solution (u, v) of (7) - (8) satisfies: $u(T,x) = v(T,x) = (0,0)$?

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THANKS FOR YOUR ATTENTION!