Averaged controllability of finitely many strings equations

Jérôme Lohéac,
joint work with Enrique Zuazua

Centre de Recherche en Automatique de Nancy, France

VIII Partial differential equations, optimal design and numerics
Benasque – 27/08/2019

Let X and U be two Hilbert spaces. Consider the parameter dependent Cauchy problems:

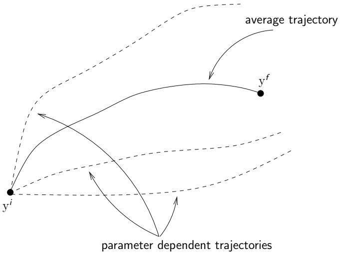
$$\dot{x}_\zeta = A_\zeta x_\zeta + B_\zeta u, \qquad x_\zeta(0) = \mathrm{x}^i_\zeta \in X, \tag{$\star$}$$

with parameter $\zeta \in \Omega$ and $(\Omega, \mathcal{F}, \mu)$ a probability space.

The aim:

given $(\mathrm{x}^i_\zeta)_{\zeta\in\Omega}$, $(\mathrm{x}^f_\zeta)_{\zeta\in\Omega}$ and $T > 0$,
find $u \in L^2([0, T], U)$ such that the solution of $x_\zeta(\cdot; u)$ of $(\star)$ satisfies:

Averaged controllability: $$\int_\Omega x_\zeta(T; u)\,\mathrm{d}\mu_\zeta = \int_\Omega x^f_\zeta\,\mathrm{d}\mu_\zeta.$$



State of art

- E. Zuazua, *Averaged control*, 2014
- M. Lazar and E. Zuazua, *Averaged control and observation of parameter dependent wave equations*, 2014
- Q. Lü and E. Zuazua, *Averaged controllability for random evolution partial differential equations*, 2016
- F. Marín, J. Martínez-Frutos and F. Periago, *Robust averaged control of vibrations for the Bernoulli-Euler beam equation*, 2017

Exact averaged controllability and observability

Definition

System $(\star)$ is said *exactly controllable in average* in time $T > 0$ if:

$$\left\{ \int_\Omega \int_0^T e^{(T-t)A_\zeta} B_\zeta u(t)\,\mathrm{d}t\mathrm{d}\mu_\zeta,\ u \in L^2([0,T],U) \right\} = X.$$

Let us consider for every $\mathrm{z}^f \in X$ the adjoint system:

$$-\dot{z}_\zeta = A_\zeta^* z_\zeta, \quad z_\zeta(T) = \mathrm{z}^f. \qquad (\overline{\mathrm{Adj}})$$

Definition

The system $(\overline{\mathrm{Adj}})$ is said *exactly observable in average* in time $T > 0$ if there exists $\bar{c}(T) > 0$ such that:

$$\bar{c}(T)\|\mathrm{z}^f\|_X^2 \leqslant \int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_X^2 \mathrm{d}t \qquad (\mathrm{z}^f \in X).$$

Duality results

Theorem

- *The system* $(\star)$ *is admissible in average if and only if* $(\overline{\mathrm{Adj}})$ *is, i.e.*

$$\forall T > 0,\ \exists \bar{C}(T) > 0,$$
$$\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\, \mathrm{d}\mu_\zeta \right\|_X^2 \mathrm{d}t \leqslant \bar{C}(T) \|\mathrm{z}^f\|_X^2 \qquad (\mathrm{z}^f \in X);$$

- *The system* $(\star)$ *is exactly controllable in average in time* $T > 0$ *if and only if* $(\overline{\mathrm{Adj}})$ *is exactly observable in average in time* T.

Kalman rank condition

Theorem (Zuazua, 2014)

Assume $\dim X < \infty$, *then the system* $(\star)$ *is controllable in average if and only if the rank condition:*

$$\operatorname{rank}\left[\int_\Omega A_\zeta^j B_\zeta \,\mathrm{d}\mu_\zeta, \quad j \in \mathbb{N}\right] = \dim X$$

is satisfied.

Perturbation result I

Theorem

Set $T > 0$, $(\Omega, \mathcal{F}, \tilde{\mu})$ *a probability space and* $\zeta_0 \in \Omega$.
Assume:

- $\{\zeta_0\} \in \mathcal{F}$ *and there exists* $C_{\zeta_0}(T), c_{\zeta_0}(T) > 0$ *such that:*
$$c_{\zeta_0}(T)\|z^f\|_X^2 \leqslant \int_0^T \left\|B_{\zeta_0}^* z_{\zeta_0}(t)\right\|_U^2 \mathrm{d}t \leqslant C_{\zeta_0}(T)\|z^f\|_X^2 \qquad (z^f \in X);$$
- *For almost every* $\zeta \in \Omega$, *there exists* $C_\zeta(T) > 0$ *such that:*
$$\int_0^T \left\|B_\zeta^* z_\zeta(t)\right\|_U^2 \mathrm{d}t \leqslant C_\zeta(T)\|z^f\|_X^2 \qquad (z^f \in X);$$
and
$$\int_\Omega \sqrt{C_\zeta(T)}\,\mathrm{d}\tilde{\mu}_\zeta < \infty.$$

Then for every $\theta \in \left[0, \left(1 + \int_\Omega \sqrt{\frac{C_\zeta(T)}{c_{\zeta_0}(T)}}\,\mathrm{d}\tilde{\mu}_\zeta\right)^{-1}\right)$, *the system* $(\star)$ *is exactly controllable in average for the probability measure:*
$$\mu = \theta\tilde{\mu} + (1-\theta)\delta_{\zeta_0}.$$

Perturbation result II

Proof: Use Minkowski inequality:

$$\left(\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t) \, \mathrm{d}\mu_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}}$$

$$= \left(\int_0^T \left\| (1-\theta) B_{\zeta_0}^* z_{\zeta_0}(t) + \theta \int_\Omega B_\zeta^* z_\zeta(t) \, \mathrm{d}\tilde{\mu}_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}}$$

$$\geqslant (1-\theta) \left(\int_0^T \left\| B_{\zeta_0}^* z_{\zeta_0}(t) \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}} - \theta \left(\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t) \, \mathrm{d}\tilde{\mu}_\zeta \right\|_U^2 \mathrm{d}t \right)^{\frac{1}{2}}$$

$$\geqslant (1-\theta) \sqrt{c_{\zeta_0}(T)} \|\mathrm{z}^f\|_X - \theta \int_\Omega \sqrt{C_\zeta(T)} \, \mathrm{d}\tilde{\mu}_\zeta \|\mathrm{z}^f\|_X$$

□

Ingham inequalities I

Consider $X = \ell^2(\mathbb{N}^*, \mathbb{C})$, $U = \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}$, with $\sum 1/\lambda_n^2 < \infty$.

Set the operator A of domain $\mathcal{D}(A) = \left\{ (a_n)_n \in \ell^2(\mathbb{N}^*),\ \sum_{n\in\mathbb{N}^*} |\lambda_n|^2 |a_n|^2 < \infty \right\} := X_1$

defined by:

$$Ae_n = 2i\pi\lambda_n e_n \qquad (n \in \mathbb{N}^*)$$

and the operator $B \in \mathcal{L}(U, X_{-1})$ defined by :

$$[Bv]_n = v \qquad (v \in \mathbb{C}).$$

Consider the system:

$$\dot{x} = Ax + Bu \qquad \text{i.e.} \qquad \dot{x}_n = 2i\pi\lambda_n x_n + u \quad (n \in \mathbb{N}^*).$$

The adjoint system is:

$$\dot{z} = -Az \qquad \text{i.e.} \qquad \dot{z}_n = 2i\pi\lambda_n z_n \quad (n \in \mathbb{N}^*) \qquad \text{thus} \quad z_n(t) = e^{2i\pi\lambda_n t} z_n(0)$$

and the observation operator:

$$\begin{array}{rcl} X_1 & \to & L^2([0,T],\mathbb{C}) \\ (a_n)_n & \mapsto & \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n t}. \end{array}$$

Ingham inequalities II

Theorem (Ingham inequalities, Ingham 1936)

Assume $\inf_{\substack{m,n\in\mathbb{N}^* \\ m\neq n}} |\lambda_m - \lambda_n| := \gamma > 0$. *Then for every* $T > 0$, *there exists* $C(T) > 0$ *such that:*

$$\int_0^T \left| \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n t} \right|^2 \mathrm{d}t \leqslant C(T) \sum_{n\in\mathbb{N}^*} |a_n|^2$$

and for every $T > \dfrac{1}{\gamma}$, *there exists* $c(T) > 0$ *such that:*

$$c(T) \sum_{n\in\mathbb{N}^*} |a_n|^2 \leqslant \int_0^T \left| \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n t} \right|^2 \mathrm{d}t.$$

Consequently, if $\gamma > 0$ and $T > \frac{1}{\gamma}$, the system $\dot{x} = Ax + Bu$ is exactly controllable.

Averaged version of Ingham inequalities I

Let us now consider the probability space $(\Omega, \mathcal{F}, \mu)$ given by $\Omega = \{\zeta_0, \dots, \zeta_K\} \subset \mathbb{R}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and μ given by $\mu(\{\zeta_k\}) := \theta_k \in [0, 1]$.
Consider the parameter dependent system:

$$\dot{x}_\zeta = \zeta A x_\zeta + Bu.$$

The goal would be to find $T > 0$ and $\bar{c}(T) > 0$ such that;

$$\bar{c}(T) \sum_{n \in \mathbb{N}^*} |a_n|^2 \leqslant \int_0^T \left| \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{N}^*} a_n e^{2i\pi\lambda_n \zeta_k t} \right|^2 \mathrm{d}t \qquad ((a_n)_n \in X).$$

Theorem

Set $\gamma > 0$ and assume $\lambda_n \in \gamma\mathbb{N}$ Then, if $T > \frac{1}{\gamma} \sum_{k=0}^K \frac{1}{|\zeta_k|}$, there exists a constant $\bar{c}(T) > 0$ such that:

$$\theta_0 \bar{c}(T) \sum_{n \in \mathbb{N}^*} |a_n|^2 \prod_{k=1}^K \sin\left(\frac{\lambda_n \pi}{\gamma} \frac{\zeta_0}{\zeta_k} \right) \leqslant \int_0^T \left| \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{N}^*} a_n e^{2i\pi\lambda_n \zeta_k t} \right|^2 \mathrm{d}t \qquad ((a_n)_n \in X).$$

Averaged version of Ingham inequalities II

Idea of the proof:

Set $f(t) = \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n\zeta_k t}$ and notice that

$$f(t+1/(\gamma|\zeta_K|)) - f(t) = \sum_{k=0}^{K-1} \theta_k \sum_{n\in\mathbb{N}^*} a_n \left(e^{2i\pi\frac{\lambda_n}{\gamma}\frac{\zeta_k}{|\zeta_K|}} - 1\right) e^{2i\pi\lambda_n\zeta t}.$$

Iterate K times and use Ingham Inequality. □

Averaged version of Ingham inequalities II

Idea of the proof:

Set $f(t) = \sum_{k=0}^{K} \theta_k \sum_{n \in \mathbb{N}^*} a_n e^{2i\pi\lambda_n \zeta_k t}$ and notice that

$$f(t + 1/(\gamma|\zeta_K|)) - f(t) = \sum_{k=0}^{K-1} \theta_k \sum_{n \in \mathbb{N}^*} a_n \left(e^{2i\pi \frac{\lambda_n}{\gamma} \frac{\zeta_k}{|\zeta_K|}} - 1 \right) e^{2i\pi\lambda_n \zeta t}.$$

Iterate K times and use Ingham Inequality. □

Corollary (With diophantine approximation, cf. Schmidt, 1970)

Assume in addition that $\frac{\zeta_1}{\zeta_0}, \ldots, \frac{\zeta_K}{\zeta_0}$ are algebraic, $\zeta_0, \ldots, \zeta_K$ are $\mathbb{Q}$-linearly independent and $\theta_0 > 0$.

Then for every $T > \frac{1}{\gamma} \sum_{k=0}^{K} \frac{1}{|\zeta_k|}$, and every $\varepsilon > 0$, there exists $\bar{c}_\varepsilon(T) > 0$ such that:

$$\bar{c}_\varepsilon(T) \sum_{n \in \mathbb{N}} \frac{|a_n|^2}{|\lambda_n|^{2(1+\varepsilon)}} \leqslant \int_0^T \left| \sum_{k=0}^{K} \theta_k \sum_{n \in \mathbb{N}^*} a_n e^{2i\pi\lambda_n \zeta t} \right|^2 \mathrm{d}t.$$

Consequently we obtained an Ingham inequality in a weighed space.

Application to the string equation I

Consider the parameter dependent string equation:

$$\begin{aligned} \ddot{w}_\zeta(t,x) &= \zeta^2 \partial_x^2 w_\zeta(t,x) && ((t,x) \in \mathbb{R}_+^* \times (0,1)), \\ w_\zeta(t,0) &= u(t) && (t \in \mathbb{R}_+^*), \\ w_\zeta(t,1) &= 0 && (t \in \mathbb{R}_+^*), \end{aligned}$$

$$w_\zeta(0,x) = \mathrm{w}^{i,0}(x) \quad \text{and} \quad \dot{w}_\zeta(0,x) = \mathrm{w}^{i,1}(x) \qquad (x \in (0,1)).$$

The adjoint problem of averaged observability is:

$$\begin{aligned} \ddot{z}_\zeta(t,x) &= \zeta^2 \partial_x^2 z_\zeta(t,x) && ((t,x) \in \mathbb{R}_+^* \times (0,1)), \\ 0 = z_\zeta(t,0) &= z_\zeta(t,1) && (t \in \mathbb{R}_+^*), \end{aligned}$$

$$z_\zeta(0,x) = \mathrm{z}^{i,0}(x) \quad \text{and} \quad \dot{z}_\zeta(0,x) = \mathrm{z}^{i,1}(x) \qquad (x \in (0,1)).$$

and the averaged observability map is:

$$(\mathrm{z}^{i,0}, \mathrm{z}^{i,1}) \mapsto -\sum_{k=0}^{K} \partial_x (A_0^{-1} \dot{z}_\zeta(t,\cdot))(0)\, \zeta_k \theta_k.$$

Application to the string equation II

Expanding $z_\zeta(t,x)$ on the Fourier basis $\sin(\pi n\zeta x)$, i.e. $z_\zeta(t,x) = \sum_{n\in\mathbb{N}^*} a_n(t)\sin(\pi n\zeta x)$ leads to an averaged observability map of the type:

$$\sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{Z}^*} a_n e^{2i\pi\lambda_n\zeta_k t},$$

with $\lambda_n = \frac{1}{2}n$.
Applying the previous corollary, we obtain:

Application to the string equation III

Proposition

Let $\varepsilon > 0$ and assume $\zeta_0, \dots, \zeta_K$ $\mathbb{Q}$-linearly independent and $\frac{\zeta_1}{\zeta_0}, \dots, \frac{\zeta_K}{\zeta_0}$ are algebraic.

Then, if $(\mathrm{w}^{i,0}_{\zeta_0}, \mathrm{w}^{i,1}_{\zeta_0}), \dots, (\mathrm{w}^{i,0}_{\zeta_K}, \mathrm{w}^{i,1}_{\zeta_K}), (\mathrm{w}^{f,0}, \mathrm{w}^{f,1}) \in X_{1+\varepsilon} \times X_\varepsilon$, for every $T > 2\sum_{k=0}^{K} \frac{1}{|\zeta_k|}$,

there exists $u \in L^2([0, T])$ such that the solution $w_\zeta(t,x) = w_\zeta(t,x;u)$ satisfies:

$$\sum_{k=0}^{K} \theta_k w_{\zeta_k}(T,x) = \mathrm{w}^{f,0}(x) \quad \text{and} \quad \sum_{k=0}^{K} \theta_k \dot{w}_{\zeta_k}(T,x) = \mathrm{w}^{f,1}(x) \qquad (x \in (0,1)).$$

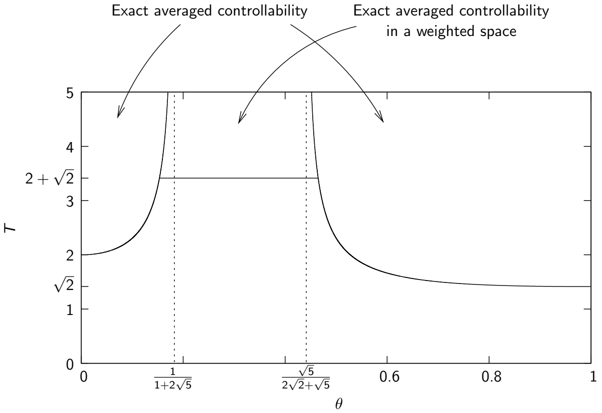
With $X_\alpha = \left\{ \varphi : x \in (0,1) \mapsto \sum_{n\in\mathbb{N}^*} a_n \sin(\pi n x), \ \sum_{n\in\mathbb{N}^*} n^{2\alpha}|a_n|^2 < \infty \right\} \qquad (\alpha \geqslant 0).$

From Dáger-Zuazua (2006),
if in addition $\frac{\zeta_l}{\zeta_k}$ are algebraic for every $k \neq l$, then there exists $u \in L^2([0, T])$ such that the solution $w_\zeta(t,x) = w_\zeta(t,x;u)$ satisfies:

$$w_{\zeta_k}(T,x) = \mathrm{w}^{f,0}(x) \quad \text{and} \quad \dot{w}_{\zeta_k}(T,x) = \mathrm{w}^{f,1}(x) \qquad (x \in (0,1), \ k \in \{0, \dots, K\}).$$

Application to the string equation IV

We can also apply the perturbation argument.
For instance, for $\zeta_0 = 1$ and $\zeta_1 = \sqrt{2}$ and measure $\mu = (1-\theta)\delta_{\zeta_0} + \theta\delta_{\zeta_1}$, we obtain the set of parameters where averaged controllability holds.



Conclusion and open questions

- Few averaged controllability results exists in PDE context.
- Averaged Ingham inequality in general means:

$$\exists c \text{ and } C > 0 \text{ s.t. } c \sum_n |a_n|^2 \leqslant \int_0^T \left| \int_\Omega \sum_n a_n e^{2i\pi\lambda_n \zeta t} \,\mathrm{d}\mu_\zeta \right|^2 \mathrm{d}t \leqslant C \sum_n |a_n|^2.$$

In particular, for $\Omega = \mathbb{R}$, we end up with:

$$c \sum_n |a_n|^2 \leqslant \int_0^T |a_n \hat{\mu}(-\lambda_n t)|^2 \,\mathrm{d}t \leqslant C \sum_n |a_n|^2.$$

That is to say, $\{\hat{\mu}(-\lambda_n \cdot)\}_n$ is a Riesz basis.

- Still for averaged Ingham inequalities, are they true for $\mathrm{d}\mu_\zeta = \frac{1}{2\varepsilon} \mathbf{1}_{(1-\varepsilon, 1+\varepsilon)}(\zeta) \,\mathrm{d}\zeta$, with $\varepsilon > 0$ small enough?

Conclusion and open questions

- Few averaged controllability results exists in PDE context.
- Averaged Ingham inequality in general means:

$$\exists c \text{ and } C > 0 \text{ s.t. } c \sum_n |a_n|^2 \leqslant \int_0^T \left| \int_\Omega \sum_n a_n e^{2i\pi \lambda_n \zeta t} \, \mathrm{d}\mu_\zeta \right|^2 \mathrm{d}t \leqslant C \sum_n |a_n|^2.$$

In particular, for $\Omega = \mathbb{R}$, we end up with:

$$c \sum_n |a_n|^2 \leqslant \int_0^T |a_n \hat{\mu}(-\lambda_n t)|^2 \, \mathrm{d}t \leqslant C \sum_n |a_n|^2.$$

That is to say, $\{\hat{\mu}(-\lambda_n \cdot)\}_n$ is a Riesz basis.

- Still for averaged Ingham inequalities, are they true for $\mathrm{d}\mu_\zeta = \frac{1}{2\varepsilon} \mathbf{1}_{(1-\varepsilon, 1+\varepsilon)}(\zeta) \, \mathrm{d}\zeta$, with $\varepsilon > 0$ small enough?

THANK YOU, FOR YOUR ATTENTION!