Measure of the compactifying effect of conservation laws

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I. Introduction. Systems of conservation laws

- We consider the class of hyperbolic systems of conservation laws :

$$u_t + f(u)_x = 0 \text{ for } (t,x) \in \mathbb{R}^+ \times \mathbb{R},$$
$$u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^d \text{ and } f : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d,$$

where for any $u \in \Omega$:

$$df(u) \text{ has } d \text{ distinct and real eigenvalues } \lambda_1 < \cdots < \lambda_n.$$

Denote $(r_i(u))$ a family of corresponding eigenvectors of $df(u)$.

- This class appears in many applications : gas dynamics (Euler equations), shallow-water flows (Saint-Venant equations), chromatography, traffic flows, etc.
- It is frequent to add conditions the characteristic fields. One says that (λ_i, r_i) is genuinely nonlinear/linearly degenerate when :

$$\forall u \in \Omega,\ r_i(u) \cdot \nabla \lambda_i(u) \neq 0 \ / \ \forall u \in \Omega,\ r_i(u) \cdot \nabla \lambda_i(u) = 0.$$

A (much) simpler particular case : scalar conservation laws

- (1-D) scalar conservation laws correspond to $d = 1$:

$$u_t + f(u)_x = 0 \text{ for } (t,x) \in \mathbb{R}^+ \times \mathbb{R},$$
$$u_{|t=0} = u_0 \text{ on } \mathbb{R},$$

  where

$$u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \text{ and } f : \mathbb{R} \to \mathbb{R}.$$

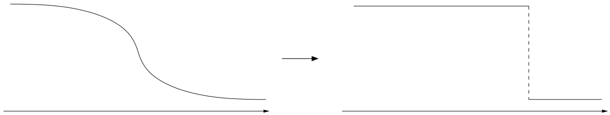
- We will mainly consider the case where the flux f is of class C^2 (though frequently, Lipschitz is sufficient).

- Genuine nonlinearity condition is transformed here into strict uniform convexity :

$$f'' \geq a > 0.$$

Singularities, entropy conditions

- It is quite classical (and easy to see using characteristics) that in general the solutions of this equation become singular in finite time.



- It is hence natural to consider possibly discontinuous weak solutions. But in this framework uniqueness is lost.

Entropy conditions

- One introduces then entropy conditions :

1. Vanishing viscosity condition : one requires that solutions can be obtained by vanishing viscosity : u is limit of u^ε, $\varepsilon \to 0^+$, where :

$$u_t^\varepsilon + f(u^\varepsilon)_x - \varepsilon u_{xx}^\varepsilon = 0.$$

2. One introduces the entropy couples $(\eta, q) : \Omega \to \mathbb{R}^2$ as functions that satisfy :

$$dq = d\eta \, df.$$

One requires that for all (η, q) with η convex, u satisfies :

$$\eta(u)_t + q(u)_x \leq 0 \text{ in the sense of measures.}$$

3. Conditions on the speed of propagation of discontinuities. Given a discontinuity separating u_l on the left and u_r on the right, moving at speed s given by Rankine-Hugoniot relations :

$$f(u_r) - f(u_l) = s(u_r - u_l),$$

one introduces Lax's inequalities :

$$\lambda_i(u_r) < s < \lambda_i(u_l),$$

so in the convex scalar case this gives :

$$u_l \geq u_r.$$

- All these conditions are essentially equivalent in the convex/GNL case.
- Regular solutions are in particular entropy solutions.

A general question

- Some of these systems present a form of nonlinear regularization mechanism.
- Many references on the subject in the scalar case : Lax, Dafermos, Lions-Perthame-Tadmor, Jabin-Perthame, De Lellis-Westdickenberg, Cheverry, etc.
- The goal of this talk is not to prove of a new regularization property, but to try to describe the compactification effect of this type of equations, which is of course connected to this regularizing effect.

II. Simplest case : convex scalar equations

- Different authors, in particular E. Hopf, P.D. Lax and O. Oleinik, have shown global existence and uniqueness of an entropy solution for initial data in $L^1 \cap L^\infty$ (or even L^1), with

$$\|u(t)\|_{L^1} \leq \|u(0)\|_{L^1}, \ \|u(t)\|_{L^\infty} \leq \|u(0)\|_{L^\infty} \text{ and } TV(u(t)) \leq TV(u(0)).$$

- Moreover, P.D. Lax has shown a regularizing effect of the associated nonlinear semi-group $S(t)$. More precisely, given a bounded set $B \subset L^1(\mathbb{R})$ and $R > 0$, one has :

$$\left\{(S(t)u_0)_{|(-R,R)}, \ u_0 \in B\right\} \text{ is relatively compact in } L^1(-R,R).$$

- The following question was raised by P.D. Lax in 2002 :

  Is it possible to give a quantitative estimate of this regularizing effect ?

- In 2005, C. De Lellis and F. Golse gave an answer to this question by using the notion of ε-entropy (a.k.a. Kolmogorov's entropy).

Kolmogorov's entropy

Definition

Let (X, d) a metric space, and let K a totally bounded subset of X.

We call an ε-covering of K, a covering of K by subsets of diameter no more than 2ε.

Let $N_\varepsilon(K)$ the minimal number of subsets in an ε-covering of K. The ε-entropy of K is defined as

$$H_\varepsilon(K \,|\, X) \doteq \log_2 N_\varepsilon(K).$$

Example. $H_\varepsilon([0, L]^n \,|\, \mathbb{R}^n) \sim -n \log_2(\varepsilon)$ as $\varepsilon \to 0^+$ (whatever L and the norm...)

Higher bound for the ε-entropy

Theorem (De Lellis-Golse, 2005)

For $L > 0$, $m > 0$ and $M > 0$, one defines

$$\mathcal{C}_{L,m,M} := \left\{ u_0 \in L^\infty(\mathbb{R}) \,/\, Supp\, u_0 \subset [-L, L],\ \|u_0\|_{L^1} \leq m,\ \|u_0\|_{L^\infty} \leq M \right\}.$$

Then for $T > 0$ and $\varepsilon > 0$ sufficiently small, the ε-entropy of $S(T)\mathcal{C}_{L,m,M}$ in $L^1(\mathbb{R})$ satisfies

$$H_\varepsilon(S(T)\mathcal{C}_{L,m,M} \mid L^1(\mathbb{R})) \leq \frac{4}{\varepsilon} \left(\frac{4L(T)^2}{a\,T} + 4L(T)\sqrt{\frac{2m}{a\,T}} \right),$$

with

$$L(T) \doteq L + 2\,c_M\,\sqrt{2mT/a} \quad \text{where} \quad c_M \doteq \max_{[-M,M]} f''.$$

(Reminder : a is such that $f'' \geq a > 0$.)

Above, $L(T)$ is an estimate of the support width at time T.

Lower bound for the ε-entropy

Theorem (Ancona-G.-Nguyen, 2012)

For $L > 0$, $m > 0$ and $M > 0$, one defines as before

$$\mathcal{C}_{L,m,M} \doteq \{u_0 \in L^\infty(\mathbb{R}) \;/\; Supp\, u_0 \subset [-L, L],\ \|u_0\|_{L^1} \leq m,\ \|u_0\|_{L^\infty} \leq M\}.$$

Then for $T > 0$ and $\varepsilon > 0$ sufficiently small, the ε-entropy of $S(T)\mathcal{C}_{L,m,M}$ in $L^1(\mathbb{R})$ satisfies

$$H_\varepsilon(S(T)\mathcal{C}_{L,m,M} \mid L^1(\mathbb{R})) \geq \frac{1}{\varepsilon} \frac{L^2}{48 \ln(2) \, |f''(0)| \, T}.$$

Remarks

- As a consequence one has

$$H_\varepsilon(S(T)\mathcal{C}_{L,m,M} \mid L^1(\mathbb{R})) \approx \frac{1}{\varepsilon}.$$

- A motivation for P.D. Lax's question is numerical analysis of these equations. Indeed, the result above gives an idea of the complexity of a numerical scheme for such an equation (whatever its nature).

  A scheme with precision ε in L^1 norm must use at least $\mathcal{O}(\frac{1}{\varepsilon})$ operations...

III. Extensions.

1. Conservation laws with source term

- A generalization of scalar conservation laws consist in scalar conservation laws with source term :

$$u_t + f(u)_x = g(t, x, u),$$

  where f is as before and g is a source term of class C^1, with at most linear growth at infinity.

- Under reasonable assumptions, S. N. Kruzkov has shown global existence and uniqueness of an entropy solution for initial data $u_0 \in L^\infty$. (Kruzkov's result is actually much more general !)

- One can have in mind a flow in presence of external force, in non-flat channels, etc.

- We denote $E(t)$ the evolution operator which maps u_0 into $u(t)$.

Assumptions

In what follows one supposes that :

$$\forall\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \qquad g(t,x,0) = 0,$$
$$\exists\ C > 0 \text{ t.q. } \forall\ (t,x,u) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R},\ |g_x(t,x,u)| \le C|u|,$$
$$\exists\ \omega \in L^1_{loc}(\mathbb{R}^+) \text{ t.q. p.p. tout } t \in \mathbb{R}^+,\ \forall\ (x,u) \in \mathbb{R}^2,\ |g_u(t,x,u)| \le \omega(t).$$

The first condition ensures that for a compactly supported initial data, the corresponding solution remains compactly supported for all times.

It can be replaced in what follows by : g is independent of x and

$$g(\cdot,0) \in L^1_{loc}(\mathbb{R}^+),$$

and obtain a similar result.

Higher ε-entropy bound for conservation laws with source term

Theorem (ibid.)

Under the above assumptions, for $T > 0$ and for $\varepsilon > 0$ sufficiently small, one has :

$$H_\varepsilon\big(E(T)(\mathcal{C}_{L,m,M}) \,|\, L^1(\mathbb{R})\big) \leq \frac{1}{\varepsilon}\,\frac{8\,L_T^2\,\Big(1 + 2(1 + aT^2K)\exp\big(\|\omega\|_{L^1}\big)\Big)}{aT},$$

where

$$K \doteq \max\big\{|g_x(s,x,u)|\,;\ (s,x) \in \Delta,\ u \in [-M_T, M_T]\big\},$$

with

$$M_T \doteq \exp\big(\|\omega\|_{L^1}\big)\,M, \quad L_T \doteq L + \|f''\|_{L^\infty(-M_T,\,M_T)}\,M_T\,T,$$

$$\Delta \doteq \Big\{(s,x) \,|\, s \in [0,T],$$
$$-L_T - (T-s)\,\|f'\|_{L^\infty(-M_T,M_T)} \leq x \leq L_T + (T-s)\,\|f'\|_{L^\infty(-M_T,M_T)}\Big\}.$$

Lower ε-entropy bounds for conservation laws with source term

Theorem (ibid.)

Under the above assumptions, for $T > 0$ and for $\varepsilon > 0$ sufficiently small, one has :

$$H_\varepsilon\big(E(T)(\mathcal{C}_{L,m,M}) \,|\, L^1(\mathbb{R})\big) \geq \frac{1}{\varepsilon}\,\frac{L^2\,\exp\big(-\|\omega\|_{L^1}\big)}{48\,\ln(2)\,|f''(0)|\,T}.$$

Remark

Hence in that case also $H_\varepsilon\big(E(T)(\mathcal{C}_{L,m,M})\big) \approx \frac{1}{\varepsilon}$.

2. Nonconvex conservation laws

- Now we consider the nonconvex case. In this situation, we use instead the following nondegeneracy condition : $f : \mathbb{R} \to \mathbb{R}$ is a smooth, non convex function with a single inflection point at zero having polynomial degeneracy, i.e. such that

$$f^{(j)}(0) \;=\; 0 \quad \text{for all} \quad j = 2, \ldots, m, \qquad f^{(m+1)}(0) \;\neq\; 0\,,$$

$$f''(u) \cdot u \cdot \mathrm{sign}\big(f^{(m+1)}(0)\big) \;>\; 0 \qquad \forall\; u \in \mathbb{R} \setminus \{0\}.$$

- In this nonconvex situation, the entropy condition becomes at the level of a discontinuity (u_ℓ, u_r) :

$$\frac{f(u_\ell) - f(u)}{u_\ell - u} \;\geq\; \frac{f(u_r) - f(u)}{u_r - u}$$

for every u between u_ℓ and u_r. (Oleinik's E-condition)

Nonconvex conservation laws, continued

Theorem (Ancona-G.-Nguyen, 2019)

For any given $L, M, T > 0$, and for every $\varepsilon > 0$ sufficiently small, the following estimates hold :

$$\mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{L,M}) \mid \mathbf{L}^1(\mathbb{R})\Big) \leq \Gamma_2^+ \cdot \frac{1}{\varepsilon^m}\,, \tag{1}$$

$$\mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{L,M}) \mid \mathbf{L}^1(\mathbb{R})\Big) \geq \Gamma_2^- \cdot \frac{1}{\varepsilon^m}\,, \tag{2}$$

where

$$\mathcal{C}_{L,M} \doteq \big\{u_0 \in L^\infty(\mathbb{R}) \;/\; Supp\, u_0 \subset [-L, L],\ \|u_0\|_{L^\infty} \leq M\big\},$$

$$\Gamma_2^+ = c_2\left(1 + L + T + \frac{L^2}{T}\right)^{m+1}$$

$$\Gamma_2^- = c_2 \cdot \frac{L^{m+1}}{T}.$$

for some constant $c_2 > 0$ depending only on f and M.

3. Systems of conservation laws

- Now we consider systems of conservation laws. Here the functional framework is different, and the standard one considers solutions with (small) total variation in space.
- This goes back to Glimm (1965), and then T.P. Liu, Bianchini-Bressan, etc.
- In that case, one can define a semigroup $S : [0, \infty[\times \mathcal{D}_0 \to \mathcal{D}_0$ defined on a closed domain $\mathcal{D}_0 \subset L^1(\mathbb{R}, \mathbb{R}^N)$, with the properties :

  (i)

$$\Big\{ v \in L^1(\mathbb{R}, \Omega) \, \Big| \, \text{Tot.Var.}(v) \leq \delta_0 \Big\} \subset \mathcal{D}_0 \subset \Big\{ v \in L^1(\mathbb{R}, \Omega) \, \Big| \, \text{Tot.Var.}(v) \leq 2\delta_0 \Big\},$$

  for suitable constant $\delta_0 > 0$.

(ii) For every $\overline{u} \in \mathcal{D}_0$, the semigroup trajectory $t \mapsto S_t\overline{u} \doteq u(t, \cdot)$ provides an entropy weak solution of the Cauchy problem, with initial data

$$u(0, \cdot) = \overline{u},$$

that satisfy

Liu stability condition. A shock discontinuity of the i-th family $(u_\ell,\, u_r)$, traveling with speed $\sigma_i[u_\ell,\, u_r]$, is *Liu admissible* if, for any state u lying on the i-th Hugoniot curve between u_ℓ and u_r, the shock speed $\sigma_i[u_\ell, u]$ of the discontinuity (u_ℓ, u) satisfies

$$\sigma_i[u_\ell,\, u] \geq \sigma_i[u_\ell,\, u_r]\,.$$

Result in the system case

Theorem (Ancona-G.-Nguyen, 2014)

Given any $L, m, M, T > 0$, for any interval $I \subset \mathbb{R}$ of length $|I| = 2L$, and for $\varepsilon > 0$ sufficiently small, the following estimates hold.

(i)

$$H_\varepsilon\Big(S_T\big(\mathcal{C}_{[L,m,M]} \cap \mathcal{D}_0\big) \mid L^1(\mathbb{R},\Omega)\Big) \geq c\,\frac{N^2 L^2}{T} \cdot \frac{1}{\varepsilon},$$

where $c > 0$ is an ugly explicit constant depending on f.

(ii)

$$H_\varepsilon\Big(S_T\big(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0\big) \mid L^1(\mathbb{R},\Omega)\Big) \leq 48N\delta_0 \cdot L_T \cdot \frac{1}{\varepsilon}\,,$$

where

$$L_T \doteq L + \frac{\Delta_\vee \lambda}{2} \cdot T, \qquad \Delta_\vee \lambda \doteq \sup\big\{\lambda_N(u) - \lambda_1(v)\,;\ u, v \in \Omega\big\}.$$

Other results

- Other results concern :
  - strictly convex (but not uniformly strictly convex) scalar equations (op. cit.)
  - Temple systems (op. cit.)
  - Hamilton-Jacobi equations (Ancona-Cannarsa-Nguyen, 2015)

IV. Ideas of proof (scalar convex case).
Higher ε-entropy bounds

- Let us begin by briefly describing De Lellis and Golse's proof of the conservative case.
- We cite two important ingredients in the proof.
- On the one side, one has the following L^1-to-L^∞ estimate :

Proposition (Lax)

If $f'' \geq a > 0$, for $u_0 \in L^1(\mathbb{R})$ and $t > 0$, one has :

$$\|S(t)u_0\|_{L^\infty} \leq \sqrt{\frac{2\|u_0\|_1}{a\,t}}.$$

- On the other side, another ingredient is Oleinik's inequality :

Theorem (Oleinik)

If $f'' \geq a > 0$, for all $u_0 \in L^\infty(\mathbb{R})$, one has, denoting $u(t,\cdot) = S(t)u_0$:

$$\forall t > 0,\ \forall x < y, \quad \frac{u(t,y) - u(t,x)}{y - x} \leq \frac{1}{a\,t}.$$

(In particular u is locally BV for $t > 0$.)

- One can see that the first ingredient can be deduced from the second one.
- A way to prove these two results is to use Lax-Oleinik's formula giving an explicit (yet nontrivial) form to solution of convex scalar conservation laws scalaires.

- One deduces from what precedes and from the finite propagation speed that

$$S(T)\mathcal{C}_{L,m,M} \subset \Big\{ u_T \in L^1(\mathbb{R}) \ / \ \|u_T\|_{L^1} \le m, \ \|u_T\|_{L^\infty} \le \sqrt{\frac{2m}{a\,T}},$$
$$\mathrm{Supp}(u_T) \subset [-L - 2c_M\sqrt{2mT/a}, L + 2c_M\sqrt{2mT/a}],$$
$$(u_T)_x \le \frac{1}{a\,T} \Big\}.$$

- In particular, denoting $q : x \mapsto x/aT$, one has

$$q - S(T)\mathcal{C}_{L,m,M} \subset \mathcal{J}_{\overline{L},\overline{V}}$$
$$\doteq \Big\{ w : [-\overline{L}/2, \overline{L}/2] \to [-\overline{V}/2, \overline{V}/2], \ w \text{ non-decreasing} \Big\},$$

for $\overline{L}$ and $\overline{V}$ that can easily be computed.

- After translation, we are hence interested in the ε-entropy of :

$$\mathcal{I}_{\overline{L},\overline{V}} \doteq \Big\{ w : [0,\overline{L}] \rightarrow [0,\overline{V}],\ w \text{ non-decreasing} \Big\}.$$

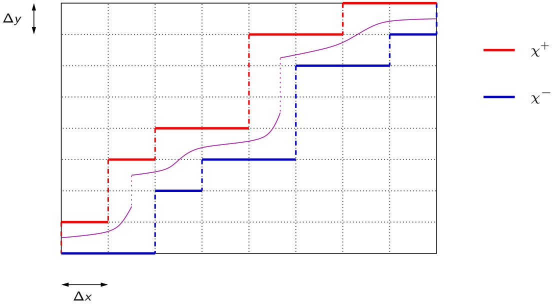
- Consequently the result is a consequence of :

Lemme (De Lellis-Golse)

For $0 \leq \varepsilon < \frac{\overline{L}\,\overline{V}}{6}$, one has :

$$H_\varepsilon(\mathcal{I}_{\overline{L},\overline{V}} \,|\, L^1(0,\overline{L})) \leq 4 \left\lfloor \frac{\overline{L}\,\overline{V}}{\varepsilon} \right\rfloor.$$

- One introduces $N \in \mathbb{N} \setminus \{0\}$, $\Delta x \doteq \overline{L}/N$ and $\Delta y \doteq \overline{V}/N$.
- One considers suitable non-decreasing step-functions χ on this grid :



- One introduces the subsets U consisting in non-decreasing functions between two such step-functions χ_- and χ^+ satisfying

$$\chi^-(k\Delta x) \leq \chi^+(k\Delta x) \leq \chi^-((k+1)\Delta x) + \Delta y.$$

- Choosing N so that these subsets are of diameter $\leq 2\varepsilon$ and counting these subsets, we reach the result.

V. Ideas of proof in the convex scalar case.
Lower ε-entropy bounds

- To establish a lower bound on $H_\varepsilon(S(T)\mathcal{C}_{L,m,M} \mid L^1(\mathbb{R}))$, we cut the proof in two parts :
  - We look for a class of functions $\mathcal{A}_T$, of simple form, and such that

    $$\mathcal{A}_T \subset S(T)\mathcal{C}_{L,m,M}.$$

  - One introduces next a finite family $\mathcal{I}$ of functions of $\mathcal{A}_T$, of cardinal N large enough, and such that for each $\overline{f} \in \mathcal{I}$,

    $$\text{Card}\,\{f \in \mathcal{I} \;/\; \|f - \overline{f}\|_{L^1} \leq 2\varepsilon\} \doteq \tilde{N}(\overline{f}),$$

    is sufficiently small. We can then conclude that the minimal number of parts in a ε-covering satisfies :

    $$N_\varepsilon \geq \frac{N}{\max_{\overline{f}} \tilde{N}_{\overline{f}}}.$$

This last point uses arguments from Bartlett-Kulkarni-Posner (1997).

Part 1. Description of certain attainable states

- We know that states of the system at time T, associated to an initial data in $\mathcal{C}_{L,m,M}$, satisfy naturally an L^1 estimate, an L^∞ estimate, Oleinik's inequality, and are compactly supported.
- A first idea is to show that, changing the constants if necessary, one can reach states that satisfy these conditions.
- More precisely, one has the following result.

Proposition

For $L, m, M, b > 0$, we fix :

$$\mathcal{A}_{[L,m,M,b]} \doteq \Big\{ u_T \in BV(\mathbb{R}) \mid Supp\,(u_T) \subset [-L, L], \\ \|u_T\|_{L^1} \leq m,\ \|u_T\|_{L^\infty} \leq M,\ Du_T \leq b \Big\},$$

Then for $h > 0$ sufficiently small, one has :

$$\mathcal{A}_{[L_T,\, 2Lh,\, h,\, (2T|f''(0)|)^{-1}]} \subset S(T)(\mathcal{C}_{L,m,M}),$$

where

$$L_T \doteq L - 2T|f''(0)|\, h.$$

Attainable states, continued

Remark

In the above statement, h is small, but not very small. If one replaces

$$\mathcal{A}_{[L_T,\,2Lh,\,h,\,(2T|f''(0)|)^{-1}]} \subset S(T)(\mathcal{C}_{L,m,M}),$$

with

$$\mathcal{A}_{[L_T,\,2Lh,\,h,\,(T\|f''\|_\infty)^{-1}]} \subset S(T)(\mathcal{C}_{L,m,M}) \text{ with } L_T \doteq L - T\|f''\|_\infty\, h,$$

the only constraint on h is $h \leq M$ and $Lh \leq m$.
But the above formula yields a better estimate in the end.

Ideas of proof.

- To prove this resultat, one shows in a first time that

$$\mathcal{A}_{[L_T,\,2Lh,\,h,\,(2T|f''(0)|)^{-1}]} \cap C^1(\mathbb{R}) \subset S(T)(\mathcal{C}_{L,m,M}),$$

- For $u_T \in \mathcal{A}_{[L_T,\, 2Lh,\, h,\, (2T|f''(0)|)^{-1}]} \cap C^1(\mathbb{R})$, one applies the local existence theory in C^1 to the initial data $u_T(-x)$.
- If one shows that the corresponding solution w exists in C^1 (without blow-up) until time T and that $w(T, -x) \in \mathcal{C}_{L,m,M}$, by invariance of the regular solutions with respect to

$$(t, x) \to (T - t, -x),$$

  one has etablished $u_T \in S(T)(\mathcal{C}_{L,m,M})$.
- The question becomes : use the assumptions on u_T to prove that the solution remains regular till $t = T$.
- It suffices to show that

$$w_x \text{ remains bounded in } L^\infty(\mathbb{R}) \text{ on any compact of } [0, T).$$

- Denoting $v \doteq w_x$, we have the equation :

$$v_t(t,x) + f'(w(t,x)) \cdot v_x(t,x) = -f''(w(t,x)) \cdot v(t,x)^2$$

- Along characteristics $x(t)$ associated to $f'(w(t,x))$, $z(t) \doteq v(t,x(t))$ satisfies

$$\dot{z}(t) = -f''(w(t,x(t))) \cdot z^2(t).$$

- It suffices to establish a lower bound for z. Oleinik's condition gives estimates on $(z(0))_-$.
- With the a priori estimates on w in L^∞, one sees that this suffices to avoid the blow up of v in C^1 before time T.
- One finally deduces by a density argument that

$$\mathcal{A}_{[L_T,\, 2Lh,\, h,\, (2T|f''(0)|)^{-1}]} \subset S(T)(\mathcal{C}_{L,m,M}),$$

thanks to the classical property of L^1 contraction of the semi-group $S(t)$:

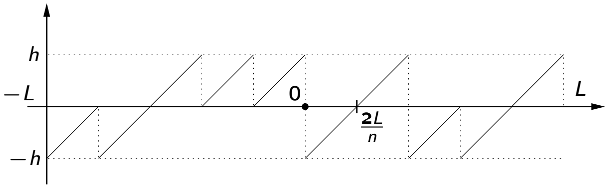
$$\|S(T)u_0 - S(T)\tilde{u}_0\|_{L^1} \leq \|u_0 - \tilde{u}_0\|_{L^1}.$$

Part 2. Description of the finite family $\mathcal{I}$

- We consider h as in the above proposition.
- One introduces for $n \geq 2$, the family of functions $\mathcal{F}_\iota : \mathbb{R} \to [-h, h]$ for $\iota \in \{-1, 1\}^n$, supported in $[-L, L]$ and defined in $[-L, L]$ by

$$\mathcal{F}_\iota(x) = \begin{cases} \frac{hn}{2L}\left(x + L - k\frac{2L}{n}\right) & \text{if } \iota_k = 1, \\ \frac{hn}{2L}\left(x + L - (k+1)\frac{2L}{n}\right) & \text{if } \iota_k = -1, \end{cases}$$

for $x \in \left[-L + k\frac{2L}{n}, -L + (k+1)\frac{2L}{n}\right)$, and $k \in \{0, \dots, n-1\}$.



(The example corresponds to $n = 10$ and $\iota = (-1, -1, 1, 1, 1, -1, 1, -1, -1, 1)$)

- The functions $\mathcal{F}_\iota$ belong to $\mathcal{A}_{[L,\, 2Lh,\, h,\, b]}$ as soon as :

$$\frac{nh}{2L} \leq b.$$

- Clearly, there are 2^n such functions.
- It remains to estimate, fixed $\bar{\iota} \in \{-1, 1\}^n$, the number of functions $\mathcal{F}_\iota$ such that :

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar{\iota}}\|_{L^1} \leq 2\varepsilon.$$

- But

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar{\iota}}\|_{L^1} = \frac{2hL}{n} \operatorname{Card}\left\{k \in \{1, \dots, n\} \mid \iota_k \neq \bar{\iota}_k\right\}.$$

- We want to count $\iota \in \{-1, 1\}^n$ such that

$$\operatorname{Card}\left\{k \in \{1, \dots, n\} \mid \iota_k \neq \bar{\iota}_k\right\} \leq \frac{n\varepsilon}{hL}.$$

Remark that this cardinal doe not depend on $\bar{\iota}$. Call it $\mathcal{C}(\varepsilon)$.

- The number of ι differing from $\bar{\iota}$ for exactly k indices is $\binom{n}{k}$.
- It follows that

$$\mathcal{C}(\varepsilon) = \sum_{k=0}^{\lfloor \frac{n\varepsilon}{hL} \rfloor} \binom{n}{k}.$$

- We can interpret the right-hand side in terms of a random walk in an elementary manner.
- If $X_1, \ldots, X_n$ are i.i.d. Bernoulli variables with $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$, then for all $\ell \leq n$ one has :

$$\mathbb{P}(X_1 + \cdots + X_n \leq \ell) = \frac{1}{2^n} \sum_{k=0}^{\ell} \binom{n}{k}.$$

- We set $S_n = X_1 + \cdots + X_n$. One uses Chernoff-Hoeffding's inequality : for $\mu > 0$,

$$\mathbb{P}(S_n - \mathbb{E}(S_n) \leq -\mu) \leq \exp\left(-\frac{2\mu^2}{n}\right),$$

- We suppose (since ε is small !) that :

$$\frac{n\varepsilon}{hL} < \frac{n}{2},$$

and we choose

$$\mu = \frac{n}{2} - \left\lfloor \frac{n\varepsilon}{hL} \right\rfloor.$$

- We obtain

$$\frac{1}{2^n}\mathcal{C}(\varepsilon) \leq \exp\left(-2\frac{\left(\frac{n}{2} - \lfloor \frac{n\varepsilon}{hL} \rfloor\right)^2}{n}\right) \leq \exp\left(-\frac{n}{2}\left(1 - \frac{\varepsilon}{hL}\right)^2\right).$$

- It remains to minimize the expression

$$\exp\left(-\frac{n}{2}\left(1-\frac{\varepsilon}{hL}\right)^2\right),$$

  with respect to n and h under the constraint

$$\frac{nh}{2L} \le b \ \text{ and } \ \frac{n\varepsilon}{hL} \le \frac{n}{2}.$$

- After computation we obtain

$$\frac{\mathcal{C}(\varepsilon)}{2^n} \le \exp\left(-\frac{1}{\varepsilon}\frac{4bL^2}{27}\right).$$

- The result follows.

Thank you for your attention !