The Turnpike Phenomenon for Problems of Optimal Boundary Control

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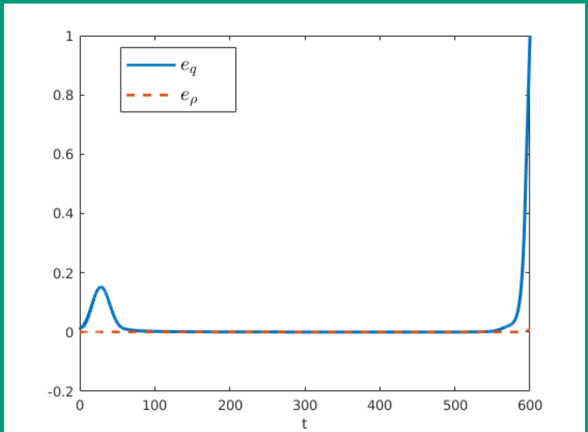

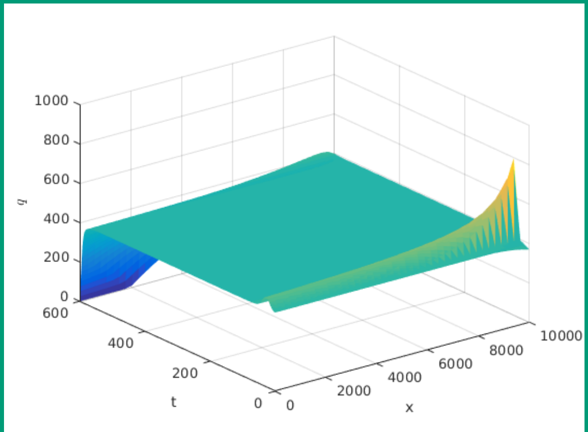

Outline

The Turnpike Phenomenon: *What is it?*

L^1 optimal Dirichlet control of the **wave** equation

L^2 optimal Neumann control of the **wave** equation

Turnpike for linear 2×2 systems: Problem definition

A turnpike property relates the dynamic and the static problem

Conclusion open problems

Inhalt

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- Turnpike results give relations between the *static optimal control* and the *dynamic optimal control*.

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- Consider a **dynamic optimal control problem** with a time interval $[0, T]$.
- If all the time-derivatives are set to zero and initial conditions and terminal conditions are canceled, this yields a **static optimal control problem**.
- Turnpike results give relations between the *static optimal control* and the *dynamic optimal control*.
- They state that for sufficiently large T, some distance between the static optimal point and the dynamic optimal point becomes small.

Example

For $T \geq 1$ we consider the problem

$$(\mathbf{OC})_T \begin{cases} \min\limits_{u \in L^2(0,T), u(t) \geq 0, y(t) \leq 0} \int\limits_0^T \frac{1}{2}|u(t)|^2 + |u(t)| + |y(t)| \, dt \text{ subject to} \\ y(0) = -1, \; y'(t) = y(t) + \exp(t)\, u(t) \\ y(T) = 0. \end{cases}$$

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The feasible set is nonempty: Define $\hat{u}(t) = \mathrm{e} - \mathrm{e}^t \geq 0$ for $t \in (0, 1)$ and $u(t) = 0$ for $t \geq 1$. Then for $t \in (0, 1)$ we have

$$\hat{y}(t) = \mathrm{e}^t \left[-1 + \int_0^t u(\tau)\, d\tau \right] = t\, \mathrm{e}^{t+1} - \mathrm{e}^{2t} \leq 0$$

and for $t \geq 1$ we have $\hat{y}(t) = 0$.

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The feasible controls are characterized by the *moment equation* $\int_0^T u(\tau)\, d\tau = 1$.

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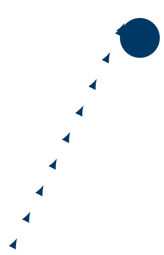
  Then instead of the moment equation, due to $y(T) \leq 0$ we have the *moment inequality*
  $$\int_0^T u(\tau)\, d\tau \leq 1.$$
- In fact, if $\exp T \geq 1 + \mathrm{e}$, $\hat{u}$ is again the optimal control!

Example

For sufficiently large T, due to the L^1-norm of y that appears in the objective function, the solution has a **finite–time turnpike structure** where the system is steered to zero in the *finite time* $t_0 = 1$ that is independent of T and remains there for all $t \in (t_0,\ T)$.

You can think of the turnpike as a point that does not move.



Inhalt

L^1-Optimal Dirichlet control of the wave equation

Norm minimal exact control, finite horizon $T \geq 1$

Let $y_0 \in L^1(0,1)$, $y_1 \in W^{-1,1}(0,1)$ and $T \geq 1$ be given. Define (**EC**) :

$$\begin{cases} \min \int_0^T |u_0(t)| + |u_1(t)|\, dt \text{ subject to} \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1) \\ y(t,0) = u_0(t),\quad y(t,1) = u_1(t),\ t \in (0,T) \\ y_{tt}(t,x) = y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1) \\ y(T,x) = 0,\ y_t(T,x) = 0,\ x \in (0,1). \end{cases}$$

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In general the optimal controls are **not unique**!

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Gugat, French-German-Spanish Conf. on Opt., Avignon 2004

Let $T \geq 1$. There exist solutions of (**EC**) that are *2–periodic* i.e. for $k \in \{1, 2, ...,\}$, $t \in (0,2)$, $t + 2k \leq T$, $l \in \{1, 2\}$ we have

$$\boxed{u_l(t+2k) = u_l(t).}$$

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The set of *all solutions* is parametrized by pairs of measurable convex combinations $(t \in (0, 1))$

$$\lambda_j^{(l)}(t) \geq 0,\ \sum_{j: t+2j \leq T} \lambda_j^{(l)}(t) = 1.$$

L^1-Optimal Dirichlet control of the wave equation

Example: Let $y_1 = 0$.

Let

$$k = \max\{j \in \{1, 2, 3, ...\} : j \leq T\}$$

and

$$\Delta = T - k \geq 0.$$

$$(0, T) = (0, \Delta) \cup (\Delta, 1) \cup (1, 1 + \Delta) \cup (2 + \Delta, 2) \cup (2, 2 + \Delta) ... \cup ((k - 1) + \Delta, k) \cup (k, k + \Delta)$$

There are $k + 1$ red intervals and k black intervals!

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$$u_0(t + j) = u_1(t + j) = (-1)^j \frac{y_0(t)}{2\,(k + 1)}.$$

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The control action can be shifted between the different time periods!

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$$d(t) = \begin{cases} (k+1) & \text{if} \quad t \in (0, \Delta), \\ k & \text{if} \quad t \in (\Delta, 1). \end{cases}$$

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The set of **all optimal controls** has the following structure:
For pairs of measurable convex combinations $l \in \{1, 2\}$, $t \in (0, 1)$
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$$u_0(t+2j) = \lambda_{2j}^{(1)}(t)\,\frac{y_0(t)}{2\,d(t)}, \quad u_0(t+2j+1) = -\lambda_{2j+1}^{(2)}(t)\,\frac{y_0(t)}{2\,d(t)},$$
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L^1-Optimal Dirichlet control of the wave equation

- The convex combinations ($t \in (0, 1)$, $l \in \{1, 2\}$)

$$\lambda_j^{(l)}(t) \geq 0, \quad \sum_{j:t+2j\leq T} \lambda_j^{(l)}(t) = 1$$

determine the *support* of the corresponding optimal control (together with the support of y_0, y_1).
It can be the whole interval $[0, T]$.

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- If for all $t \in (0, 1)$, the $\lambda_j^{(l)}(t)$ are **equal** for all j, we obtain *periodic* controls.
In fact, this yields $\lambda_j^{(l)}(t) = \frac{1}{d(t)}$.
These are the optimal controls with **minimal L^2-norm.**

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  In this case, the *support* of the corresponding optimal control can be constrained to a subinterval of $[0, T]$ of minimal length 1.
- Thus we have optimal controls with support $(0, 1)$.
  These controls steer the system to rest at the time $t = 1$.

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Adding a tracking-term in the goal function

Finite horizon $T \geq 1$, $\gamma > 0$. Define (**P**):

$$
\begin{cases}
\min\limits_{u_0, u_1 \in L^1(0,T)} \int_0^T |u_0(t)| + |u_1(t)| \, dt \\
+ \quad \gamma \int_1^T \int_0^1 |y(t,x)| \, dx \, dt \\
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\text{subject to} \\
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Solution of (P)

The nonsmooth problem (**P**) has a *unique* solution.
The unique solution of (**P**) steers the state to rest at time $t = 1$.
Then the control is switched off.

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The unique solution of (**P**) steers the state to rest at time $t = 1$.
Then the control is switched off.
Here we have an extreme (finite time) turnpike structure!
For $t > 1$ we have

$$u_0(t) = u_1(t) = 0,$$

$y(t, x) = 0$, $x \in (0, 1)$
and $y_t(t, x) = 0$.

This is possible due to
exact controllability!

Optimal Dirichlet L^1-control of the wave equation

Proof.

Problem (**EC**) has a unique solution where the support of the controls is in $(0, 1)$.

Optimal Dirichlet L^1-control of the wave equation

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Problem (**EC**) has a unique solution where the support of the controls is in $(0, 1)$.
This optimal control (u_0^*, u_1^*) steers the state to rest at time $t = 1$.

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Proof.

Problem (**EC**) has a unique solution where the support of the controls is in $(0, 1)$.
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Let ν(**EC**) denote the optimal value of (**EC**).

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Let $\nu(\textbf{EC})$ denote the optimal value of (**EC**).
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Since the objective function of (**EC**) is $\leq$ the objective function of (**P**), we have

$$\nu(\textbf{EC}) \leq \nu(\textbf{P}).$$

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The objective value of (**P**) for $(u_0^*,\ u_1^*)$ is

$$\nu(\mathbf{EC}) + \gamma \int_1^T \int_0^1 |y(t,\ x)|\, dx\, dt = \nu(\mathbf{EC}) \geq \nu(\mathbf{P}).$$

Hence $\nu(\mathbf{EC}) = \nu(\mathbf{P})$.

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We can replace (**P**) with the problem without terminal constraints.

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We can replace (**P**) with the problem without terminal constraints.
If T is sufficiently large, the solution should stay the same - however, the proof is not written.

Inhalt

The Turnpike Phenomenon: *What is it?*

L^1 optimal Dirichlet control of the **wave** equation

L^2 optimal Neumann control of the **wave** equation

Turnpike for linear 2×2 systems: Problem definition

A turnpike property relates the dynamic and the static problem

Conclusion open problems

L^2-optimal Neumann control of the wave equation

Norm minimal exact control, finite horizon $T \geq 2$

Let $y_0 \in H^1(0,1)$ with $y_0(0) = 0$, $y_1 \in L^2(0,1)$ and $T \geq 2$ be given. Define (**EC**) :

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\begin{cases}
\min \|u\|^2_{L^2(0,T)} \text{ subject to} \\
y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1) \\
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Gugat, Arab. J. Math. 2015 *Open Access*

Let $T \in \mathbb{N}$ be even. The **unique** solution of (**EC**) is *4–periodic* and

$$u(t) = \begin{cases} \frac{y_0'(1-t) - y_1(1-t)}{T}, & t \in (0,1), \\ \frac{y_0'(t-1) + y_1(t-1)}{T}, & t \in (1,2). \end{cases}$$

For $k \in \{1, 2, ..., (T-2)/2\}$, $t \in (0,2)$ we have

$$\boxed{u(t + 2k) = (-1)^k u(t).}$$

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For $t_0 = 0$ (*Moving horizon*) this yields the well-known feedback law

$$y_x(t_0, 1) = -\frac{1}{T-1}\, y_t(t_0, 1).$$

Optimal Neumann control of the wave equation

Adding a tracking-term in the goal function

Finite horizon $T \geq 2$, $\gamma > 0$. Define (**P**):

$$\begin{cases} \min_{u \in L^2(0,T)} \int_0^T (y_x(t,0))^2 + \gamma\, u^2(t)\, dt \ \text{subject to} \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1) \\ y(t,0) = 0,\ \boxed{y_x(t,1) = u(t)},\ t \in (0,T) \\ y_{tt}(t,x) = y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1), \\ y(T,x) = 0,\ y_t(T,x) = 0,\ x \in (0,1). \end{cases}$$

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For $t \in (0, 2)$, $k \in \mathbb{N}_0$ and $t + 2k \leq T$:

$$\boxed{u(t + 2k) = \left(z_\gamma^k - \frac{1}{z_\gamma^k}\right) \frac{1 + z_\gamma}{1 - z_\gamma} H(t).}$$

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This is an *exponential turnpike* structure!

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Inhalt

Turnpike for linear systems: Problem definition

For $d_- < 0 < d_+$, define the 2×2 matrix

$$D(x) = \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix}.$$

Let a 2×2 matrix $M(x)$ and $\eta_0 \in (-\infty, 0]$ be given.

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Consider the system equation

$$(pde): \; r_t + D\, r_x = \eta_0\, M\, r,$$

where for $t \in (0, T)$ and $x \in (0, L)$ the state is given by $r(t, x) = \begin{pmatrix} r_+(t, x) \\ r_-(t, x) \end{pmatrix}.$

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Initial conditions ($t = 0$)

For $x \in [0, L]$: $r_+(0, x) = 0$, $r_-(0, x) = 0$.

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$$D(x) = \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix}.$$

Let a 2×2 matrix $M(x)$ and $\eta_0 \in (-\infty, 0]$ be given.

Dirichlet boundary control ($x \in \{0, L\}$)

$r_+(t, 0) = u_+(t), \; r_-(t, L) = u_-(t)$ with boundary controls $u_+, u_- \in L^2(0, T)$.

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Objective Function

$$J(u, r) = \int_0^T f_0(u_+(t), r_-(t, 0))\, dt + \int_0^T f_L(u_-(t), r_+(t, L))\, dt$$

with strictly convex quadratic functions f_0, f_L.

Turnpike for linear systems: Problem definition

For $d_- < 0 < d_+$, define the 2 × 2 matrix

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Dynamic optimal control problem

Optimal boundary control problem

$$\begin{cases} \min_{u \in (L^2(0,T))^2} J(u, r) \\ \text{subject to (pde), initial c. and b.c.} \end{cases}$$

Stability Assumptions

For real numbers μ_+, μ_- define

$$E(x) = \begin{pmatrix} \exp(-\mu_+ \, x) & 0 \\ 0 & \exp(\mu_- \, x) \end{pmatrix}.$$

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Assume that there exist $\mu_+ > 0$, $\mu_- > 0$ and $\nu_a < 0$ ($\mu_+ < 0$, $\mu_- < 0$ and $\nu_0 > 0$ respectively) such that for all $x \in [0, L]$

$$\sup_{v:\, v^\top E(x)\, v = 1} v^\top \left[E'(x) D(x) + E(x) D'(x) - 2|\eta_0| E(x) M(x)\right] v \le \nu_a < 0 \text{ and respectively}$$

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$$\inf_{v:\, v^\top E(x)\, v = 1} v^\top \left[E'(x) D(x) - E(x) D'(x) + 2\,|\eta_0| E(x)\, M(x)\right] v \ge \nu_0 > 0.$$

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Can be checked with a 1-d optimization problem!

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Can be checked with a 1-d optimization problem!

1. If $M(x)$ is a diagonal matrix or if $|\eta_0|$ is sufficiently small or if $L > 0$ is **sufficiently small**, both conditions hold.

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($\mu_+ < 0$, $\mu_- < 0$ and $\nu_0 > 0$ respectively) such that for all $x \in [0, L]$

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$$\inf_{v:\, v^\top E(x)\, v=1} v^\top \left[E'(x)D(x) - E(x)D'(x) + 2\,|\eta_0| E(x)\, M(x)\right] v \geq \nu_0 > 0.$$

Can be checked with a 1-d optimization problem!

1. If $M(x)$ is a diagonal matrix or if $|\eta_0|$ is sufficiently small or if $L > 0$ is **sufficiently small**, both conditions hold.
2. If $M^\top = M$, both conditions equivalent with $\nu_0 = -\nu_a$.

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$$\text{(bvp)} \begin{cases} D\, R_x^{(\sigma)}(x) = \eta_0\, M(x)\, R^{(\sigma)}(x), \\ R_+^{(\sigma)}(0) = u_+^{(\sigma)}, \\ R_-^{(\sigma)}(L) = u_-^{(\sigma)}. \end{cases}$$

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Static optimal control problem

$$\begin{cases} \min_{u^{(\sigma)} \in \mathbb{R}^2} J_0(u^{(\sigma)},\ R^{(\sigma)}(x)) \\ \text{subject to (bvp).} \end{cases}$$

Inhalt

The dyn. and stat. problems have a turnpike prop.

SICON 2019 (with F. Hante)

For a finite time horizon $T > 0$, let the **optimal dynamic control** be

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For the state we have

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Inhalt

Conclusion

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  In the applications, the turnpike results allow to obtain
  L^1*-optimal controls* by *finite time stabilizing feedback controlers* (LIONEL ROSIER)
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Suspension Bridge

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NATURWISSENSCHAFTLICHE FAKULTÄT

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- In the L^2–case, the finite-time turnpike does not occur.
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- **Thank you for your attention!**