Local control of the non-isentropic Navier-Stokes equations

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Non-isentropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } (0,T)\times\Omega \\ \rho(\partial_t u + (u\cdot\nabla)u) + \nabla(p(\rho,\theta)) & \\ \qquad\qquad - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) = 0 & \text{in } (0,T)\times\Omega \\ C_v\rho(\partial_t\theta + u\cdot\nabla\theta) - \lambda \operatorname{div}(u_{\mathcal{S}})^2 - 2\mu D(u) : D(u) & \\ \qquad\qquad - p(\theta,\rho)\operatorname{div}(u) - \kappa\Delta\theta = 0 & \text{in } (0,T)\times\Omega, \end{cases} \tag{1}$$

where ρ is the density, u the velocity, θ is the temperature, and p is the pressure which is a function of ρ and θ obeying, for instance, the ideal gas law

$$p(\theta,\rho) = R\theta\rho. \tag{2}$$

Control Problem

We will consider the local exact controllability around constant states $(\overline{\rho}, \overline{u}, \overline{\theta}) \in \mathbb{R}_{+}^{*} \times (\mathbb{R}^{d} \setminus \{0\}) \times \mathbb{R}$. The controls applied to the system (1) in velocity and temperature are the values of $u_{\mathcal{S}}$ and $\theta_{\mathcal{S}}$ on the whole boundary

$$\theta|_{\partial\Omega}(t,x) = f_{\theta}(t,x), \tag{3}$$
$$u|_{\partial\Omega}(t,x) = f_{u}(t,x). \tag{4}$$

For what concerns the density we control only on the part of the region where the flow is inward $\Gamma = \{(t,x) \in [0,T] \times \partial\Omega| \;\; u(t,x) \cdot n(x) < 0\}$ where $\vec{n}$ is the exterior unit normal to $\partial\Omega$

$$\rho|_{\Gamma}(t,x) = f_{\rho}(t,x). \tag{5}$$

For the density we control only on the part of the region where the flow is inward $\Gamma = \{(t,x) \in [0,T] \times \partial\Omega| \quad u(t,x) \cdot n(x) < 0\}$ where $\vec{n}$ is the exterior unit normal to $\partial\Omega$

$$\rho|_{\Gamma}(t,x) = f_{\rho}(t,x). \tag{6}$$

The control problem is to find boundary controls such that the reach the constant state in time T

$$\rho(T) = \bar{\rho}$$

$$u(T) = \bar{u}$$

$$\theta(T) = \bar{\theta}$$

Previous results

- S. Ervedoza, O. Glass, and S. Guerrero: Local exact controllability of the isentropic Navier-Stokes equations
- Chowdhury, M. Ramaswamy, and J.-P. Raymond: controllability of the linear system in one dimension around zero velocity (isentropic).
- D. Maity. controllability of the linear system in one dimension around zero velocity (non-isentropic).

Theorem 1

Let $d \in \{2,3\}$, $\overline{\rho} > 0$ and $\overline{u} \in \mathbb{R}^d \setminus \{0\}$. Let $L > 0$ be larger than the thickness of Ω in the direction $\overline{u}/|\overline{u}|$, and assume

$$T > L/|\overline{u}|. \tag{7}$$

Then there exists $\delta > 0$ such that for all $(\rho_0, u_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$ satisfying

$$\|(\rho_0, u_0, \theta_0)\|_{H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)} \leq \delta, \tag{8}$$

there exists a solution (ρ, u, θ) of (1) satisfying the initial condition

$$\rho_{\mathcal{S}}(0,x) = \overline{\rho}+\rho_0(x), \quad u_{\mathcal{S}}(0,x) = \overline{u}+u_0(x), \quad \theta_{\mathcal{S}}(0,x) = \overline{\theta}+\theta_0(x) \qquad \text{in } \Omega, \tag{9}$$

and the final condition

$$\rho(T,x) = \overline{\rho}, \quad u(T,x) = \overline{u}, \quad \theta(T,x) = \overline{\theta} \quad \text{in } \Omega. \tag{10}$$

Theorem 1 (cont)

Furthermore, we can choose the control so that for the controlled trajectory (ρ, u, θ) one has the following regularity:

$$\begin{aligned} &\rho \in C([0,T]; H^2(\Omega)) \\ &u \in L^2(0,T; H^3(\Omega)) \cap C^0([0,T]; H^2(\Omega)) \\ &\theta \in L^2(0,T; H^3(\Omega)) \cap C^0([0,T]; H^2(\Omega)). \end{aligned} \tag{11}$$

Theorem 2

Let $d = 1$, $\overline{\rho} > 0$ and $\overline{u} \in \mathbb{R}^d \setminus \{0\}$. Let $L > 0$ be larger than the thickness of Ω in the direction $\overline{u}/|\overline{u}|$, and assume

$$T > L/|\overline{u}|. \tag{12}$$

Then there exists $\delta > 0$ such that for all $(\rho_0, u_0, \theta_0) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ satisfying

$$\|(\rho_0, u_0, \theta_0)\|_{H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)} \leq \delta, \tag{13}$$

there exists a solution $(\rho_{\mathcal{S}}, u_{\mathcal{S}}, \theta_{\mathcal{S}})$ of (1) satisfying the initial condition

$$\rho_{\mathcal{S}}(0,x) = \overline{\rho} + \rho_0(x), \quad u_{\mathcal{S}}(0,x) = \overline{u} + u_0(x), \quad \theta_{\mathcal{S}}(0,x) = \overline{\theta} + \theta_0(x) \qquad \text{in } \Omega, \tag{14}$$

and the final condition

$$\rho_{\mathcal{S}}(T,x) = \overline{\rho}, \quad u_{\mathcal{S}}(T,x) = \overline{u}, \quad \theta_{\mathcal{S}}(T,x) = \overline{\theta} \quad \text{in } \Omega. \tag{15}$$

Theorem 2 (cont)

Furthermore, we can choose the control so that for the controlled trajectory $(\rho_{\mathcal{S}}, u_{\mathcal{S}}, \theta_{\mathcal{S}})$ one has the following regularity:

$$
\begin{aligned}
&\rho_{\mathcal{S}} \in C([0,T]; H^1(\Omega)) \\
&u_{\mathcal{S}} \in L^2(0,T; H^2(\Omega)) \cap C^0([0,T]; H^1(\Omega)) \\
&\theta_{\mathcal{S}} \in L^2(0,T; H^2(\Omega)) \cap C^0([0,T]; H^1(\Omega)).
\end{aligned}
\tag{16}
$$

We transform the boundary control problem in an interior control problem without boundary by embedding the domain Ω in the flat torus $\mathbb{T}_L$.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = \check{v}_\rho & \text{in } (0,T)\times\mathbb{T}_L \\ \rho(\partial_t u + (u\cdot\nabla)u) + \nabla(p(\rho,\theta)) & \\ \qquad - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u = \check{v}_u & \text{in } (0,T)\times\mathbb{T}_L \\ C_v\rho(\partial_t\theta + u\cdot\nabla\theta) - \lambda \operatorname{div}(u)^2 - 2\mu D(u) : D(u) & \\ \qquad - p(\theta_{\mathcal{S}},\rho)\operatorname{div}(u) - \kappa\Delta\theta = \check{v}_\theta & \text{in } (0,T)\times\mathbb{T}_L \end{cases} \tag{17}$$

with $\check{v}_\rho$, $\check{v}_u$ and $\check{v}_\theta$ supported in $\mathbb{T}_L \setminus \overline{\Omega}$.

Because we are interested in local controllability around $(\bar{\rho}, \bar{u}, \bar{\theta})$ we consider the translations

$$\check{\rho} := \rho - \overline{\rho}, \quad \check{u} := u - \overline{u}, \quad \check{\theta} := \theta - \overline{\theta}.$$

Also we separate the system into the linear* and nonlinear terms.

$$\begin{cases} \partial_t \check{\rho} + (\bar{u} + \check{u}) \cdot \nabla \check{\rho} + \bar{\rho}\, \text{div}(\check{u}) = \check{v}_\rho + \check{f}_\rho(\check{\rho}, \check{u}, \check{\theta}) & \text{in } (0,T) \times \mathbb{T} \\ \bar{\rho}(\partial_t \check{u} + (\bar{u} + \check{u}) \cdot \nabla \check{u}) - \mu \Delta \check{u} - (\mu + \lambda) \nabla\, \text{div}\, \check{u} & \\ \qquad + \bar{p}_\rho \nabla \check{\rho} + \bar{p}_\theta \nabla \check{\theta} = \check{v}_u + \check{f}_u(\check{\rho}, \check{u}, \check{\theta}) & \text{in } (0,T) \times \mathbb{T} \\ C_v \bar{\rho}(\partial_t \check{\theta} + (\bar{u} + \check{u}) \cdot \nabla \check{\theta}) - \kappa \Delta \check{\theta} + \bar{p}\, \text{div}(\check{u}) = \check{v}_\theta + \check{f}_\theta(\check{\rho}, \check{u}, \check{\theta}) & \text{in } (0,T) \times \mathbb{T} \end{cases} \tag{18}$$

where the nonlinear terms are

$$\check{f}_\rho(\check{\rho}, \check{u}, \check{\theta}) = -\check{\rho}\,\mathrm{div}(\check{u}) \tag{19}$$

$$\check{f}_u(\check{\rho}, \check{u}, \check{\theta}) = -\check{\rho}(\partial_t \check{u} + (\bar{u} + \check{u}) \cdot \nabla \check{u}) - \nabla(p(\bar{\rho} + \check{\rho}, \bar{\theta} + \check{\theta}) - \bar{p}_\rho \rho - \bar{p}_\theta \theta) \tag{20}$$

$$\begin{aligned} \check{f}_\theta(\check{\rho}, \check{u}, \check{\theta}) = -C_v \check{\rho}(\partial_t \check{\theta} + (\bar{u} + \check{u}) \cdot \nabla \check{\theta}) + \lambda\,\mathrm{div}(\check{u})^2 + 2\mu D(\check{u}) : \nabla \check{u} \\ - (p(\bar{\theta} + \check{\theta}, \bar{\rho} + \check{\rho}) - \bar{p})\,\mathrm{div}(\check{u}) \end{aligned} \tag{21}$$

In order to take care of these nonlinear terms, we consider the following change of coordinates

$$\frac{dX_w}{dt}(t,\tau,x) = w(t, X_w(t,\tau,x)), \quad t \in [0,T], \qquad X_w(\tau,\tau,x) = x. \tag{22}$$

$$Y_{\check{u}}(t,x) = X_{\bar{u}+\check{u}}(t,T,X_{\bar{u}}(T,t,x)) \tag{23}$$

and the variables in this coordinates:

$$\rho(t,x) = \check{\rho}(t, Y_{\check{u}}(t,x)), \quad u(t,x) = \check{u}(t, Y_{\check{u}}(t,x)), \quad \theta(t,x) = \check{\theta}(t, Y_{\check{u}}(t,x)). \tag{24}$$

In these coordinates the system is:

$$\begin{cases} \partial_t \rho + \bar{u}\cdot\nabla\rho + \bar{\rho}\,\mathrm{div}(u) = v_\rho \chi + f_\rho(\rho,u,\theta) & \text{in } (0,T)\times\mathbb{T}_L, \\ \bar{\rho}(\partial_t u + \bar{u}\cdot\nabla u) - \mu\Delta u - (\mu+\lambda)\nabla\,\mathrm{div}\,u & \\ \qquad\qquad + \bar{p}_\rho\nabla\rho + \bar{p}_\theta\nabla\theta = v_u\chi + f_u(\rho,u,\theta) & \text{in } (0,T)\times\mathbb{T}_L, \\ C_v\bar{\rho}(\partial_t\theta + \bar{u}\cdot\nabla\theta) - \kappa\Delta\theta + \bar{p}\,\mathrm{div}(u) = v_\theta\chi + f_\theta(\rho,u,\theta) & \text{in } (0,T)\times\mathbb{T}_L, \end{cases} \tag{25}$$

Study of the linear system

$$\begin{cases} \partial_t \rho + \bar{u} \cdot \nabla \rho + \bar{\rho} \,\mathrm{div}(u) = v_\rho \chi + \widehat{f_\rho} & \text{in } (0, T) \times \mathbb{T}_L, \\ \bar{\rho}(\partial_t u + \bar{u} \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \,\mathrm{div}\, u & \\ \qquad\qquad + \bar{p}_\rho \nabla \rho + \bar{p}_\theta \nabla \theta = v_u \chi + \widehat{f_u} & \text{in } (0, T) \times \mathbb{T}_L, \\ C_v \bar{\rho}(\partial_t \theta + \bar{u} \cdot \nabla \theta) - \kappa \Delta \theta + \bar{p} \,\mathrm{div}(u) = v_\theta \chi + \widehat{f_\theta} & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \tag{26}$$

The controllability of the previous system is equivalent to the observability property of the adjoint system:

$$\begin{cases} -(\partial_t \sigma + \bar{u}\cdot\nabla\sigma) - \bar{p}_\rho \operatorname{div}(z) = g_\sigma & \text{in } (0,T)\times\mathbb{T}_L, \\ -\bar{\rho}(\partial_t z + (\bar{u}\cdot\nabla)z) - \mu\Delta z - (\mu+\lambda)\nabla(\operatorname{div} z) & \\ \qquad\qquad - \bar{\rho}\nabla\sigma - \bar{p}\nabla\eta = g_z & \text{in } (0,T)\times\mathbb{T}_L, \\ -C_v\bar{\rho}(\partial_t\eta + \bar{u}\cdot\nabla\eta) - \kappa\Delta\eta - \bar{p}_\theta \operatorname{div}(z) = g_\eta & \text{in } (0,T)\times\mathbb{T}_L. \end{cases} \tag{27}$$

If we take divergence on the equation of z we obtain:

$$\begin{cases} -(\partial_t\sigma + \bar{u}\cdot\nabla\sigma) - \bar{p}_\rho \operatorname{div}(z) = g_\sigma & \text{in } (0,T)\times\mathbb{T}_L, \\ -\bar{\rho}(\partial_t \operatorname{div}(z) + \bar{u}\cdot\nabla \operatorname{div}(z)) & \\ \qquad\qquad -\nu\Delta(\operatorname{div}(z) + \frac{\bar{\rho}}{\nu}\sigma + \frac{\bar{p}}{\nu}\eta) = \operatorname{div}(g_z) & \text{in } (0,T)\times\mathbb{T}_L, \\ -C_v\bar{\rho}(\partial_t\eta + \bar{u}\cdot\nabla\eta) - \kappa\Delta\eta - \bar{p}_\theta \operatorname{div}(z) = g_\eta & \text{in } (0,T)\times\mathbb{T}_L, \end{cases} \tag{28}$$

Introducing the variable(similar to the effective viscous flux introduced by P.-L. Lions)

$$q := \operatorname{div}(z) + \frac{\bar{\rho}}{\nu}\sigma. \tag{29}$$

$$
\begin{cases}
-(\partial_t \sigma + \bar{u}\cdot\nabla\sigma) = \bar{p}_\rho(q - \frac{\bar{\rho}}{\nu}\sigma) + g_\sigma & \text{in } (0,T)\times\mathbb{T}_L, \\
-\bar{\rho}(\partial_t q + \bar{u}\cdot\nabla q) - \nu\Delta q = \bar{p}\Delta\eta & \\
\qquad + \frac{\bar{\rho}^2}{\nu}\bar{p}_\rho(q - \frac{\bar{\rho}}{\nu}\sigma) + \operatorname{div}(g_z) + \frac{\bar{\rho}^2}{\nu}g_\sigma & \text{in } (0,T)\times\mathbb{T}_L, \\
-C_v\bar{\rho}(\partial_t\eta + \bar{u}\cdot\nabla\eta) - \kappa\Delta\eta = \bar{p}_\theta(q - \frac{\bar{\rho}}{\nu}\sigma) + g_\eta & \text{in } (0,T)\times\mathbb{T}_L,
\end{cases}
\tag{30}
$$

- The observability of this system is equivalent to the previous one because if we have estimates on (σ, q, η), we can use the equation

$$
-\overline{\rho}(\partial_t z + \overline{u}\cdot\nabla z) - \mu\Delta z = g_z + \bar{\rho}\frac{\mu}{\nu}\nabla\sigma + (\lambda+\mu)\nabla q + \bar{p}\nabla\eta \tag{31}
$$

- Importantly, we got rid of almost all the higher order coupling term (except for the diffusion)
- If $\nu \neq \frac{\kappa}{C_v}$ we can diagonalize the diffusion part and eliminate completely the higher order coupling.

By taking the adjoint again, we obtain that the observability of the previous system is equivalent to the controllability of the following system:

$$\begin{cases} \partial_t r + \bar{u}\cdot\nabla r = -\frac{\bar{\rho}}{\nu}(\bar{p}_\rho r + \frac{\bar{\rho}^2}{\nu}\bar{p}_\rho y + \bar{p}_\theta h) + f_r + v_r\chi_0 & \text{in } (0,T)\times \\ \bar{\rho}(\partial_t y + \bar{u}\cdot\nabla y) - \nu\Delta y = (\bar{p}_\rho r + \frac{\bar{\rho}^2}{\nu}\bar{p}_\rho y + \bar{p}_\theta h) + f_y + v_y\chi_0 & \text{in } (0,T)\times \\ C_v\bar{\rho}(\partial_t h + \bar{u}\cdot\nabla h) - \kappa\Delta h = \bar{p}\Delta y + f_h + v_h\chi_0 & \text{in } (0,T)\times \\ (r(0,\cdot), y(0,\cdot), h(0,\cdot) = (r_0, y_0, h_0) & \text{in} \\ (r(T,\cdot), y(T,\cdot), h(T,\cdot)) = (0,0,0) & \text{in} \end{cases} \quad (32)$$

Controllability of $(r, y, h) \in L^2(H^2) \times L^2(H^4) \times L^2(H^3)$ with $(f_r, f_y, f_h) \in L^2(H^2) \times L^2(H^2) \times L^2(H^1)$

$$\Downarrow$$

Observability of $(\sigma, q, \eta) \in L^2(H^{-2}) \times L^2(H^{-2}) \times L^2(H^{-1})$ with $(g_\sigma, g_q, g_\eta) \in L^2(H^{-2}) \times L^2(H^{-4}) \times L^2(H^{-3})$

$$\Downarrow$$

Observability of $(\sigma, z, \eta) \in L^2(H^{-2}) \times L^2(H^{-1}) \times L^2(H^{-1})$ with $(g_\sigma, g_z, g_\eta) \in L^2(H^{-2}) \times L^2(H^{-3}) \times L^2(H^{-3})$

$$\Downarrow$$

Controllability of $(\rho, u, \theta) \in L^2(H^2) \times L^2(H^3) \times L^2(H^3)$ with $(f_\rho, f_u, f_\theta) \in L^2(H^2) \times L^2(H^1) \times L^2(H^1)$

Carleman weight

We use a Carleman weight introduced by Badra, Ervedoza and Gerrero. We take $\psi = \psi(t,x) \in C^2([0,T] \times \mathbb{T}_L, [6,7])$ that satisfies the transport equation

$$\partial_t \psi + \overline{u} \cdot \nabla \psi = 0 \text{ in } (0,T) \times \mathbb{T}_L, \tag{33}$$

and for which, there exists a subset $\omega \subset\subset \{\chi_0 = 1\}$ such that ψ does not have critical points in $[0,T] \times (\mathbb{T}_L \setminus \omega)$. Now we choose $T_0 > 0,\ T_1 > 0$ and $\varepsilon > 0$ small enough so that

$$T_0 + 2T_1 < T - \frac{L_0 + 12\varepsilon}{|\overline{u}|}. \tag{34}$$

Carleman weight

Now for any real number $\alpha \geq 2$, we introduce the weight function in time $\zeta(t)$ defined by

$$\zeta = \zeta(t) \quad \text{such that} \quad \begin{cases} \forall t \in [0, T_0], \, \zeta(t) = 1 + \left(1 - \dfrac{t}{T_0}\right)^{\alpha}, \\ \forall t \in [T_0, T - 2T_1], \, \zeta(t) = 1, \\ \forall t \in [T - T_1, T), \, \zeta(t) = \dfrac{1}{T - t}, \\ \zeta \text{ is increasing on } [T - 2T_1, T - T_1], \\ \zeta \in C^2([0, T)). \end{cases} \tag{35}$$

Then we consider the following weight function $\varphi = \varphi(t, x)$:

$$\varphi(t, x) = \zeta(t) \left(\lambda_0 e^{12\lambda_0} - \exp(\lambda_0 \psi(t, x))\right), \tag{36}$$

where s, λ_0 are positive parameters with $s \geq 1$, $\lambda_0 \geq 1$. The parameter α is chosen as

$$\alpha = s\lambda_0^2 e^{2\lambda_0} \geq 2, \tag{37}$$

Carleman estimates for transport equation

The equation

$$\begin{cases} \partial_t r + \overline{u} \cdot \nabla r + \dfrac{\overline{p}_\rho \overline{\rho}}{\nu} r = \tilde{f}_r + v_r \chi_0, & \text{in } (0, T) \times \mathbb{T}_L, \\ r(0, \cdot) = r_0, & \text{in } \mathbb{T}_L, \end{cases} \tag{38}$$

is null controllable and the controlled trajectory satisfies the following estimate

$$\|\zeta^{-1} r e^{s\varphi}\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \|\zeta^{-1} v_r e^{s\varphi}\|_{L^2(0,T;L^2(\mathbb{T}_L))} \leq C \left(\|\zeta^{-1} \tilde{f}_r e^{s\varphi}\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \|r_0 e^{s\varphi(0)}\|_{L^2(\mathbb{T}_L)} \right). \tag{39}$$

Carleman estimates for heat equation

The equation

$$\begin{cases} \dfrac{\overline{\rho}}{\nu}\partial_t y - \Delta y = \tilde{f}_y + v_y \chi_0, & \text{in } (0,T) \times \mathbb{T}_L, \\ y(0,\cdot) = y_0, & \text{in } \mathbb{T}_L, \end{cases} \tag{40}$$

is null controllable and the controlled trajectory satisfies the following estimate

$$\begin{aligned} s^{\frac{3}{2}} \left\| y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-\frac{3}{2}} \chi_0 v_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ + s^{1/2} \left\| \zeta^{-1} \nabla y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ \leq C \left\| \zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + C s^{1/2} \left\| y_0 e^{s\varphi(0)} \right\|_{L^2(\mathbb{T}_L)}. \end{aligned} \tag{41}$$

Proposition

Let us fix an initial condition $(r_0, y_0, h_0) \in L^2(\mathbb{T}_L) \times L^2(\mathbb{T}_L) \times L^2(\mathbb{T}_L)$. There exist $C > 0$ and $s_0 \geq 1$ large enough such that for all $s \geq s_0$, if f_r, f_y and f_h satisfy the estimates

$$\left\|\zeta^{-1} f_r e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\|\zeta^{-1} f_y e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\|\zeta^{-\frac{3}{2}} f_h e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} < \infty, \quad (42)$$

there exists a controlled trajectory (r, y, h) solving (32)

Proposition (cont)

and satisfying the following estimate:

$$
\begin{aligned}
&s^{\frac{2}{3}}\left\|\zeta^{-1} r e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s\left\|h e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\|\zeta^{-1}\nabla h e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
&\quad + s^{\frac{11}{6}}\left\|\zeta^{\frac{1}{2}} y e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}}\left\|\zeta^{-\frac{1}{2}}\nabla y e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
&\quad + s^{-\frac{1}{6}}\left\|\zeta^{-\frac{3}{2}}\Delta y e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
&\leq C\left(s^{\frac{2}{3}}\left\|\zeta^{-1} f_r e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{1}{3}}\left\|\zeta^{-1} f_y e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))}\right. \\
&\quad \left. + s^{-1/2}\left\|\zeta^{-\frac{3}{2}} f_h e^{s\varphi}\right\|_{L^2(0,T;L^2(\mathbb{T}_L))}\right) \\
&\quad + C\left(s^{\frac{2}{3}}\left\|r_0 e^{s\varphi(0)}\right\|_{L^2(\mathbb{T}_L)} + s^{\frac{5}{6}}\left\|y_0 e^{s\varphi(0)}\right\|_{L^2(\mathbb{T}_L)} + \left\|h_0 e^{s\varphi(0)}\right\|_{L^2(\mathbb{T}_L)}\right).
\end{aligned}
\tag{43}
$$

Idea of the proof: Use a fixed point argument with the map $(\tilde{r}, \tilde{y}, \tilde{h}) \to (r, y, h)$ solving

$$\begin{cases} \partial_t r + \bar{u} \cdot \nabla r = -\frac{\bar{\rho}}{\nu}(\bar{p}_\rho \tilde{r} + \frac{\bar{\rho}^2}{\nu}\bar{p}_\rho \tilde{y} + \bar{p}_\theta \tilde{h}) + f_r + v_r \chi_0 & \text{in } (0,T) \times \\ \bar{\rho}(\partial_t y + \bar{u} \cdot \nabla y) - \nu \Delta y = (\bar{p}_\rho \tilde{r} + \frac{\bar{\rho}^2}{\nu}\bar{p}_\rho \tilde{y} + \bar{p}_\theta \tilde{h}) + f_y + v_y \chi_0 & \text{in } (0,T) \times \\ C_v \bar{\rho}(\partial_t h + \bar{u} \cdot \nabla h) - \kappa \Delta h = \bar{p} \Delta \tilde{y} + f_h + v_h \chi_0 & \text{in } (0,T) \times \\ (r(0,\cdot), y(0,\cdot), h(0,\cdot) = (r_0, y_0, h_0) & \text{in} \\ (r(T,\cdot), y(T,\cdot), h(T,\cdot)) = (0,0,0) & \text{in} \end{cases} \tag{44}$$

and proof that is a contraction for the norm:

$$s^{\frac{2}{3}} \left\| \zeta^{-1} r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s \left\| h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \nabla h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{11}{6}} \left\| \zeta^{\frac{1}{2}} y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}} \left\| \zeta^{-\frac{1}{2}} \nabla y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \tag{45}$$

Proposition

There exists $s_0 \geq 1$, such that for all $s \geq s_0$, for all $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0) \in H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$, $\widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta$ such that $\widehat{f}_\rho e^{s\Phi} \in L^2(0, T; H^2(\mathbb{T}_L))$ and $\widehat{f}_u e^{7s\Phi/6}, \widehat{f}_\theta e^{s\Phi} \in L^2(0, T; H^1(\mathbb{T}_L))$, there exist control functions v_ρ, v_u, v_θ and a corresponding controlled trajectory (ρ, u, θ) solving (26) with initial data $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$, satisfying the control problem and depending linearly on the data $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, \widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta)$.

Proposition (cont)

Besides, we have the estimate:

$$
\begin{aligned}
&\left\|(\rho e^{6s\Phi/7}, u e^{6s\Phi/7}, \theta e^{6s\Phi/7})\right\|_{L^2(0,T;H^2(\mathbb{T}_L))\times L^2(0,T;H^3(\mathbb{T}_L))\times L^2(0,T;H^3(\mathbb{T}_L))} \\
&+\left\|(\chi v_\rho e^{6s\Phi/7}, \chi v_u e^{6s\Phi/7}, \chi v_\theta e^{6s\Phi/7})\right\|_{L^2(0,T;H^2(\mathbb{T}_L))\times L^2(0,T;H^1(\mathbb{T}_L))\times L^2(0,T;H^1(\mathbb{T}_L))} \\
&\quad\leq C\left\|(\widehat{f}_\rho e^{s\Phi}, \widehat{f}_u e^{7s\Phi/6}, \widehat{f}_\theta e^{s\Phi})\right\|_{L^2(0,T;H^2(\mathbb{T}_L))\times L^2(0,T;H^1(\mathbb{T}_L))\times L^2(0,T;H^1(\mathbb{T}_L))} \\
&\qquad + C\left\|(\widehat{\rho}_0 e^{s\Phi(0)}, \widehat{u}_0 e^{7s\Phi(0)/6}, \widehat{\theta}_0 e^{s\Phi(0)})\right\|_{H^2(\mathbb{T}_L)\times H^2(\mathbb{T}_L)\times H^2(\mathbb{T}_L)}. \qquad (46)
\end{aligned}
$$

where

$$\Phi(t) := \zeta(t)\lambda_0 e^{12\lambda_0}. \qquad (47)$$

Idea of the proof of the controllability of the original problem

Using Schauder Fixed point theorem to the map

$$\begin{cases} \partial_t \rho + \bar{u}\cdot\nabla\rho + \bar{\rho}\,\mathrm{div}(u) = v_\rho \chi + f_\rho(\widehat{\rho},\widehat{u},\widehat{\theta}) & \text{in } (0,T)\times\mathbb{T}_L, \\ \bar{\rho}(\partial_t u + \bar{u}\cdot\nabla u) - \mu\Delta u - (\mu+\lambda)\nabla\,\mathrm{div}\,u + \bar{p}_\rho\nabla\rho & \\ +\,\bar{p}_\theta\nabla\theta = v_u\chi + f_u(\widehat{\rho},\widehat{u},\widehat{\theta}) & \text{in } (0,T)\times\mathbb{T}_L, \\ C_v\bar{\rho}(\partial_t\theta + \bar{u}\cdot\nabla\theta) - \kappa\Delta\theta + \bar{p}\,\mathrm{div}(u) = v_\theta\chi + f_\theta(\widehat{\rho},\widehat{u},\widehat{\theta}) & \text{in } (0,T)\times\mathbb{T}_L, \end{cases} \tag{48}$$

in the set

$$C_R = \{(\rho,u,\theta) \text{ with } \rho\in E, u\in F, \theta\in F \\ \left\|(\rho e^{5s\Phi/6}, u e^{5s\Phi/6}, \theta e^{5s\Phi/6})\right\|_{E\times F^d\times F} \le R\}.$$

where

$$E = L^{\infty}(0, T; H^2(\mathbb{T}_L)) \cap H^1(0, T; L^2(\mathbb{T}_L)) \quad (49)$$

$$F = L^2(0, T; H^3(\mathbb{T}_L)) \cap L^{\infty}(0, T; H^2(\mathbb{T}_L)) \cap H^1(0, T; H^1(\mathbb{T}_L)) \quad (50)$$

we can estimate the nonlinear terms as

$$\left\| \left(f_{\rho}(\widehat{\rho}, \widehat{u}) e^{s\Phi}, f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}) e^{7s\Phi/6}, f_{\theta}(\widehat{\rho}, \widehat{u}, \widehat{\theta}) e^{s\Phi} \right) \right\|_{L^2(H^2(\mathbb{T}_L)) \times L^2(H^1(\mathbb{T}_L)) \times L^2(H^1(\mathbb{T}_L$$

$$\leq CR^2, \quad (51)$$

which allows to prove that the fixed point map is well defined from C_R to C_R. And finally proving compactness of the map gives us a fixed point.

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Thank you for your attention!