A Finite Element Method for Elliptic Distributed Optimal Control Problems with Pointwise Control and State Constraints

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VIII PDEs, Optimal design and Numerics
Benasque

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Outline

- The Optimal Control Problem
- Finite Element Methods
- A Priori Error Analysis
- Numerical Results
- Post-processing

The Control Problem

$$
\begin{aligned}
\text{minimize} \quad & J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L_2(\Omega)} \\
& (y,u) \in H^1_0(\Omega) \times L_2(\Omega) \\
& \int_\Omega \nabla y \cdot \nabla v \, dx = \int_\Omega uv \, dx \quad \forall v \in H^1_0(\Omega) \\
& \psi_1 \le y \le \psi_2 \quad \text{a.e. in} \quad \Omega \\
& \phi_1 \le u \le \phi_2 \quad \text{a.e. in} \quad \Omega
\end{aligned}
$$

$\Omega \subset \mathbb{R}^2$: bounded convex polygonal domain

$y_d \in L_2(\Omega)$: desired state

$\beta > 0$: regularization parameter

$\psi_1 < \psi_2$ in $\bar{\Omega}$ and $\psi_1 < 0 < \psi_2$ on $\partial\Omega$

$\phi_1 < \phi_2$ in $\bar{\Omega}$

The Control Problem (Standard approach)

To eliminate y and solve a minimization problem for u

$$\text{minimize } J(u) := \frac{1}{2}\|Su - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L_2(\Omega)}$$

$$\text{subject to} \quad u \in L_2(\Omega), \quad \phi_1 \le u \le \phi_2, \quad \psi_1 \le Su \le \psi_2$$

$$S : \text{Control to State Mapping}$$

- K. Deckelnick and M. Hinze (2007)
- C. Meyer (2008)
- S. Cherednichenko and A. Rösch (2008)
- E. Casas, M. Mateos, and B. Vexler (2014)

The Control Problem (Our approach)

$-\Delta y = u$, by elliptic regularity $y \in H^2(\Omega) \cap H_0^1(\Omega)$, we can reformulate the problem eliminating u and solve for y

$$\underset{y \in K}{\text{minimize}} \quad J(y) := \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)}$$

$$K := \{v \in V : \psi_1 \leq v \leq \psi_2 \;\text{ and }\; \phi_1 \leq -\Delta v \leq \phi_2 \;\text{ a.e. in } \Omega\}$$

$$V := H^2(\Omega) \cap H_0^1(\Omega)$$

Slater Condition There exists $\tilde{y} \in H^2(\Omega) \cap H_0^1(\Omega)$ such that
(i) $\psi_1 < \tilde{y} < \psi_2$ on $\overline{\Omega}$
(ii) $\phi_1 \leq -\Delta\tilde{y} \leq \phi_2$ on Ω

$-\Delta y = u$, by elliptic regularity $y \in H^2(\Omega) \cap H_0^1(\Omega)$, we can reformulate the problem eliminating u and solve for y

$$\underset{y \in K}{\text{minimize}} \quad J(y) := \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)}$$

$$K := \{v \in V : \psi_1 \leq v \leq \psi_2 \;\text{ and }\; \phi_1 \leq -\Delta v \leq \phi_2 \;\text{ a.e. in } \Omega\}$$

$$V := H^2(\Omega) \cap H_0^1(\Omega)$$

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(ii) $\phi_1 \leq -\Delta \tilde{y} \leq \phi_2$ on Ω

- Unlike the existing results, this approach allows us to analyze the state error in $H^2(\Omega)$ like norm

The Control Problem

$$\underset{y \in K}{\text{minimize}} \quad J(y) \quad := \quad \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)}$$

The Control Problem

$$\underset{y\in K}{\text{minimize}} \quad J(y) \quad := \quad \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)}$$

$\Leftrightarrow$

$$\underset{y\in K}{\text{minimize}} \quad J(y) \quad := \quad \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)} + \frac{1}{2}\|y\|^2_{L_2(\Omega)} - (y_d, y)$$

The Control Problem

$$\underset{y\in K}{\text{minimize}} \quad J(y) \quad := \quad \frac{1}{2}\|y-y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)}$$

$\Leftrightarrow$

$$\underset{y\in K}{\text{minimize}} \quad J(y) \quad := \quad \frac{\beta}{2}\|\Delta y\|^2_{L_2(\Omega)} + \frac{1}{2}\|y\|^2_{L_2(\Omega)} - (y_d, y)$$

$\Leftrightarrow$

$$\underset{y\in K}{\text{minimize}} \quad J(y) := \frac{1}{2}a(y,y) - (y_d, y)$$

$$a(w,v) = \beta\int_\Omega D^2 w : D^2 v\,dx + \int_\Omega wv\,dx \qquad \forall\, w,v \in V$$

$$D^2 w : D^2 v = \sum_{1\le i,j\le 2}\Big(\frac{\partial^2 w}{\partial x_i \partial x_j}\Big)\Big(\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)$$

- $a(\cdot,\cdot)$ is symmetric, bounded and coercive over V

To find $y \in K$ such that

$$a(y, z - y) \geq (y_d, z - y) \quad \forall z \in K$$

where

$$a(w, v) = \beta \int_{\Omega} D^2 w : D^2 v \, dx + \int_{\Omega} wv \, dx \qquad \forall \, w, v \in V$$
$$K = \{v \in V : \psi_1 \leq v \leq \psi_2 \text{ and } \phi_1 \leq -\Delta v \leq \phi_2 \text{ a.e. in } \Omega\}$$

- K is nonempty, closed and convex
- $a(\cdot, \cdot)$ is bounded and coercive over V

($\implies$ unique solution of the continuous problem)

Existence and Uniqueness

Theorem

(Stampacchia) Let V be a Hilbert space and K be closed and convex subset of V. Let $a(\cdot,\cdot)$ be a continuous, V-elliptic bilinear form and $f(\cdot)$ be a continuous linear functional on V. Then there exists a unique solution $u \in K$ such that

$$a(u, v-u) \geq f(v-u) \quad \textit{for all} \quad v \in K$$

KKT System Using KKT theory, there exists $\lambda \in L_2(\Omega)$ and $\mu \in \mathbb{M}(\Omega)$ such that

$$
\begin{aligned}
a(y,z) + \int_\Omega \lambda \Delta z \, dx - \int_\Omega y \, d\mu = (y_d, z) \quad \forall z \in V \\
(\lambda, \Delta y + \phi_j) \le 0 \ \text{ a.e. in } \ \Omega \text{ for } j = 1,2 \\
(\lambda, (\Delta y + \phi_1)(\Delta y + \phi_2)) = 0 \\
< \mu, y - \psi_j > \le 0 \ \text{ a.e. in } \ \Omega \text{ for } j = 1,2 \\
< \mu, (y - \psi_1)(y - \psi_2) > = 0
\end{aligned}
$$

$\mathbb{M}(\Omega)$: space of regular Borel measures

KKT System Using generalized KKT theory, there exists $\lambda \in L_2(\Omega)$ and $\mu \in \mathbb{M}(\Omega)$ such that

$$a(y,z) + \int_\Omega \lambda \Delta z \, dx - \int_\Omega y \, d\mu = (y_d, z) \quad \forall z \in V$$

$$\begin{aligned}
&\lambda \geq 0 && \text{if} && -\Delta y = \phi_1 && \text{(Lower control contact set)}\\
&\lambda \leq 0 && \text{if} && -\Delta y = \phi_2 && \text{(Upper control contact set)}\\
&\lambda = 0 && \text{otherwise} && &&\\
&\mu \geq 0 && \text{if} && y = \psi_1 && \text{(Lower state contact set)}\\
&\mu \leq 0 && \text{if} && y = \psi_2 && \text{(Upper state contact set)}\\
&\mu = 0 && \text{otherwise} && &&
\end{aligned}$$

Regularity

- $y \in H^3_{loc}(\Omega) \cap H^{2+\alpha}(\Omega)$ $\quad (0 < \alpha \leq 1)$
- $u = -\Delta y \in H^1(\Omega) \cap L_\infty(\Omega)$
- $\lambda \in W^{1,p}(\Omega)$, $p < 2$
- $\mu \in W^{-1,s}(\Omega)$, $s < 2$

Casas-Mateos-Vexler (2014)
Casas (1985)
Blum-Rannacher (1980)

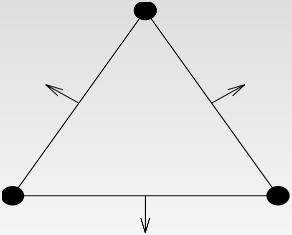
$\mathcal{T}_h$: regular triangulation of Ω

Morley Finite Element

$$\tilde{V}_h := \{v \in L_2(\Omega) : v|_T \in P_2(T), \quad v \text{ is continuous at the vertices and normal derivative of } v \text{ is continuous at the midpoints of the edges}\}$$

$$V_h = \{v \in \tilde{V}_h : v \text{ vanishes at the vertices of } \partial\Omega\}$$

(Nonconforming element)



6 dof

$\mathcal{T}_h$: regular triangulation of Ω

Define the discrete space K_h as

$$K_h := \{v \in V_h : \psi_1(p) \le v(p) \le \psi_2(p) \quad \forall\, p \in \mathcal{V}_h,$$
$$\int_T \phi_1 \; dx \le -\int_T \Delta v \; dx \le \int_T \phi_2 \; dx \quad \forall\, T \in \mathcal{T}_h \;\}$$
$$= \{v \in V_h : I_h\psi_1 \le I_h v \le I_h\psi_2 \;\text{ and }\; Q_h\phi_1 \le Q_h(-\Delta v) \le Q_h\phi_2 \;\}$$

where

- I_h : Nodal interpolation operator for the conforming P_1 finite element space associated with $\mathcal{T}_h$
- Q_h : L_2 orthogonal projection onto the space of piecewise constant functions associated with $\mathcal{T}_h$

Discrete Problem

$$\underset{z \in K_h}{\text{minimize}} \quad \frac{\beta}{2}\|D_h^2 z\|_{L_2(\Omega)}^2 + \frac{1}{2}\|z\|_{L_2(\Omega)}^2 - \int_\Omega y_d z \, dx$$

To find $y_h \in K_h$ such that

$$a_h(y_h, z_h - y_h) \geq (y_d, z_h - y_h) \quad \forall z_h \in K_h$$

where

$$a_h(w, v) = \beta \sum_{T \in \mathcal{T}_h} \int_T D^2 w : D^2 v \, dx + \int_\Omega wv \, dx \qquad \forall \, w, v \in V_h$$

- $a_h(\cdot, \cdot)$ is bounded and coercive on V_h

($\implies$ unique solution for the discrete problem)

Theorem

There exists a positive constant C independent of h such that

$$\|y - y_h\|_h \leq Ch^{\min(1-\epsilon,\alpha)}$$

$$\begin{aligned} \|v\|_h^2 &= a_h(v,v) \quad \text{for } v \in V_h \\ h &= \max_{T\in\mathcal{T}_h} diamT \\ y &\in H^3_{loc}(\Omega) \cap H^{2+\alpha}(\Omega) \qquad (0 < \alpha \leq 1) \\ \lambda &\in W^{1,2-\epsilon}(\Omega) \end{aligned}$$

Theorem

There exists a positive constant C independent of h such that

$$\|y - y_h\|_h \leq Ch^{\min(1-\epsilon,\alpha)}$$

Corollary

$$\|y - y_h\|_{L_\infty(\Omega)} + \Big(\sum_{T \in \mathcal{T}_h} |y - y_h|^2_{H^1(T)}\Big)^{1/2} \leq Ch^{\min(1-\epsilon,\alpha)}$$

(higher order convergence observed in practice)

- We can remove ϵ if the supports of λ and μ are disjoint or if $\mu \in H^{-1}(\Omega)$ (Casas-Mateos-Vexler (2014))

- The Lagrange multiplier $\lambda \in W^{1,p}(\Omega)$ for any $p < 2$

Ingredients of Proof

- The Lagrange multiplier $\lambda \in W^{1,p}(\Omega)$ for any $p < 2$
- Interpolation operator preserves discrete constraints

Interpolation Operator

$\Pi_h : V \to V_h$ defined as

$$\Pi_h v(p) = v(p) \quad \forall p \in \mathcal{V}_h$$
$$\int_e \frac{\partial \Pi_h v}{\partial n} ds = \int_e \frac{\partial v}{\partial n} ds \quad \forall e \in \mathcal{E}_h$$

Note that $\Pi_h v \in K_h$ for $v \in K$

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Note that $\Pi_h v \in K_h$ for $v \in K$

$$\int_T \Delta \Pi_h v \, dx = \int_{\partial T} \frac{\partial \Pi_h v}{\partial n} \, ds = \int_{\partial T} \frac{\partial v}{\partial n} \, ds = \int_T \Delta v \, dx$$
$$\implies \int_T \phi_1 \, dx \leq - \int_T \Delta \Pi_h v \, dx \leq \int_T \phi_2 \, dx$$
$$\text{Also,} \quad \psi_1(p) \leq \Pi_h v(p) \leq \psi_2(p)$$

$\mathcal{V}_h$: set of vertices of $\mathcal{T}_h$ $\qquad$ $\mathcal{E}_h$: set of edges of $\mathcal{T}_h$

Interpolation Operator

$\Pi_h : V \to V_h$ defined as

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$$\implies \int_T \phi_1 \, dx \leq -\int_T \Delta \Pi_h v \, dx \leq \int_T \phi_2 \, dx$$
$$\text{Also,} \quad \psi_1(p) \leq \Pi_h v(p) \leq \psi_2(p)$$

$$\|v - \Pi_h v\|_h \leq C h^\alpha \|v\|_{H^{2+\alpha}(\Omega)} \quad \forall v \in H^{2+\alpha}(\Omega)$$

(Interpolation estimates)

Ingredients of Proof

- The Lagrange multiplier $\lambda \in W^{1,p}(\Omega)$ for any $p < 2$
- Interpolation operator preserves discrete constraints
- Enriching map preserves discrete constraints

Enriching Map

$E_h : V_h \to V_{HCT}$, for any $v \in V_h$, define

$$E_h v(p) = v(p), \quad \nabla E_h v(p) = \frac{1}{|\mathcal{T}_p|} \sum_{T \in \mathcal{T}_p} \nabla v|_T(p) \qquad p \in \mathcal{V}_h$$

$$\int_e \frac{\partial E_h v}{\partial n} ds = \int_e \frac{\partial v}{\partial n} ds \qquad \forall e \in \mathcal{E}_h$$

$|\mathcal{T}_p|$: the cardinality of $\mathcal{T}_p$. Clearly $E_h v \in V_{HCT}$ for every $v \in V_h$

Enriching Map

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$$\int_T \Delta E_h v \, dx = \int_{\partial T} \frac{\partial E_h v}{\partial n} \, ds = \int_{\partial T} \frac{\partial v}{\partial n} \, ds = \int_T \Delta v \, dx$$

$$\Longrightarrow \int_T \phi_1 \, dx \leq - \int_T \Delta E_h v \, dx \leq \int_T \phi_2 \, dx$$

Also, $\quad \psi_1(p) \leq E_h v(p) \leq \psi_2(p)$

Enriching Map

$E_h : V_h \to V_{HCT}$, for any $v \in V_h$, define

$$E_h v(p) = v(p), \quad \nabla E_h v(p) = \frac{1}{|\mathcal{T}_p|} \sum_{T \in \mathcal{T}_p} \nabla v|_T(p) \qquad p \in \mathcal{V}_h$$

$$\int_e \frac{\partial E_h v}{\partial n} ds = \int_e \frac{\partial v}{\partial n} ds \qquad \forall e \in \mathcal{E}_h$$

$|\mathcal{T}_p|$: the cardinality of $\mathcal{T}_p$. Clearly $E_h v \in V_{HCT}$ for every $v \in V_h$

$$\int_T \Delta E_h v \, dx = \int_{\partial T} \frac{\partial E_h v}{\partial n} \, ds = \int_{\partial T} \frac{\partial v}{\partial n} \, ds = \int_T \Delta v \, dx$$

$$\Longrightarrow \int_T \phi_1 \, dx \leq - \int_T \Delta E_h v \, dx \leq \int_T \phi_2 \, dx$$

Also, $\quad \psi_1(p) \leq E_h v(p) \leq \psi_2(p)$

$$\sum_{T \in \mathcal{T}_h} (h_T^{-4} \|E_h v - v\|^2_{L_2(T)} + h_T^{-2} \|\nabla(E_h v - v)\|^2_{L_2(T)}) + \|v - E_h v\|^2_h \leq C \|v\|^2_h$$

Example 1

$$J(y,u) = \frac{1}{2}\int_\Omega (y - y_d)^2 \; dx + \frac{\beta}{2}\int_\Omega u^2 \; dx$$

subject to constraints:

$$\int_\Omega \nabla y \cdot \nabla v \; dx = \int_\Omega uv \; dx \quad \forall v \in H_0^1(\Omega)$$

$$\psi_1 \leq y \leq \psi_2, \quad \phi_1 \leq u \leq \phi_2$$

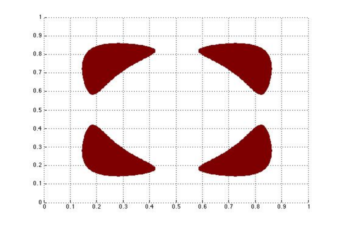
with the following data:

$$\Omega = (0,1)^2 \quad \beta = 1e-3 \quad y_d = 2$$

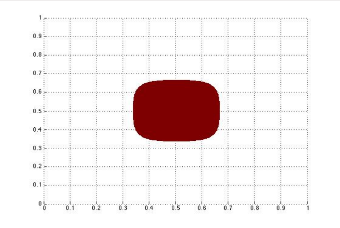
$$\psi_1 = -\infty \qquad \psi_2 = 1 \qquad \phi_1 = -1 \qquad \phi_2 = 25$$

h	$\|y - y_h\|_h$	order	$\|y - y_h\|_1$	order	$\|y - y_h\|_{L^\infty(\Omega)}$	order
1/2	$9.5793\times10^{+0}$	–	1.0446×10^{-0}	–	2.5466×10^{-1}	–
1/4	$1.0661\times10^{+1}$	-0.19	6.9316×10^{-1}	0.71	1.1975×10^{-1}	1.31
1/8	$1.5387\times10^{+1}$	-0.58	4.7208×10^{-1}	0.61	1.1414×10^{-1}	0.08
1/16	$8.7327\ \times10^{-0}$	0.86	1.1994×10^{-1}	2.07	3.1482×10^{-2}	1.95
1/32	$4.5462\ \times10^{-0}$	0.96	2.9576×10^{-2}	2.07	7.5706×10^{-3}	2.10
1/64	$2.3095\ \times10^{-0}$	0.99	7.4146×10^{-3}	2.02	1.9696×10^{-3}	1.97
1/128	$1.1599\ \times10^{-0}$	0.99	1.8979×10^{-3}	1.98	5.0768×10^{-4}	1.97
1/512	$5.8053\ \times10^{-1}$	1.00	4.6538×10^{-4}	2.03	1.2460×10^{-4}	2.03

Table: Errors and orders of convergence of state for Morley element



Upper control active set



State active set

Example 2 (Cherednichenko-Rösch, 2008)

$$J(y,u) = \frac{1}{2}\int_\Omega (y - y_d)^2 \, dx + \frac{\beta}{2}\int_\Omega (u - u_d)^2 \, dx$$

subject to constraints:

$$\int_\Omega \nabla y \cdot \nabla v \, dx = \int_\Omega (u + f) v \, dx \quad \forall v \in H_0^1(\Omega)$$

$$\psi_1 \le y \le \psi_2 \qquad \text{a.e. in } \Omega$$

$$\phi_1 \le u \le \phi_2 \qquad \text{a.e. in } \Omega$$

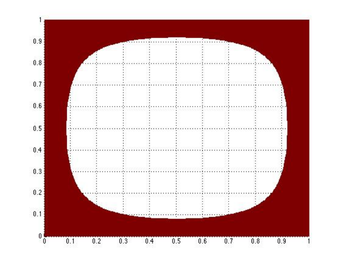
$$\Omega = (0,1)^2 \quad \beta = 0.1 \qquad y = \sin(\pi x_1)\sin(\pi x_2)$$

$$\phi_1 = 0 \qquad \phi_2 = 100 \qquad \psi_2 = +\infty$$

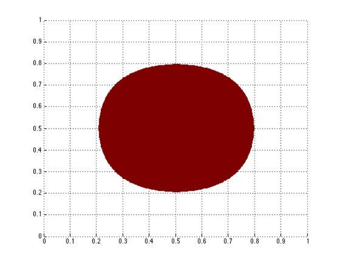
$$\psi_1 = \begin{cases} y & \text{if } y \ge 0.6 \\ 2y - 0.6 & \text{if } y < 0.6 \end{cases}$$

h	$\|y - y_h\|_h$	order	$\|y - y_h\|_1$	order	$\|y - y_h\|_{L^\infty(\Omega)}$	order
1/2	5.4755×10^{-0}	–	5.9315×10^{-1}	–	1.7941×10^{-1}	–
1/4	3.5980×10^{-0}	0.73	2.4522×10^{-1}	1.54	4.0612×10^{-2}	2.59
1/8	1.9986×10^{-0}	0.93	7.6709×10^{-2}	1.84	9.4203×10^{-3}	2.31
1/16	1.0542×10^{-0}	0.97	2.4978×10^{-2}	1.70	3.1295×10^{-3}	1.67
1/32	5.3380×10^{-1}	1.00	7.9547×10^{-3}	1.69	9.9299×10^{-4}	1.69
1/64	2.6651×10^{-1}	1.01	2.3288×10^{-3}	1.79	3.0314×10^{-4}	1.73
1/128	1.3249×10^{-1}	1.01	6.1150×10^{-4}	1.94	8.4879×10^{-5}	1.85
1/512	6.6049×10^{-2}	1.01	1.5926×10^{-4}	1.95	2.1254×10^{-5}	2.00

Table: Error and orders of convergence of state for Morley element



Lower control active set



Lower state active set

Example 3 (Rösch-Wachsmuth, 2012)

$$J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 \; dx + \frac{\beta}{2}\int_\Omega u^2 \; dx$$

subject to constraints:

$$\int_\Omega \nabla y \cdot \nabla v \; dx + \int_\Omega yv \; dx = \int_\Omega uv \; dx \quad \forall v \in H_0^1(\Omega)$$

$$\psi_1 \leq y \leq \psi_2$$

$$\phi_1 \leq u \leq \phi_2$$

Find

$y \in K = \{v \in H^2(\Omega) \cap H_0^1(\Omega) : \psi_1 \leq v \leq \psi_2, \quad \phi_1 \leq -\Delta v + v \leq \phi_2\}$

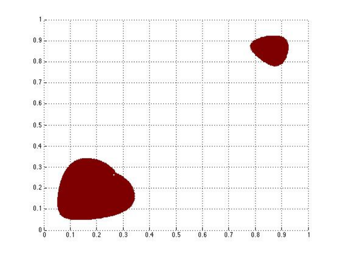
considering the following data

$$\Omega = (0,1)^2 \qquad \beta = 1e-3 \qquad y_d = 10(1-x-y)^3$$

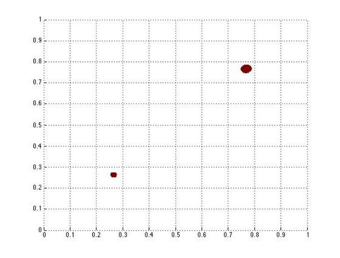
$$\psi_1 = -0.35 \quad \psi_2 = 0.4 \qquad \phi_1 = -25 \qquad \phi_2 = 20$$

h	$\|y - y_h\|_h$	order	$\|y - y_h\|_1$	order	$\|y - y_h\|_{L^\infty(\Omega)}$	order
1/4	$1.4925 \times 10^{+1}$	–	1.6597×10^{-0}	–	3.7484×10^{-1}	–
1/8	$1.2723 \times 10^{+1}$	0.28	8.8981×10^{-1}	1.09	1.6277×10^{-1}	1.45
1/16	9.2676×10^{-0}	0.50	3.2279×10^{-1}	1.61	1.1957×10^{-1}	0.49
1/32	6.7719×10^{-0}	0.47	1.1236×10^{-1}	1.59	4.0609×10^{-2}	1.63
1/64	3.4895×10^{-0}	0.98	2.8264×10^{-2}	2.04	9.7672×10^{-3}	2.10
1/128	1.7613×10^{-0}	1.00	7.3155×10^{-3}	1.97	2.5029×10^{-3}	1.99
1/256	8.8010×10^{-1}	1.01	1.8193×10^{-3}	2.02	6.7383×10^{-4}	1.90
1/512	4.3941×10^{-1}	1.01	4.7584×10^{-4}	1.94	1.6125×10^{-4}	2.07

Table: Error and orders of convergence of state for Morley element



Lower and upper control active set



Lower and upper state active set

Problem 1

$$\min_{(y,u)\in H_0^1(\Omega)\times L_2(\Omega)} \quad J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2\ dx + \frac{\beta}{2}\int_\Omega u^2\ dx$$

subject to constraints:

$$-\Delta y + y = u \quad \text{in} \ \ \Omega$$

$$\psi_1 \le y \le \psi_2, \quad \phi_1 \le u \le \phi_2 \quad \text{a.e. in} \ \ \Omega$$

Problem 2

$$\min_{(y,u)\in H_0^1(\Omega)\times L_2(\Omega)} \quad J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2\ dx + \frac{\beta}{2}\int_\Omega (u-u_d)^2\ dx$$

subject to constraints:

$$-\Delta y = u + f \quad \text{in} \ \ \Omega$$

$$\psi_1 \le y \le \psi_2, \quad \phi_1 \le u \le \phi_2 \qquad \text{a.e. in} \ \ \Omega$$

Post-processing

Procedure 1

$$u = -\Delta y, \quad u_h = -\Delta_h y_h$$

- The control error $\|u - u_h\|_{L_2(\Omega)}$ have the same convergence behaviour as state error in energy norm i.e. $\|y - y_h\|_h$

Procedure 2

$$\int_\Omega \nabla y \cdot \nabla v \; dx = \int_\Omega uv \; dx \quad \forall v \in H_0^1(\Omega)$$

because $u = -\Delta y$.
Discrete approximation

For Morley FEM: Find $u_h \in V_h^1$ such that

$$\int_\Omega u_h v_h \; dx = \sum_{T \in \mathcal{T}_h} \int_T \nabla y_h \cdot \nabla v_h \; dx \quad \forall v_h \in V_h^1$$

$$V_h^1 = \{v \in C^0(\bar{\Omega}) : v|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h, v = 0 \text{ on } \partial\Omega\}$$

Error Estimate

Theorem

There exists a positive constant C independent of h such that

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch^{\min(1-\epsilon,\alpha)}$$

Theorem

There exists a positive constant C independent of h such that

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch^{\min(1-\epsilon,\alpha)}$$

H^1 Estimate

$$\begin{aligned}\|u - u_h\|_{H^1(\Omega)} &\leq |u - \Pi_h u|_{H^1(\Omega)} + |\Pi_h u - u_h|_{H^1(\Omega)} \\ &\leq |u - \Pi_h u|_{H^1(\Omega)} + Ch^{-1}\|\Pi_h u - u_h\|_{L_2(\Omega)} \\ &\leq |u - \Pi_h u|_{H^1(\Omega)} + Ch^{-1}(\|\Pi_h u - u_h\|_{L_2(\Omega)} + \|u - u_h\|_{L_2(\Omega)}\end{aligned}$$

Example 1

$$J(y,u) = \frac{1}{2}\int_\Omega (y - y_d)^2 \ dx + \frac{\beta}{2}\int_\Omega u^2 \ dx$$

subject to constraints:

$$\int_\Omega \nabla y \cdot \nabla v \ dx = \int_\Omega uv \ dx \quad \forall v \in H_0^1(\Omega)$$

$$\psi_1 \le y \le \psi_2, \quad \phi_1 \le u \le \phi_2$$

with the following data:

$$\Omega = (0,1)^2 \quad \beta = 1e-3 \quad y_d = 2$$

$$\psi_1 = -\infty \quad \psi_2 = 1 \quad \phi_1 = -1 \quad \phi_2 = 25$$

h	$\|u-u_h\|_1$	order	$\|u-u_h\|_{L^2(\Omega)}$	order
1/2	$1.1425\ \times 10^{+2}$	–	8.8227×10^{-0}	–
1/4	$1.0700\times 10^{+2}$	0.11	6.9196×10^{-0}	0.42
1/8	$6.6056\times 10^{+1}$	0.76	2.2119×10^{-0}	1.81
1/16	$3.8279\times 10^{+1}$	0.82	7.5924×10^{-1}	1.62
1/32	$2.9417\times 10^{+1}$	0.39	3.0341×10^{-1}	1.35
1/64	$2.0410\times 10^{+1}$	0.53	1.0478×10^{-1}	1.55
1/128	$1.3183\times 10^{+1}$	0.63	3.5453×10^{-2}	1.57

Table: Control errors and orders of convergence of procedure 2 for Morley FEM

- Developed a convergence analysis of Morley finite element method for elliptic optimal control problems with state and control constraints. The idea is to reduce the minimization problem into fourth order variational inequality and then use the complementarity form of the variational inequality.
- Obtained convergence of the state error in H^2 type norm
- Discussed post-processing methods to obtain the approximation of the control variable
- Presented numerical results for the proposed finite element method

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Thank You