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Regularity for minimizers in 2d of the optimal p-compliance problem with length penalization

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Let Ω be an open bounded set in $\mathbb{R}^N$ (where $N \geq 2$) and let $f \in L^{p'}(\Omega)$ with $1 < p < \infty$, $(1/p) + (1/p') = 1$. We consider the problem

$$\min\{C_p(\Sigma) + \lambda \mathcal{H}^1(\Sigma) : \Sigma \in \mathcal{K}(\Omega)\},$$

where $\mathcal{K}(\Omega)$ is a class of compact connected subsets of the closure of Ω, $\lambda > 0$ is a fixed constant and C_p is the p-compliance functional which for a given $\Sigma \in \mathcal{K}(\Omega)$ is defined as the maximum value of the functional

$$\int_\Omega f u \,\mathrm{d}x - \frac{1}{p} \int_{\Omega \setminus \Sigma} |\nabla u|^p \,\mathrm{d}x$$

on the Sobolev space $W_0^{1,p}(\Omega \setminus \Sigma)$.

The physical interpretation of the problem in two-dimensions is the following: we can think of Ω as a membrane that attached along $\Sigma \cup \partial\Omega$ (where Σ can be seen as the "glue line") to the some fixed base and subjected to the given force f. Then the displacement u_Σ of the membrane satisfies the p-Poisson equation

$$\begin{cases} -\Delta_p u & = & f \text{ in } \Omega \setminus \Sigma \\ u & = & 0 \text{ on } \Sigma \cup \partial\Omega, \end{cases}$$

and the rigidity of the membrane is measured through the p-compliance functional

$$C_p(\Sigma) = \frac{1}{p'} \int_\Omega f u_\Sigma \,\mathrm{d}x.$$

We are looking for the best location of the "glue line" Σ in $\overline{\Omega}$ in order to maximize the rigidity of Ω, subject to the force f, and at the same time to minimize the quantity or cost of the glue.

We extend a recent result of A. Chambolle, J. Lamboley, A. Lemenant and E. Stepanov (SIAM J. Math. Anal., 49(2), 1166–1224), proving that

every minimizer in the given bounded domain in $\mathbb{R}^2$ for a given $p \in (1, \infty)$ contains no loops and is a locally $C^{1,\alpha}$ regular curve outside of a set with Hausdorff dimension strictly less then one. In view of the fact that there is no monotonicity of energy for $p \in (1, 2) \cup (2, +\infty)$, we introduce a new approach that allows us establish the desired decay for the potential u_Σ. The proof of the importance of the connectivity condition in the statement of the problem in N-dimensions and for $p > N - 1$ is provided.