Controlllability of low Reynolds numbers swimmers of ciliated type

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VIII Partial differential squations, optimal design and numerics
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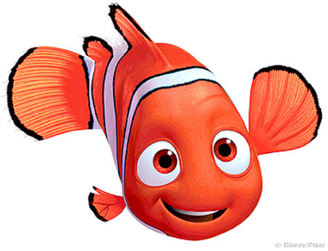
Motivations

Definition (To swimm)

To propel oneself in water by natural means (such as movements of the limbs, fins, or tail).

The action of swimming is seen as a control problem. Given two points, can the fish reach one from another?

The displacement of the fish is due to fluid-structure interactions.



The fluid

The Reynolds number: $Re = \dfrac{\rho U L}{\mu}$.

$Re \ll 1$		$Re \gg 1$ → Re
Stokes Equations	Navier-Stokes Equations	Euler Equations (perfect fluid), turbulence

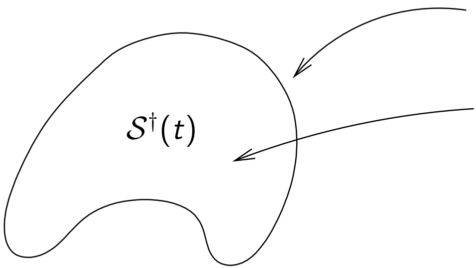
Examples :

	$L\,(cm)$	$U\,(cm.s^{-1})$	$T\,(s)$	Re
Bacteria	10^{-5}	10^{-3}	10^{-4}	10^{-5}
Spermatozoon	10^{-3}	10^{-2}	10^{-2}	10^{-3}
fish	50	100	0.5	5.10^{4}
Pigeon	25	10^{3}	5.10^{-1}	10^{5}

The coupled problem

Fluid domain $\mathcal{F}^\dagger(t)$

Navier-Stokes equation

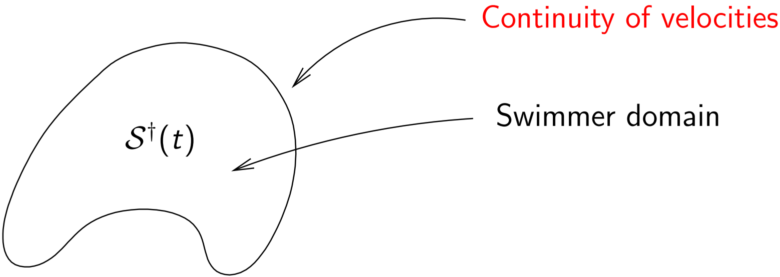


$+$ Newton's principle, $ma = \Sigma F$

The coupled problem

Fluid domain $\mathcal{F}^\dagger(t)$

Steady-state Stokes equation



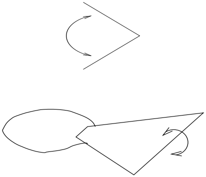
$+$ Forces equilibrium, $0 = \Sigma \boldsymbol{F}$

The deformations I

All deformations are not interesting for the motion...

Theorem (Scallop Theorem, Purcell 1977)

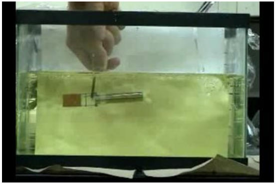
If the swimmer's deformation is reversible in time, then the displacement of the swimmer is null.



No motion

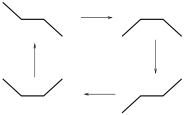
$\Rightarrow$

in Stokes fluid

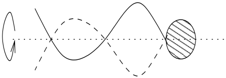


Taylor's experience

The deformations II



Purcell's swimmer



Helical deformation

Motion

$\Rightarrow$

in Stokes fluid

Taylor's experience

References I

- Swimmer models:
  - Experiences:
    - G. Taylor,
      Analysis of the swimming of microscopic organisms, 1951
  - Exhibition of physical specificities:
    - E. M. Purcell,
      Life at low Reynolds number, 1977
    - S. Childress,
      Mechanics of swimming and flying, 1981
    - A. Shapere and F. Wilczek,
      Geometry of self-propulsion at low Reynolds number and
      Efficiencies of self-propulsion at low Reynolds number, 1989

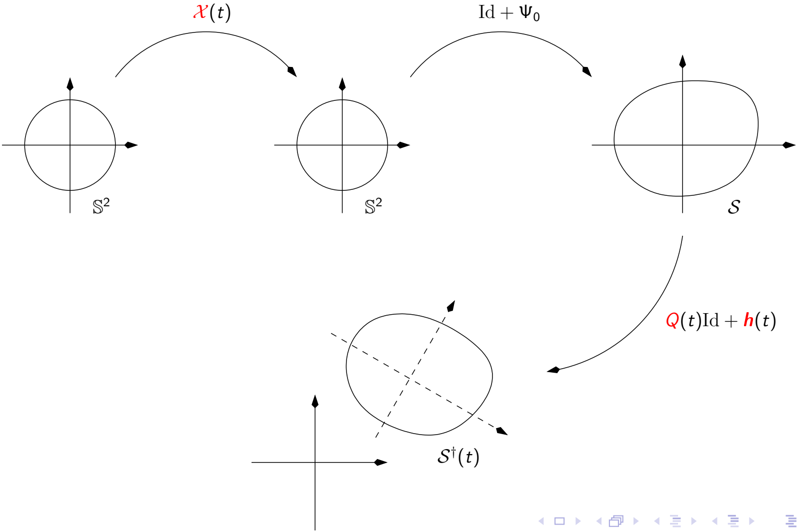
References II

- Controllability results:
  - With Euler fluid:
    - T. Chambrion and A. Munnier,
      Locomotion and control of a self-propelled shape-changing body in a fluid, 2011
  - With Navier-Stokes fluid:
    - ...
  - With Stokes fluid:
    - F. Alouges, A. DeSimone and A. Lefebvre,
      Optimal strokes for low Reynolds number swimmers: an example, 2008
    - J. San Martín, T. Takahashi and M. Tucsnak,
      A control theoretic approach to the swimming of microscopic organisms, 2007
      An optimal control approach to ciliary locomotion, 2016

- Number of controls:
  - J. San Martín, T. Takahashi and M. Tucsnak (2007) $\longrightarrow$ 6
  - F. Alouges, A. DeSimone et al. (2013) $\longrightarrow$ 4
  - J. L. and A. Munnier (2014) $\longrightarrow$ 4

The swimmer mechanism

The swimmer is located by the position of it's mass center $\boldsymbol{h}(t) \in \mathbb{R}^3$ and an angular position $Q(t) \in SO(3)$.



Construction of $\mathcal{X}$

Given $\Theta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)$, we define the map

$$\begin{array}{ccl} \mathbb{S}^2 & \longrightarrow & \mathbb{S}^2 \\ y & \longmapsto & \exp_y(\Theta(y)) = \cos|\Theta(y)|\, y + \operatorname{sinc}|\Theta(y)|\, \Theta(y). \end{array}$$

Consider $\delta = (\delta_1, \ldots, \delta_d) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, for every $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$, we set

$$\mathcal{X}_\delta(s) = \exp_y\left(\sum_{j=1}^{d} s_j \delta_j(y)\right).$$

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$$\mathcal{X}_\delta(s) = \exp_y\left(\sum_{j=1}^d s_j \delta_j(y)\right).$$

Given δ, we define $\tilde{\mathcal{J}}(\delta) \subset \mathbb{R}^d$, the set of $s \in \mathbb{R}^d$ such that $\mathcal{X}_\delta(s)$ is a C^1-diffeomorphism of $\mathbb{S}^2$, and $\mathcal{J}(\delta)$ the connected component of $\tilde{\mathcal{J}}(\delta)$ containing 0.

Lemma

$\mathcal{J}(\delta)$ is a nonempty open subset of $\mathbb{R}^d$.

Hypothesis on Ψ_0

Set

$$C_0^k(\mathbb{R}^3) = \left\{ \varphi \in C^k(\mathbb{R}^3) \ \middle| \ \lim_{|x|\to\infty} \partial_{x_i}^j \varphi(x) = 0, \ i \in \{1,2,3\}, \ j \in \{0,\dots,k\} \right\}$$

endowed with the norm

$$\|\varphi\|_{C_0^k(\mathbb{R}^3)} = \sum_{j=0}^{k} \sum_{i=1}^{3} \sup_{x\in\mathbb{R}^3} |\partial_{x_i}^j \varphi(x)|.$$

We define

$$\tilde{\mathcal{D}}_0^k = \left\{ \Psi \in C_0^k(\mathbb{R}^3)^3 \ \middle| \ \mathrm{Id}_{\mathbb{R}^3} + \Psi \text{ is a } C^1\text{-diffeomorphism of } \mathbb{R}^3 \right\}$$

and $\mathcal{D}_0^k$ the connected component of $\tilde{\mathcal{D}}_0^k$ containing 0.

Lemma

$\mathcal{D}_0^k$ is a nonempty open subset of $C_0^k(\mathbb{R}^3)^3$.

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Lemma

$\mathcal{D}_0^k$ is a nonempty open subset of $C_0^k(\mathbb{R}^3)^3$.

We assume $\Psi_0 \in \mathcal{D}_0^k$, so that the shape of the swimmer is

$$\mathcal{S}_c = (\mathrm{Id}_{\mathbb{R}^3} + \Psi_0)(\mathbb{S}^2),$$

with $c = (\Psi_0, \delta)$ the *swimmer configuration*.

Global deformation

Finally, we set

$$X_c(s) = (\mathrm{Id}_{\mathbb{R}^3} + \Psi_0) \circ \mathcal{X}_\delta(s) \qquad (c = (\Psi_0, \delta),\ s \in \mathcal{J}(\delta)).$$

$X_c(s)$ is a C^k-diffeomorphism from $\mathbb{S}^2$ to $\mathcal{S}_c$.

Proposition

The set

$$\mathcal{A}^k(d) = \left\{(\Psi_0, \delta, s) \in D_0^k \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d \ \mid\ s \in \mathcal{J}(\delta)\right\}$$

is a nonempty connected open set of $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$.

Rigid motion of the swimmer

While immersed in the fluid, the swimmer can rotate and translate. The physical deformation of the swimmer is

$$X^{\dagger}(h, Q, s)(y) = QX_c(s)(y) + h.$$

We also set

$$\mathcal{S}^{\dagger}(h, Q) = Q\mathcal{S}_c + h.$$

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The deformation velocity is then

$$\begin{aligned} v^{\dagger}(t, x) &= \dot{Q}Q(x-h) + \dot{h} + Q\frac{\mathrm{d}X_c(s)}{\mathrm{d}t}\left(X_c(s)^{-1}\left(Q^{\top}(x-h)\right)\right) \\ &= \dot{Q}Q(x-h) + \dot{h} + Q\left(\sum_{j=1}^{d} \dot{s}_j \partial_{s_j} X_c(s)\left(X_c(s)^{-1}\left(Q^{\top}(x-h)\right)\right)\right) \\ & \qquad (x \in \mathcal{S}^{\dagger}(h, Q)). \end{aligned}$$

The coupled problem I

$$
\begin{aligned}
-\Delta u^{\dagger} + \nabla p^{\dagger} &= 0 && \text{in } \mathcal{F}^{\dagger}(h, Q),\\
\operatorname{div} u^{\dagger} &= 0 && \text{in } \mathcal{F}^{\dagger}(h, Q),\\
\lim_{|x|\to\infty} u^{\dagger}(t, x) &= 0, &&\\
u^{\dagger} &= v^{\dagger} && \text{on } \mathcal{S}^{\dagger}(h, Q),\\
\int_{\mathcal{S}^{\dagger}(h,Q)} \sigma(u^{\dagger}, p^{\dagger})\mathrm{n}^{\dagger}\,\mathrm{d}\Gamma &= 0, &&\\
\int_{\mathcal{S}^{\dagger}(h,Q)} (x-h)\times\sigma(u^{\dagger}, p^{\dagger})\mathrm{n}^{\dagger}\,\mathrm{d}\Gamma &= 0, &&
\end{aligned}
$$

with $\sigma(u^{\dagger}, p^{\dagger}) = \nabla u^{\dagger} + \nabla u^{\dagger\top} - p^{\dagger}\mathrm{I}_3$, the Cauchy stress tensor.

The coupled problem II

By the change of variables, $u^{\dagger}(t,x) = Qu\left(t, Q(t)^{\top}(x-h(t))\right)$ and $p^{\dagger}(t,x) = p\left(t, Q(t)^{\top}(x-h(t))\right)$, we obtain,

$$\begin{aligned}
-\Delta u + \nabla p &= 0 && \text{in } \mathcal{F}_c,\\
\operatorname{div} u &= 0 && \text{in } \mathcal{F}_c,\\
\lim_{|x|\to\infty} u(t,x) &= 0, &&\\
u &= \ell + \omega \times x + \sum_{j=1}^{d} \dot{s}_j \partial_{s_j} X_c(s) \circ X_c(s)^{-1} && \text{on } \mathcal{S}_c,\\
\int_{\mathcal{S}_c} \sigma(u,p)\mathrm{n}\, \mathrm{d}\Gamma &= 0, &&\\
\int_{\mathcal{S}_c} x \times \sigma(u,p)\mathrm{n}\, \mathrm{d}\Gamma &= 0, &&
\end{aligned}$$

with

$$\dot{h} = Q\ell \quad \text{and} \quad \dot{Q} = Q\mathbb{A}(\omega),$$

where $\mathbb{A}(\omega)$ is such that $\mathbb{A}(\omega)x = \omega \times x$.

A geometric control problem I

Define for $i \in \{1,2,3\}$ and $j \in \{1,\dots,d\}$ the solutions $\left\{\begin{array}{l} u_c^i \\ u_c^{i+3} \\ v_c^j(s) \end{array}\right.$ of the Stokes system

with boundary conditions $\left\{\begin{array}{rl} u_c^i &= e_i \\ u_c^{i+3} &= e_i \times x \\ v_c^j(s) &= D_c^j(s) = \partial_{s_j} X_c(s) \circ X_c(s)^{-1} \end{array}\right.$ on $\mathcal{S}_c$.

Then we have,

$$u(t,x) = \sum_{i=1}^{3} \ell_i(t) u_c^i(x) + \sum_{i=1}^{3} \omega_i(t) u_c^{i+3}(x) + \sum_{j=1}^{d} \dot{s}_j(t) v_c^j(s(t))(x).$$

A geometric control problem II

Noticing that,

$$\int_{\mathcal{S}_c} e_i \cdot \sigma(u,p)\mathrm{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}_c} \mathrm{D}(u) : \mathrm{D}(u_c^i) \quad \text{and}$$

$$\int_{\mathcal{S}_c} (e_i \times x) \cdot \sigma(u,p)\mathrm{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}_c} \mathrm{D}(u) : \mathrm{D}(u_c^{i+3}) \qquad (i \in \{1,2,3\}),$$

the relations $\int_{\mathcal{S}_c} \sigma(u,p)\mathrm{n}\,\mathrm{d}\Gamma = 0$ and $\int_{\mathcal{S}_c} x \times \sigma(u,p)\mathrm{n}\,\mathrm{d}\Gamma = 0$ read as

$$K_c \begin{pmatrix} \ell \\ \omega \end{pmatrix} = N_c(s)\dot{s},$$

with

$$K_c = 2\left(\int_{\mathcal{F}_c} \mathrm{D}(u^i) : \mathrm{D}(u^j)\right)_{i,j\in\{1,\dots,6\}} \quad \text{and} \quad N_c(s) = -2\left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i) : \mathrm{D}(v_c^j(s))\right)_{\substack{i\in\{1,\dots,6\}\\ j\in\{1,\dots,d\}}}.$$

A geometric control problem III

Lemma

The mapping $(c, s) \in \mathcal{A}^k(d) \mapsto (K_c, N_c(s)) \in \mathcal{M}_6(\mathbb{R}) \times \mathcal{M}_{6,d}(\mathbb{R})$ *is analytic.*
In addition, for every c, K_c *is positive definite.*

Consequently, the system can be rewritten as

$$\dot{h} = Q\ell, \tag{1a}$$

$$\dot{Q} = Q\mathbb{A}(\omega), \tag{1b}$$

$$\dot{s} = \lambda, \tag{1c}$$

$$\begin{pmatrix} \ell \\ \omega \end{pmatrix} = K_c^{-1} N_c(s)\lambda, \tag{1d}$$

with state $(h, Q, s) \in \mathbb{R}^3 \times SO(3) \times \mathbb{R}^d$ and control $\lambda \in \mathbb{R}^d$.

Well-posedness

Proposition

For every $(h_0, Q_0, s_0) \in \mathbb{R}^3 \times SO(3) \times \mathcal{J}(\delta)$, *the system* (1) *endowed with the initial condition* $(h, Q, s)(0) = (h_0, Q_0, s_0)$ *and control* $\lambda \in L^1_{loc}(\mathbb{R}_+)^d$ *(resp.* $\in C^{p-1}(\mathbb{R}_+)^d$*) admits a unique maximal solution* (h, Q, s) *which is absolutely continuous (resp. of class* C^p*).*

Furthermore, given $T > 0$*, if for every* $t \in [0, T]$, $s_0 + \int_0^t \lambda(\tau)\,\mathrm{d}\tau \in \mathcal{J}(\delta)$*, then the solution* (h, Q, s) *is well-defined on* $[0, T]$*.*

formulation in a Lie group I

Define the Lie group

$$E(3,d) = \left\{ P(h,Q,s),\ (h,Q,s) \in \mathbb{R}^3 \times SO(3) \times \mathbb{R}^d \right\},$$

with $P(h,Q,s) = \begin{pmatrix} \begin{pmatrix} Q & h \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \mathrm{I}_d & s \\ 0 & 1 \end{pmatrix} \end{pmatrix} \in \mathcal{M}_{d+5}(\mathbb{R})$.

Its Lie algebra is

$$\mathfrak{e}(3,d) = \left\{ \mathfrak{p}(\ell,\omega,\lambda),\ (\ell,\omega,\lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^d \right\},$$

with $\mathfrak{p}(\ell,\omega,\lambda) = \begin{pmatrix} \begin{pmatrix} \mathbb{A}(\omega) & \ell \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \end{pmatrix}$

formulation in a Lie group II

Defining $\mathfrak{f}_c^j(s) = \mathfrak{p}\left(\ell_c^j(s), \omega_c^j(s), e_j\right) = \begin{pmatrix} \begin{pmatrix} \mathbb{A}(\omega_c^j(s)) & \ell_c^j(s) \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & e_j \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in \mathfrak{e}(3, d)$,

with $\begin{pmatrix} \ell_c^j(s) \\ \omega_c^j(s) \end{pmatrix} = K_c^{-1} N_c(s) e_j$, the system writes

$$\frac{\mathrm{d}\, P(h, Q, s)}{\mathrm{d}t} = I(Q) \sum_{j=1}^{d} \mathfrak{f}_c^j(s)\lambda_j = \sum_{j=1}^{d} f_c^j(h, Q, s)\lambda_j,$$

with $I(Q) = P(0, Q, 0) = \begin{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \mathrm{I}_d & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$

Lie brackets computations I

Remind that

$$[f_c^j, f_c^i](h, Q, s) = \mathrm{D}f_c^i(h, Q, s) \cdot f_c^j(h, Q, s) - \mathrm{D}f_c^j(h, Q, s) \cdot f_c^i(h, Q, s).$$

We have,

$$[f_c^j, f_c^i] = I(Q)\tilde{\mathfrak{p}}\left(\partial_{s_j} V_c^i - \partial_{s_i} V_c^j + V_c^j \wedge V_c^i\right),$$

$$[f_c^k, [f_c^j, f_c^i]] = I(Q)\tilde{\mathfrak{p}}\left(\partial_{s_k}\left(\partial_{s_j} V_c^i - \partial_{s_i} V_c^j\right) + \partial_{s_k} V_c^j \wedge V_c^i + V_c^j \wedge \partial_{s_k} V_c^i \right.$$
$$\left. + V_c^k \wedge \left(\partial_{s_j} V_c^i - \partial_{s_i} V_c^j + V_c^j \wedge V_c^i\right)\right),$$

with $V_c^i(s) = \begin{pmatrix} \ell_c^i(s) \\ \omega_c^i(s) \end{pmatrix} = K_c^{-1} N_c(s) e_i$, $V_c^j \wedge V_c^i = \begin{pmatrix} \omega_c^j \times \ell_c^i - \omega_c^i \times \ell_c^j \\ \omega_c^j \times \omega_c^i \end{pmatrix}$ and $\tilde{\mathfrak{p}}\left(\begin{pmatrix} \ell \\ \omega \end{pmatrix}\right) = \mathfrak{p}(\ell, \omega, 0) = \begin{pmatrix} \mathbb{A}(\omega) & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Lie brackets computations II

Lemma

$$\dim \mathrm{Lie}_{(h,Q,s)} \left\{ f_c^1, \ldots, f_c^d \right\} = \dim \mathrm{Lie}_{(0,\mathrm{I}_3,s)} \left\{ f_c^1, \ldots, f_c^d \right\}.$$

Furthermore,

$$\begin{aligned} d+6 &\geqslant \dim \mathrm{Lie}_{(0,\mathrm{I}_3,s)} \left\{ f_c^1, \ldots, f_c^d \right\} \\ &\geqslant d + \dim \Big(\mathrm{Span} \left\{ \tilde{\mathfrak{p}}^{-1} \left([f_c^j, f_c^i](0,\mathrm{I}_3,s) \right),\ i,j \in \{1,\ldots,d\} \right\} \\ &\qquad + \mathrm{Span} \left\{ \tilde{\mathfrak{p}}^{-1} \left([f_c^k,[f_c^j, f_c^i]](0,\mathrm{I}_3,s) \right),\ i,j,k \in \{1,\ldots,d\} \right\} \Big). \end{aligned}$$

Lie brackets computations III

In order to compute these brackets, one needs to compute the derivatives of $s \mapsto V_c^j(s) = K_c^{-1} N_c(s) e_j$. But

$$\partial_s^\alpha N_c(s) e_j = -2 \left(\partial_s^\alpha \left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i : \mathrm{D}(v_c^j(s)) \right) \right)_{i \in \{1,\dots,6\}}$$

$$= -2 \left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i : \mathrm{D}(\partial_s^\alpha v_c^j(s)) \right)_{i \in \{1,\dots,6\}} = - \begin{pmatrix} \displaystyle\int_{\mathcal{S}_c} \sigma(w, r) \mathrm{n} \, \mathrm{d}\Gamma \\ \displaystyle\int_{\mathcal{S}_c} x \times \sigma(w, r) \mathrm{n} \, \mathrm{d}\Gamma \end{pmatrix},$$

with (w, r) solution of

$$\begin{aligned} -\Delta w + \nabla r &= 0 && \text{in } \mathcal{F}_c, \\ \operatorname{div} w &= 0 && \text{in } \mathcal{F}_c, \\ w = \partial_s^\alpha D_c^j(s) &= \partial_s^\alpha \left(\partial_{s_j} X_c(s) \circ X_c(s)^{-1} \right) && \text{on } \mathcal{S}_c, \\ \lim_{|x| \to \infty} w(x) &= 0. \end{aligned}$$

Main Theorem

Theorem

Given $d \geqslant 3$, $\varepsilon, \eta > 0$, $\bar{c} = (\bar{\Psi}_0, \bar{\delta}) \in \mathcal{SC}^2(d)$, $T > 0$ *and* $(\bar{h}, \bar{Q}, \bar{s}) \in C^0([0,T], \mathbb{R}^3 \times SO(3) \times \mathcal{J}(\bar{\delta}))$.
There exists $c = (\Psi_0, \delta) \in \mathcal{SC}^\infty(d)$ *such that*

$$\|c - \bar{c}\| < \varepsilon,$$

and there exists $s \in C^\infty([0,T], \mathbb{R}^d)$, *with*

$$s(t) \in \mathcal{J}(\delta), \quad s(0) = \bar{s}(0), \quad s(T) = \bar{s}(T) \quad \text{and} \quad |s(t) - \bar{s}(t)| \leqslant \eta \qquad (t \in [0,T]),$$

such that the corresponding position (h, Q) *of the swimmer, with the initial conditions*

$$h(0) = \bar{h}(0), \quad Q(0) = \bar{Q}(0)$$

satisfies

$$\sup_{t \in [0,T]} \left(|h(t) - \bar{h}(t)| + |Q(t) - \bar{Q}(t)| \right) < \eta$$

together with

$$h(T) = \bar{h}(T), \quad Q(T) = \bar{Q}(T).$$

Schedule of the proof

1. Pick $d = 3$ and Consider the case $\Psi_0 = 0$.
   - According to Lamb (1932) in the case of a Sphere, the solution of the Stokes problem can be explicitly computed in terms of a sum of functions involving spherical harmonics.
   - We make a particular choice of δ, based on spherical harmonics.
   - To compute the drag and torque forces, we use the strategy proposed in Brenner (1963).
   - Finally, we check that $\dim \mathrm{Lie}_{(0,I_3,0)}\{f^1_{(0,\delta)}, f^2_{(0,\delta)}, f^3_{(0,\delta)}\} = 3 + 6$ for a correct choice of vector fields δ_1, δ_2 and δ_3.
     That is to say that we check that the dimension of the Lie algebra evaluated at $s = 0$ is maximal.
2. To conclude we use the analytic properties of the vector fields with respect to s and c, the fact that the dimension of the Lie algebra is independent of h and Q and Chow Theorem.

Open problems

- Controllability with 2 parameters?
- Swimmer controllability in a bounded domain?

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THANK YOU, FOR YOUR ATTENTION!