The behavior of an elastic body subjected to a strong oscillating magnetic field.

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We are interested in the homogenization problem

$$\begin{cases} \rho\partial_{tt}^2 u_\varepsilon - \operatorname{div} Ae(u_\varepsilon) + B_\varepsilon \times \partial_t u_\varepsilon = f_\varepsilon \ \text{ in } \ Q_T \\ u_\varepsilon = 0 \ \ on \ \ (0,T) \times \partial\Omega \\ u_\varepsilon(0,x) = u_\varepsilon^0, \quad \partial_t u_\varepsilon(0,x) = u_\varepsilon^1. \end{cases}$$

$\Omega \subset \mathbb{R}^3$ bounded open, $Q_T = (0,T) \times \Omega$, A positive tensor

$$f_\varepsilon \to f \ \ in\ L^1\big(0,T; L^2(\Omega)\big)^3$$

$$u_\varepsilon^0 \rightharpoonup u^0 \text{ in } H_0^1(\Omega)^3, \ \ u_\varepsilon^1 \rightharpoonup u^1\ in\ L^2(\Omega)^3$$

$$B_\varepsilon(t,x) = F_\varepsilon(x) + G_\varepsilon(t,x) + H_\varepsilon(t,x)$$

$$F_\varepsilon \rightharpoonup 0 \ \text{ in } \ W^{-1,p}(\Omega)^3, \qquad G_\varepsilon \overset{*}{\rightharpoonup} 0 \ \text{ in } L^\infty(Q_T)^3$$

$$H_\varepsilon \to H \ \text{ in } \ H^1\big(0,T; W^{-1,p}(\Omega)\big)^3, \quad p > 3.$$

Remark. We can consider $\Omega \subset \mathbb{R}^N$, replacing

$$B_\varepsilon \times \partial_t u_\varepsilon$$

by

$$\hat{B}_\varepsilon \partial_t u_\varepsilon$$

whith $\hat{B}_\varepsilon : \Omega \to \mathbb{R}^{N\times N}$ skew-symmetric.

The present of $H_\varepsilon(t,x)$ does not vary the structure of the limit of the equation.

The most interesting terms are $F_\varepsilon, G_\varepsilon$.

We start by studying the influence of F_ε.

Lemma. Let u_ε be the solution of

$$\begin{cases} -\operatorname{div} Ae(u_\varepsilon) + F_\varepsilon \times z_\varepsilon = f_\varepsilon \ \text{ in } \ \Omega \\ u_\varepsilon = 0 \ \ on \ \ \partial\Omega, \end{cases}$$

where

$F_\varepsilon \rightharpoonup 0$ in $W^{-1,p}(\Omega)^3$, $p > 3$, $\quad z_\varepsilon \rightharpoonup z$ in $H^1(\Omega)$, $\quad f_\varepsilon \to f$ in $H^{-1}(\Omega)$.

Define w_ε^j as the solutions of

$$\begin{cases} -\operatorname{div} Ae\left(w_\varepsilon^j\right) + F_\varepsilon \times e_j = 0 \ \text{ in } \ \Omega \\ w_\varepsilon^j = 0 \ \ on \ \ \partial\Omega, \end{cases}$$

and $M \in L^{\frac{p}{2}}(\Omega)^{3\times 3}$ by

$$Ae\left(w_\varepsilon^j\right) : e\left(w_\varepsilon^k\right) \rightharpoonup M e_j \cdot e_k \ \text{ in } \ L^{\frac{p}{2}}(\Omega).$$

Then, defining u by

$$u_\varepsilon \rightharpoonup u \ \text{ in } \ H_0^1(\Omega)^3$$

we have

$$u_\varepsilon - u - \sum_{i=1}^{3} w_\varepsilon^j z_j \to 0 \text{ in } H_0^1(\Omega)^3$$

$$Ae(u_\varepsilon) : e(u_\varepsilon) \stackrel{*}{\rightharpoonup} Ae(u) : e(u) + Mu \cdot u \ \text{ in the measures.}$$

Remark. The lemma provides a corrector result, i.e. a strong approximation in $H_0^1(\Omega)^3$ of u_ε. It does not give a limit equation for u.

Indeed: $F_\varepsilon \rightharpoonup 0$ in $W^{-1,p}(\Omega)^3$, $p > 3$, $z_\varepsilon \rightharpoonup z$ in $H^1(\Omega)$ does not permit to pass to the limit in $F_\varepsilon \times z_\varepsilon$ and then in

$$\begin{cases} -\operatorname{div} Ae(u_\varepsilon) + F_\varepsilon \times z_\varepsilon = f_\varepsilon \text{ in } \Omega \\ u_\varepsilon = 0 \ on \ \partial\Omega, \end{cases}$$

Remark. The lemma is related with a result of L. Tartar,1977. He considers the Navier-Stokes problem

$$\begin{cases} -\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + F_\varepsilon \times u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon \text{ in } \Omega \\ u_\varepsilon = 0 \ on \ \partial\Omega, \end{cases}$$

The term $F_\varepsilon \times u_\varepsilon$ represents here a Coriolis force.

Using the functions w_ε^j (oscillating functions method) we get the limit problem

$$\begin{cases} -\Delta u + (u \cdot \nabla) u + Mu + \nabla p = f \text{ in } \Omega \\ u = 0 \ on \ \partial\Omega. \end{cases}$$

Theorem: The solution u_ε of

$$\begin{cases} \rho\partial_{tt}^2 u_\varepsilon - \operatorname{div} Ae(u_\varepsilon) + B_\varepsilon \times \partial_t u_\varepsilon = f_\varepsilon \ \text{ in } \ Q_T \\ u_\varepsilon = 0 \ \ on \ \ (0,T) \times \partial\Omega \\ u_\varepsilon(0,x) = u_\varepsilon^0, \quad \partial_t u_\varepsilon(0,x) = u_\varepsilon^1, \end{cases}$$

with $B_\varepsilon(t,x) = F_\varepsilon(x) + G_\varepsilon(t,x) + H_\varepsilon(t,x)$, satisfies

$$u_\varepsilon \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty\left(0,T;H_0^1(\Omega)\right)^3 \cap W^{1,\infty}\left(0,T;L^2(\Omega)\right)^3$$

$$G_\varepsilon \times \partial_t u_\varepsilon^0 \ \overset{*}{\rightharpoonup} g \text{ in } L^\infty\left(0,T;L^2(\Omega)\right)^3$$

$$\exists\zeta:\Omega \to \mathbb{R}^3, \ F_\varepsilon \times u_\varepsilon^0 \rightharpoonup M\zeta \text{ in } H^{-1}(\Omega)^3, \ M\zeta \in L^{\frac{2p}{p-2}}(\Omega)^3, \ \ M\zeta\cdot\zeta \in L^1(\Omega).$$

$$\begin{cases} (\rho I + M)\partial_{tt}^2 u - \operatorname{div} Ae(u) + H \times \partial_t u + g = f \ \text{ in } \ Q_T \\ u = 0 \ \ on \ \ (0,T) \times \partial\Omega \\ u(0,x) = u^0, \quad \partial_t u(0,x) = (\rho I + M)^{-1}(\rho u^1 + M\zeta), \end{cases}$$

Proposition: We have

$$\rho|u_\varepsilon^1|^2 + Ae(u_\varepsilon^0):e(u_\varepsilon^0) \overset{*}{\rightharpoonup}$$

$$\mu^0 + Ae(u^0):e(u^0) + (\rho I + M)^{-1}(\rho u^1 + M\zeta)\cdot(\rho u^1 + M\zeta)$$

in the measures, with μ^0 a nonnegative measure.

Assume the initial conditions *well posed* (related to Francfort-Murat, 1992)

$$u_\varepsilon^1 \rightharpoonup u^1 \text{ in } H^1(\Omega),$$

$$-\mathrm{div}\big(Ae(u_\varepsilon^0)\big) + F_\varepsilon \times u_\varepsilon^1 \ \text{ compact in } \ H^{-1}(\Omega)^N.$$

Then,

$$M\zeta = Mu^1$$

$$\rho|u_\varepsilon^1|^2 + Ae(u_\varepsilon^0):e(u_\varepsilon^0) \overset{*}{\rightharpoonup} \rho|u^1|^2 + Ae(u^0):e(u^0) + Mu^1\cdot u^1$$

in the measures

Theorem: Assume the initial conditions *well posed* and $\partial_t G_\varepsilon$ bounded in $L^1(0,T;L^\infty(\Omega))^{3\times 3}$. Then

$$g = 0$$

and the following corrector result holds

$$\partial_t u_\varepsilon \sim \partial_t u_0 \quad \text{in} \quad L^2(0,T;L^2(\Omega))^3$$

$$e(u_\varepsilon) \sim e(u) + \sum_{j=1}^{N} e(w_\varepsilon^j)\partial_t u_{0,j} \quad \text{in} \quad L^2(0,T;L^2(\Omega))^{3\times 3}.$$

Remark: If these conditions do not hold, we still have

$$e\left(\int_{t_1}^{t_2} u_\varepsilon\, dt\right) \sim e\left(\int_{t_1}^{t_2} u\, dt\right) + \sum_{j=1}^{N} e(w_\varepsilon^j)\left(u_{0,j}(t_2) - u_{0,j}(t_1)\right)$$

in $L^2(0,T;L^2(\Omega))^{3\times 3}$, $\forall t_1, t_2, 0 < t_1 < t_2$.

The proof of the theorem consists in integrating in time and then to use this result. Integrating in time, we do not see the oscillations in time.

Example:

$$G_\varepsilon(t,x) = H_\varepsilon(t,x) = 0,\ \ F_\varepsilon(x) = \frac{1}{\varepsilon}F\left(\frac{x}{\varepsilon}\right),\ \ F \in L^p_\#(Y)^3,\ \ \int_Y F dy = 0.$$

The limit problem reads as

$$\begin{cases} (\rho I + M)\partial^2_{tt}u - \operatorname{div} Ae(u) = f \ \text{ in } \ Q_T \\ u = 0 \ \ on \ \ (0,T) \times \partial\Omega \\ u(0,x) = u^0, \quad \partial_t u(0,x) = (\rho I + M)^{-1}(\rho u^1 + M\zeta), \end{cases}$$

$$M e_j \cdot e_k = \int_Y Ae(w^j) : e(w^k) dy,$$

with

$$\begin{cases} -\operatorname{div} Ae(w^j) + F \times e_j = 0 \ \text{ in } \ \mathbb{R}^3 \\ w \in H^1_\#(Y)^3. \end{cases}$$

The magnetic field induces an increasing of mass in the homogenized equation. The new mass is anisotropic.

What about the structure of g?

Related problem: JCD, J. Couce-Calvo, F. Maestre, J.D. Martín-Gómez, 2014.
Corrector for

$$\begin{cases} \partial_t(\rho_\varepsilon \partial_t u_\varepsilon) - \operatorname{div}_x(A_\varepsilon \nabla_x u_\varepsilon) + B_\varepsilon \cdot \nabla_{t,x} u_\varepsilon = f_\varepsilon \text{ in } (0,T) \times \mathbb{R}^N \\ u_\varepsilon(0,x) = u_\varepsilon^0, \ \partial_t u_\varepsilon(0,x) = v_\varepsilon^0 \end{cases}$$

with

$$\rho_\varepsilon(x,t) = \rho^0\left(\frac{x}{\varepsilon}\right) + \varepsilon\rho^1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad A_\varepsilon(x,t) = A^0\left(\frac{x}{\varepsilon}\right) + \varepsilon A^1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

$$B_\varepsilon(x,t) = B\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad f_\varepsilon(x,t) = f\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

$$u_\varepsilon^0 = u^0(x) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right), \ \ v_\varepsilon^0 = v^0\left(x, \frac{x}{\varepsilon}\right).$$

The functions are periodic in x/ε and almost periodic in t/ε.
M. Brassart, M. Lenczner, 2010 consider the case $\rho^1 = 0, A^1 = 0, B = 0$.

Even if the coefficients do not oscillate in time, the corrector is

$$u_\varepsilon(t,x) \sim u_0(t,x) + \varepsilon u_1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

i.e. the oscillations in space of the coefficients introduce oscillations in time for the solution.

If $B = 0$, the homogenized equation is

$$\begin{cases} \partial_t\left(\overline{\rho^0}\partial_t u_0\right) - \mathrm{div}_x(A_H \nabla_x u_0) = \overline{f} \text{ in } (0,T) \times \mathbb{R}^N \\ u_0(0,x) = u^0, \ \ \partial_t u_\varepsilon(0,x) = \overline{v^0}, \end{cases}$$

$$\overline{\rho^0} = \int_Y \rho^0(y)dy, \ \ \overline{f}(t,x) = \int_Y M_s\big(f(t,x,s,y)\big)dy, \ \ \overline{v^0}(x) = \int_Y v^0(x,y)dy,$$

A_H the usual homogenized matrix associated to A^0.

This can be obtained using the asymptotic expansion

$$u_\varepsilon(t,x) \sim u_0(t,x) + \varepsilon u_1\left(t, x, \frac{x}{\varepsilon}\right).$$

This does not provide a corrector. This differs from parabolic problems.

Example: JCD, J. Couce-Calvo, F. Maestre, J.D. Martín-Gómez, 2014.

$$\begin{cases} \partial_{tt}^2 u_\varepsilon - \partial_{xx}^2 u_\varepsilon + 2\cos\dfrac{2\pi(t+x)}{\varepsilon}\partial_x u_\varepsilon = f \ \text{ in } \ (0,T)\times\mathbb{R} \\ u_0(0,x) = u^0, \ \ \partial_t u_\varepsilon(0,x) = v^0. \end{cases}$$

The limit problem is

$$\begin{cases} \partial_{tt}^2 u_0 - \partial_{xx}^2 u_0 + 2\displaystyle\int_0^t g(t-s)\partial_x u_0(r, x+t-r)dr = f \ \text{ in } \ (0,T)\times\mathbb{R} \\ u_0(0,x) = u^0, \ \ \partial_t u_0(0,x) = v^0. \end{cases}$$

$$g(s) = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{t^{2k}}{4^k k!\,(k+1)!}$$

Returning to the elasto-magnetic problem:

$$\begin{cases} \rho\partial_{tt}^2 u_\varepsilon - \operatorname{div} Ae(u_\varepsilon) + B_\varepsilon \times \partial_t u_\varepsilon = f_\varepsilon \ \text{ in } \ Q_T \\ u_\varepsilon = 0 \ \ on \ \ (0,T)\times\partial\Omega \\ u_\varepsilon(0,x) = u_\varepsilon^0, \quad \partial_t u_\varepsilon(0,x) = u_\varepsilon^1, \end{cases}$$

Limit equation

$$\begin{cases} (\rho I + M)\partial_{tt}^2 u - \operatorname{div} Ae(u) + H \times \partial_t u + g = f \ \text{ in } \ Q_T \\ u = 0 \ \ on \ \ (0,T)\times\partial\Omega \\ u(0,x) = u^0, \quad \partial_t u(0,x) = (\rho I + M)^{-1}(\rho u^1 + M\zeta), \end{cases}$$

$$G_\varepsilon \times \partial_t u_\varepsilon^0 \ \stackrel{*}{\rightharpoonup} g \text{ in } L^\infty\big(0,T; L^2(\Omega)\big)^3$$

$$\rho|u_\varepsilon^1|^2 + Ae(u_\varepsilon^0):e(u_\varepsilon^0) \stackrel{*}{\rightharpoonup}$$

$$\mu^0 + Ae(u^0):e(u^0) + (\rho I + M)^{-1}(\rho u^1 + M\zeta)\cdot(\rho u^1 + M\zeta)$$

Theorem: $\exists \mathfrak{S}: L^1\left(0,T;L^2(\Omega)\right)^3 \to L^\infty\left(0,T;L^2(\Omega)\right)^3$, linear, continuous, s.t.
$\forall x \in \Omega,\ \forall S \in (0,T),$ a.e. $s \in (0,S)$

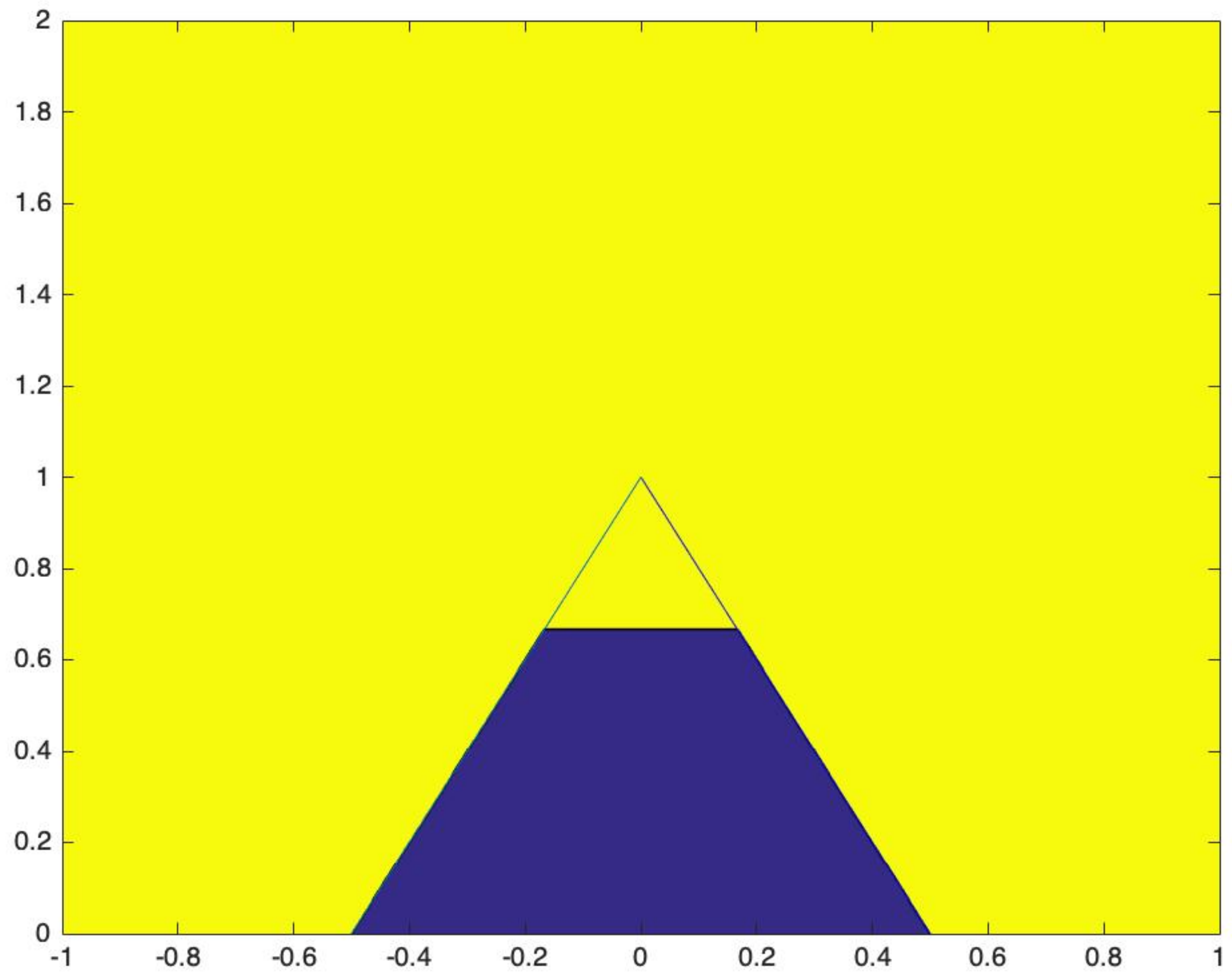
$$\int_{B(x,c(S-s))} |\mathfrak{S}w|^2 dx \leq C\left(\int_0^S \left(\int_{B(x,c(S-t))} |w|^2 dx\right)^{\frac{1}{2}} dt\right)^2$$

$$0 \leq \int_0^S \int_{B(x,c(S-s))} \mathfrak{S}w \cdot w dx\, dt$$

$$\int_{B(x,c(S-s))} |g - \mathfrak{S}\partial_t u|^2 dx \leq C\ \mu^0(\bar{B}(x,S))$$

with

$$c = \sqrt{\frac{|A|}{\rho}}.$$



2
1.8
1.6
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1
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0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Corollary: If the initial data is *well posed* the limit problem is

$$\begin{cases} (\rho I + M)\partial_{tt}^2 u - \operatorname{div} Ae(u) + H \times \partial_t u + \mathfrak{S}\partial_t u = f \quad \text{in } \; Q_T \\ u = 0 \;\; on \;\; (0,T) \times \partial\Omega \\ u(0,x) = u^0, \quad \partial_t u(0,x) = u^1. \end{cases}$$

Example with initial data not well posed. Assume

$$u_\varepsilon(0,x) = u^0, \quad \partial_t u_\varepsilon(0,x) = u^1.$$

$\exists \mathcal{F}: L^2(\Omega)^3 \to L^\infty\left(0,T; L^2(\Omega)\right)^3$ with

$$\int_{B(x,c(S-s))} |\mathcal{F}v|^2 \, dx \leq C \int_{B(x,cS)} (\rho I + M)^{-1} M v \cdot v dx$$

such that the limit problem is

$$\begin{cases} (\rho I + M)\partial_{tt}^2 u - \operatorname{div} Ae(u) + H \times \partial_t u + \mathfrak{S}\partial_t u = f + \mathcal{F}u^1 \quad \text{in } \; Q_T \\ u = 0 \;\; on \;\; (0,T) \times \partial\Omega \\ u(0,x) = u^0, \quad \partial_t u(0,x) = \rho(\rho I + M)^{-1} u^1. \end{cases}$$

Remark: In order to get a nonlocal term in the limit we need to take magnetic fields oscillating in space and time simultaneously.

We shown $B_\varepsilon(t,x) = F_\varepsilon(x) \rightharpoonup F$ in $W^{-1,p}(\Omega)^3$, provides an increasing of mass but not a non-local term in the limit.

Analogously, assuming

$$B_\varepsilon \in C^1([0,T])^3,\ B_\varepsilon, B'_\varepsilon\ parallel,$$
$$B_\varepsilon(0) = 0, e^{-\rho^{-1}B_\varepsilon \times} \to \mathcal{M}^{-1} \text{ in } C^{\frac{1}{0}}([0,T])^{3\times 3}$$

The limit equation of

$$\begin{cases} \rho\partial_{tt}^2 u_\varepsilon - \operatorname{div} Ae(u_\varepsilon) + B_\varepsilon \times \partial_t u_\varepsilon = f_\varepsilon \text{ in } Q_T \\ u_\varepsilon = 0\ \ on\ \ (0,T)\times\partial\Omega \\ u_\varepsilon(0,x) = u_\varepsilon^0, \quad \partial_t u_\varepsilon(0,x) = u_\varepsilon^1. \end{cases}$$

is

$$\begin{cases} \rho\mathcal{M}^t\mathcal{M}\partial_{tt}^2 u - \operatorname{div} Ae(u) + \rho\mathcal{M}^t\mathcal{M}\partial_t u = f + \mathcal{F}u^1 \text{ in } Q_T \\ u = 0\ \ on\ \ (0,T)\times\partial\Omega \\ u(0,x) = u^0, \quad \partial_t u(0,x) = \mathcal{M}^{-1}(0)u^1. \end{cases}$$