Thermoelasticity : from exponential to polynomial decay

S. NAFIRI

Team of Modeling and Numerics (MoNum)
Ecole Hassania des Travaux Publics (EHTP)

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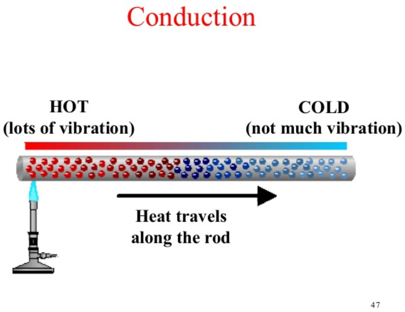
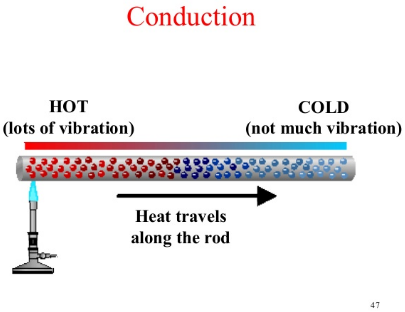
Outline

1. Introduction and Motivation
2. Asymptotic behavior of thermoelasticity
3. Approximation and simulation of thermoelastic system
   - Exponential decay
   - Polynomial decay
4. Impact of **B.C** on the behavior of solutions
5. Conclusion and open problems

Introduction and motivation

Experiment


Conduction
HOT
(lots of vibration)
COLD
(not much vibration)
Heat travels
along the rod
47



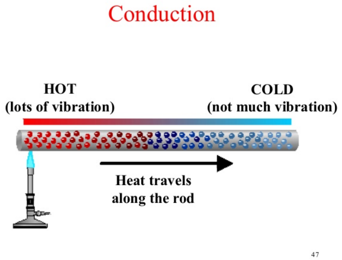
It is well known from experiment that the deformation of a body is inseparably connected with a change of its heat content and therefore with a change of the temperature distribution in the body=thermal expansion property

It is natural...

The first works on thermal stresses and thermoelasticity :

- **FOURIER, J-B.J**, Théorie Analytique de la Chaleur. Paris, Didot, 1822.
- **Duhamel, J.-M.-C.**, Second mémoire sur les phénomènes thermo-mécaniques, J. de l'École Polytechnique, tome 15, cahier 25, 1837, pp. 1–57.

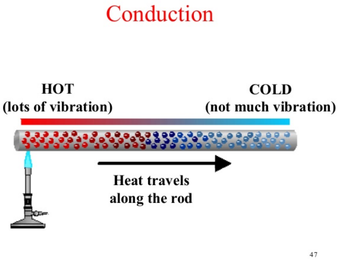
Thermal expansion



$$u_{tt}(x,t) - u_{xx}(x,t) = 0$$

$$\theta_t(x,t) - \theta_{xx}(x,t) = 0$$

Thermal expansion



$$(S)\begin{cases} u_{tt} - u_{xx} + \gamma\theta_x & = 0 \\ \theta_t - \theta_{xx} + \gamma u_{tx} & = 0 \end{cases}$$

Linear Thermoelasticity

$$(S)\begin{cases} u_{tt} - u_{xx} + \gamma\theta_x & = 0, \quad \Omega \times (0, +\infty), \\ \theta_t - \theta_{xx} + \gamma u_{tx} & = 0, \quad \Omega \times (0, +\infty), \end{cases}$$

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- **A. DAY**, Heat Conduction with Linear Thermoelasticity, Springer-Verlag, New York, 1985.
- **CARLSON, D. E.**, Linear thermoelasticity. In Handbuch der Physik. Bd.Vla/2, edited by C. Truesdell. Berlin, Springer, 1972.
- **C. Dafermos**, On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity, Arch. Rational. Mech. Anal. 29, 241-271 (1968)
- **Clarence Zener**, Internal Friction in Solids. I. Theory of Internal Friction in Reeds, Phys. Rev. 52, 230–235 (1937) ; 53, 90–99 (1938).

Asymptotic behavior of thermoelasticity

Strong stability

$$(S)\begin{cases} u_{tt}-u_{xx}+\gamma\theta_x & =0, \quad \Omega\times(0,+\infty),\\ \theta_t-\theta_{xx}+\gamma u_{tx} & =0, \quad \Omega\times(0,+\infty),\\ u=0=\theta & =0, \quad \partial\Omega \\ +I.C \end{cases}$$

$$E(t) \underset{t\to\infty}{\to} 0 \text{ (Dafermos 1968)}$$

C. M. Dafermos, On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. Arch. Rat. Mech. Anal., 29, 1968, pp. 241-271.

Remark : No decay rate was given.

Exponential decay remains open for some time (24 years !)

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$\exists C,\alpha>0$ s.t :

$$\|T(t)y\|_{D(A)}\leqslant Ce^{-\alpha t}\|y\|_{D(A)} \quad \text{(Slemrod 1981)}$$

- **Slemrod, M**, Global existence, uniqueness, and asymptotic stability of classical smooth solutionsin one-dimensional non-linear thermoelasticity,Arch. Rational Mech. Anal., 76(1981), 97-133.
- **J.E.M. Rivera**, Energy decay rate in linear thermoelasticity, Funkcial Ekvac., Vol. 35 (1992), pp. 19-30.

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$\exists C, \alpha > 0$ s.t :

$$\|T(t)y\|_H \leqslant Ce^{-\alpha t}\|y\|_H \quad \text{(Hansen 1992)}$$

- **S. W. Hansen**, Exponential energy decay in a linear thermoelastic rod. J. Math. Anal. Appli.,167, 1992, pp. 429-442.
- **Z. Liu and S.M. Zheng**, Exponential stability of the semigroup associated with a thermoelastic system, Quart. Appl. Math. 51 (1993), pp. 535-545.
- **Z.Y. Liu and S. Zheng**, Semigroups Associated with Dissipative Systems. Chapman & Hall/CRC Research Notes in Mathematics Series (1999).

Techniques used in litterature

- The energy method (Slemrod 1981)
- The spectral analysis method (Rivera, Shibata 1990)
- Fourier series expansion method and decoupling technique (Hansen 1992)
- Combination of semigroup theory and energy method (Gibson, Rosen and Tao 1992)
- Control theory approach and a uniqueness continuation theorem (Kim 1992)
- Contradiction argument (Gearhart-Prüss) and PDE technique (Liu & Zheng 1993)

$$(S)\begin{cases} u_{tt} + Au + \gamma A^{\alpha}\theta & = 0 \\ \theta_t + A^{\beta}\theta + \gamma A^{\alpha}u_t & = 0 \end{cases}$$

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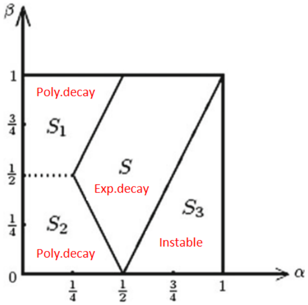
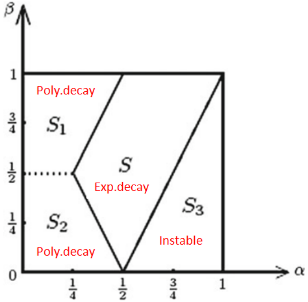


F. Ammar-Khodja, A. Benabdallah and D. Teniou, Dynamical stabilizers and coupled systems. ESAIM Proceedings 2, (1997), pp 253-262.

F. Ammar-Khodja, A. Bader, A. Benabdallah, Dynamic stabilization of systems via decoupling techniques. ESAIM Control Optim. Calc. Var. 4, (1999), pp. 577-593.



F. Alabau, P. Cannarsa, V. Komornik, Indirect internal stabilization of weakly coupled evolution equations, J.evol.equ. (2002) 2 : 127.

J. Hao and Z. Liu, Stability of an abstract system of coupled hyperbolic and parabolic equations. Zeitschrift für angewandte Mathematik und Physik, 64, (2013), pp. 1145-1159.

Exponential decay

Theorem : Gearhart 1978 and Prüss 1984

Let H be a Hilbert space and A the generator of a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$. $\{T(t)\}_{t\geqslant 0}$ is exponentially stable

$$\Updownarrow$$

$$\sup\{Re\lambda, \lambda \in \sigma(A)\} < 0 := s(A) < 0 + \sup_{Re\lambda\geqslant 0} \{\|(\lambda I - A)^{-1}\|\} < \infty$$

Theorem : Huang Falun 1985

In the previous result if $\{T(t)\}_{t\geqslant 0}$ is a contraction semigroup, then $T(\cdot)$ is exponentially stable $\Leftrightarrow i\mathbb{R} \subset \rho(A)$ and $\sup_{\beta\in\mathbb{R}}\{\|(i\beta I - A)^{-1}\|\} < \infty$

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Remark : There is also Lyapunov theory.

Question :

If a system of PDE decays exponentially to zero, what are conditions for which its numerical approximation still decreases exponentially to zero uniformly with respect to the step size ?

Approximation and simulation of thermoelastic system

Theorem : Z. Liu & S. Zheng 1994, SIAM J. CONTROL AND OPTIMIZATION

Let $T_n(\cdot)$, $(n = 1, \cdots)$ be a sequence of c_0-semigroups of operators on the Hilbert spaces H_n and let $\mathcal{A}_n$ be the corresponding infinitesimal generators. Then $T_n(\cdot)$ are uniformly exponentially stable iff

1. $\sup\limits_{n\in\mathbb{N}}\{Re\lambda, \lambda \in \sigma(\mathcal{A}_n)\} < 0 := \sigma_0 < 0\,;$
2. $\exists\sigma \in (\sigma_0, 0)$ s.t :
$$\sup_{Re\lambda\geqslant\sigma, n\in\mathbb{N}}\{\|(\lambda I_n - \mathcal{A}_n)^{-1}\|\} = M_0 < \infty$$
3. $\exists M_1 > 0$ s.t : $\|T_n(t)\|_{\mathcal{L}(H_n,H_n)} \leqslant M_1 < \infty, \quad \forall t > 0, \quad n \in \mathbb{N}$

Theorem : Liu & Zheng 1994, SIAM J. CONTROL AND OPTIMIZATION

If the family $\{T_n(\cdot)\}$, $(n = 1, \cdots)$ of c_0-semigroups is of contraction, on the Hilbert spaces H_n and $\mathcal{A}_n$ be the corresponding infinitesimal generators. Then $T_n(\cdot)$ are uniform. exponentially stable $\Leftrightarrow$
$\forall n \in \mathbb{N}, \quad i\mathbb{R} \subset \rho(\mathcal{A}_n)$ and $\sup\limits_{\beta\in\mathbb{R}, n\in\mathbb{N}}\{\|(i\beta I_n - \mathcal{A}_n)^{-1}\|\} < \infty$

Exponential decay of approximate thermoelaticity

$$
(S)\begin{cases} u_{tt}-u_{xx}+\gamma\theta_x & =0, \quad \Omega\times(0,+\infty),\\ \theta_t-\theta_{xx}+\gamma u_{tx} & =0, \quad \Omega\times(0,+\infty),\\ u=0=\theta & =0, \quad \partial\Omega\\ +I.C \end{cases}
$$

- **J. S. GIBSON, I. G. ROSEN, AND G. TAO**, Approximation in control of thermoelastic systems, SIAM J. Control. Optim., 30 (1992), pp. 1163-1189.
- **Z. Y. Liu and S. Zheng**, Uniform exponential stability and approximation in control of a thermoelastic system. SIAM J. Control Optim. 32, (1994), pp. 1226-1246.

Numerical simulation for exp.case : FDM, FEM, MFEM

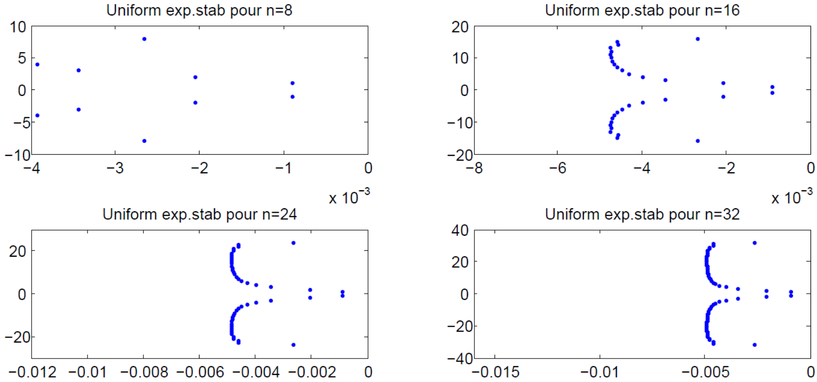

Table – Distance between $\sigma(A_n)$ and the imaginary axis for the spectral method in the case of Dirichlet-Dirichlet boundary conditions.

n	min{-Reλ, $\lambda \in \sigma(A_n)$}
8	8.9227$\times 10^{-4}$
16	8.9383$\times 10^{-4}$
24	8.9402$\times 10^{-4}$
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Theorem : Hansen 1992

If $\gamma < 1/2$. Eigenvalues of the generators A_n $(z_n' = A_n z_n, z_n(0) = z_{n0})$ satisfy

$$\sup_{\lambda \in \sigma(A_n) - \{0\}} Re\lambda \leqslant -\frac{\gamma^2}{2}.$$

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Farid Ammar Khodja, Assia Benabdallah, and Djamel Teniou, Stability of coupled systems, Abstr. Appl. Anal. Volume 1, Number 3 (1996), 327-340.

Uniform exponential case

- **H. T. Banks, K. Ito and C.Wang**. Exponentially stable approximations of weakly damped wave equations. Internat. Ser. Numer. Math. 100, Birkhäuser, (1991), pp. 1-33.
- **J. A. Infante and E. Zuazua**. Boundary observability for the space semi-discretizations of the $1-d$ wave equation. ESAIM : Mathematical Modelling and Numerical Analysis, 33, (1999), pp. 407-438.
- **L. I. Ignat and E. Zuazua**, A two-grid approximation scheme for nonlinear Schrödinger equations : dispersive properties and convergence, C. R. Math. Acad. Sci. Paris, 341, (2005), pp. 381-386.
- **K. Ramdani, T. Takahashi and M. Tucsnak**, Uniformly exponentially stable approximations for a class of second order evolution equations application to LQR problems. ESAIM Control. Optim. Calc. Var., 13, (2007), pp. 503-527.
- **S. Ervedoza and E. Zuazua**, Uniform exponential decay for viscous damped systems. Progr. Nonlinear Differential Equations Appl. 78, (2009), pp. 95-112.

Wave eqn.

(WE) with internal damping

$$(S)\begin{cases} u_{tt} - \Delta u + \gamma u_t & = 0, \Omega \\ u & = 0, \partial\Omega \\ +I.C & \end{cases}$$

(WE) with boundary damping

$$(S)\begin{cases} u_{tt} - \Delta u + u_t & = 0, \Omega \\ u & = 0, \partial\Omega_1 \\ \frac{\partial u}{\partial \nu} + \gamma u_t & = 0, \partial\Omega_2 \\ +I.C & \end{cases}$$

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Remark

For the wave equation with internal or boundary friction damping, the dissipation is relatively strong so that the energy method can be applied to obtain the exponential stability as well as the uniformly exponential stability for the approximation. However, the dissipation in the thermoelastic system, due to heat conduction, is much weaker.

Polynomial decay

Theorem. (Borichev and Tomilov, 2010)

Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent :

$$(i) \qquad \|(isI - A)^{-1}\| = O(|s|^{\alpha}), \qquad s \to \infty.$$

$$(ii) \qquad \|T(t)(-A)^{-\alpha}\| = O(t^{-1}), \qquad t \to \infty.$$

$$(iii) \qquad \|T(t)(-A)^{-1}\| = O(t^{\frac{-1}{\alpha}}), \qquad t \to \infty.$$

- **Z. Liu and B. Rao**, Characterization of polynomial decay rate for the solution of linear evolution equation, Zeitschrift für angewandte Mathematik und Physik ZAMP, 56, (2005), pp. 630-644.
- **Bátkai, A., Engel, K.-J., Prüss, J., Schnaubelt, R.**, Polynomial stability of operator semigroups. Math.Nachr. 279, 1425-1440 (2006).

Polynomial decay

- **C. J. K. Batty and T. Duyckaerts**, Non-uniform stability for bounded semigroups on Banach spaces, J. Evol. Equ., 8(4), pp.765-780, 2008.
- **Borichev Alex, Tomilov Yu**, Optimal polynomial decay of functions and operator semigroups (2010).

Polynomial decay

$$(S)\begin{cases} u_{tt}+u_{xx}+\gamma\theta_x &= 0\\ \theta_t+\theta_{xx}+\gamma u_{tx} &= 0\end{cases} \Longrightarrow \quad (S)\begin{cases} u_{tt}+u_{xx}+\gamma\theta &= 0\\ \theta_t+\theta_{xx}+\gamma u_t &= 0\end{cases}$$

- **F. A. Khodja, A. Benabdallah and D. Teniou**, Dynamical stabilizers and coupled systems, ESAIM Proceedings, Vol. 2 (1997), 253-262.
- **Z. Liu and B. Rao**, Frequency domain approach for the polynomial stability of a system of partially damped wave equations, (2006).
- **Louis Tebou**, Stabilization of some coupled hyperbolic/parabolic equations. Discrete & Continuous Dynamical Systems B, 2010, 14 (4) : 1601-1620.
- **J. Hao and Z. Liu**, Stability of an abstract system of coupled hyperbolic and parabolic equations. Zeitschrift für angewandte Mathematik und Physik,64, (2013), pp. 1145-1159.

Uniform Polynomial Stability of $T_n(\cdot)$

Theorem : S. N and L. Maniar 2016

Let $T_n(t)$ $(n = 1, \dots)$ be a uniformly bounded sequence of C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators, such that $i\mathbb{R} \subset \rho(A_n)$ and $\sup\limits_{n \in \mathbb{N}} ||A_n^{-1}|| < \infty$. Then for a fixed $\alpha > 0$ the following conditions are equivalent :

1. $$\sup_{s,\ n \in \mathbb{N}} |s|^{-\alpha} \|R(is, A_n)\| < \infty.$$
2. $$\sup_{t \geqslant 0,\ n \in \mathbb{N}} \|tT_n(t) A_n^{-\alpha}\| < \infty.$$
3. $$\sup_{t \geqslant 0,\ n \in \mathbb{N}} \|t^{\frac{1}{\alpha}} T_n(t) A_n^{-1}\| < \infty.$$

L. Maniar and S. Nafiri, Approximation and uniform polynomial stability of C_0-semigroups, ESAIM : COCV 22 (2016), pp. 208–235.

$0 \leq \tau < \frac{1}{2}$:

$$\left\{ \begin{array}{l} \ddot{u}_n + \rho B_n u_n - \mu B_n^\tau \theta_n = 0, \\ \dot{\theta}_n + \kappa B_n \theta_n + \sigma B_n^\tau \dot{u}_n = 0, \\ u_n(0) = u_{0n},\ \dot{u}_n(0) = u_{1n},\ \theta_n(0) = \theta_{0n}, \end{array} \right.$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} x_n' = \mathcal{A}_{\tau,n} x_n, \\ x_n(0) = x_{n0}, \end{array} \right.$$

Hypothesis

- H_n family of Hilbert spaces.
- $B_n : D(B_n) \subset H_n \to H_n$, selfadjoint, positive definite, B_n^{-s} compact for positive s, $0 \in \rho(B_n)$ and $\sup\limits_{n \in \mathbb{N}} ||B_n^{\frac{-1}{2}}|| < \infty$.
- $\mathcal{H}_n = D(B_n^{\frac{1}{2}}) \times H_n \times H_n$.

Consequences

- $\mathcal{A}_{\tau,n}$ generates a family of C_0-semigroupes of contraction $S_{\tau,n}(t)$.
- $i\mathbb{R} \subset \rho(\mathcal{A}_{\tau,n}), \quad n \in \mathbb{N}$.
- $\sup_{n\in\mathbb{N}} ||\mathcal{A}_{\tau,n}^{-1}|| < \infty$.

Theorem

Assume $0 \leq \tau < \frac{1}{2}$. Then, the semigroup generated by $\mathcal{A}_{\tau,n}$ is uniform. poly. stable with order at most $\alpha = 2(1-2\tau)$.

$\mathcal{A}_{\tau,n}$ verifies the hypothesis of the main theorem, then

$$\sup_{|\beta|\geq 1,\; n\in\mathbb{N}} \frac{1}{|\beta|^{2(1-2\tau)}} \|(i\beta I_{3n} - \mathcal{A}_{\tau,n})^{-1}\| < \infty.$$

$$\Updownarrow$$

$$\exists C > 0, \alpha > 0 : \|tT_n(t)(-A_n)^{-\alpha}\| \leq C, \qquad \forall t > 0, \forall n \in \mathbb{N}.$$

Application 2

$$(S)\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \gamma\theta(x,t) = 0 \quad in\ (0,\pi)\times(0,\infty), \\ \theta_t(x,t) - k\theta_{xx}(x,t) - \gamma u_t(x,t) = 0 \quad in\ (0,\pi)\times(0,\infty), \\ u(x,t)\mid_{x=0,\pi} = 0 = \theta(x,t)\mid_{x=0,\pi} \qquad on\ (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x) \quad on\ (0,\pi), \end{cases}$$

- F. A. Khodja, A. Benabdallah and D. Teniou, Dynamical stabilizers and coupled systems, ESAIM Proceedings, Vol. 2 (1997), 253-262.
- Z. Liu and B. Rao : Frequency domain approach for the polynomial stability of a system of partially damped wave equations, (2006).

By introducing new variable (velocity)

$$v = u_t, \tag{1}$$

system (S) can be reduced to the following abstract first order evolution equation :

$$(S)\begin{cases} \frac{dz}{dt} = \mathcal{A}z \\ z(0) = z_0 \end{cases}$$

with

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ D^2 & 0 & -\gamma \\ 0 & \gamma & kD^2 \end{pmatrix}$$

$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ the state space equipped with the norm

$$\|z\|_{\mathcal{H}} = \left(\|Dz_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|z_3\|_{L^2}^2\right)^{\frac{1}{2}},$$

Here we have used the notation $D = \partial/\partial x,\ D^2 = \partial^2/\partial x^2$.

Approximations by a spectral method

Let

$$E_j = \begin{pmatrix} \phi_j \\ 0 \\ 0 \end{pmatrix}, \quad E_{n+j} = \begin{pmatrix} 0 \\ \psi_j \\ 0 \end{pmatrix}, \quad E_{2n+j} = \begin{pmatrix} 0 \\ 0 \\ \xi_j \end{pmatrix}, \qquad j = 1, ..., n$$

be a basis for the finite dimensional space $\mathcal{H}_n = H_1^n(\Omega) \times H_2^n(\Omega) \times H_3^n(\Omega) \subset H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \subset \mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. The inner product on $\mathcal{H}_n$ is the one induced by the $\mathcal{H}$ product. We consider the approximation to system (S) of the form

$$z_n = \sum_{j=1}^{3n} \tilde{z}_j(t) E_j(x),$$

which is required to satisfy the following variational system :

$$(\dot{z}_n, E_i)_{\mathcal{H}} = (\mathcal{A} z_n, E_i)_{\mathcal{H}}, \qquad i = 1, ..., 3n.$$

$$M_n \dot{\tilde{z}}_n = \begin{bmatrix} M_n^{(1)} & & \\ & M_n^{(2)} & \\ & & M_n^{(3)} \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}}_n^{(1)} \\ \dot{\tilde{z}}_n^{(2)} \\ \dot{\tilde{z}}_n^{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \tilde{D}_n^T & 0 \\ -\tilde{D}_n & 0 & -\gamma \tilde{F}_n \\ 0 & \gamma \tilde{F}_n^T & -G_n \end{bmatrix} \begin{bmatrix} \tilde{z}_n^{(1)} \\ \tilde{z}_n^{(2)} \\ \tilde{z}_n^{(3)} \end{bmatrix} = \tilde{A}_n \tilde{z}_n$$

with

$$(M_n^{(1)})_{ij} = (D\phi_i, D\phi_j)_{L^2}, \quad (M_n^{(2)})_{ij} = (\psi_i, \psi_j)_{L^2}, \quad (M_n^{(3)})_{ij} = (\xi_i, \xi_j)_{L^2},$$

$$(\tilde{D}_n)_{ij} = (D\phi_i, D\psi_j)_{L^2}, \quad (\tilde{F}_n)_{ij} = (\xi_i, \psi_j)_{L^2}, \quad (G_n)_{ij} = (D\xi_i, D\xi_j)_{L^2}$$

and

$$\tilde{z}_n^{(i)} = (\tilde{z}_{(i-1)n+1}, ..., \tilde{z}_{in})^T, \qquad i = 1, 2, 3.$$

By construction, the matrix $M_n^{(i)}$ is symmetric and positive definite. Therefore, there exists a lower triangle matrix $L_n^{(i)}$ such that

$$M_n^{(i)} = (L_n^{(i)})^T (L_n^{(i)}),$$

and denote $L_n \tilde{z}_n$ by $\bar{z}_n$,
then

$$\dot{\bar{z}}_n = A_n \bar{z}_n$$

with

$$A_n = \begin{bmatrix} 0_{\mathbb{C}^n} & (L_1^T)^{-1} \tilde{D}_n^T L_2^{-1} & 0_{\mathbb{C}^n} \\ -(L_2^T)^{-1} \tilde{D}_n L_1^{-1} & 0_{\mathbb{C}^n} & -\gamma (L_2^T)^{-1} \tilde{F}_n L_3^{-1} \\ 0_{\mathbb{C}^n} & \gamma (L_3^T)^{-1} \tilde{F}_n^T L_2^{-1} & -(L_3^T)^{-1} G_n L_3^{-1} \end{bmatrix}.$$

It is easy to see that

$$(A_n \bar{z}_n, \bar{z}_n)_{\mathbb{C}^{3n}} = -(G_n L_3^{-1} \bar{z}_n^{(3)}, L_3^{-1} \bar{z}_n^{(3)})_{\mathbb{C}^n} \leq 0$$

provided that G_n is semipositive definite.
$\Longrightarrow A_n$ generates a $C0$-semigroup $T_n(t)$ of contraction on $\mathcal{H}_n$

Let
$\phi_j = \sqrt{\frac{2}{\pi}} \frac{1}{j} \sin jx, \quad \psi_j = \sqrt{\frac{2}{\pi}} \sin jx, \quad \xi_j = \sqrt{\frac{2}{\pi}} \sin jx, \qquad j = 1, ..., n.$
the eigenvalues of (S), then

$$A_n = \begin{bmatrix} 0 & D_n & 0 \\ -D_n & 0 & -\gamma \\ 0 & \gamma & -D_n^2 \end{bmatrix}$$

with

$$D_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & n \end{bmatrix}.$$

Uniform Polynomial Stability with spectral element method

Theorem

The semigroups generated by A_n are uniformly polynomially stable. Moreover, we have :

$$\sup_{|\beta|\geq 1,\ n\in\mathbb{N}} \frac{1}{\beta^2}\|(i\beta I - A_n)^{-1}\| < \infty,$$

$$\Updownarrow$$

$$\sup_{t\geqslant 0,\ n\in\mathbb{N}} \|t^{\frac{1}{2}} T_n(t) A_n^{-1}\| < \infty.$$

Finite difference semi-discretization

$$x_0 = 0 < x_1 = \Delta < \cdots < x_{n-1} = (n-1)\Delta < x_n = \pi$$

$$\begin{cases} \ddot{u}_j(t) + \frac{1}{\Delta^2}[u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)] + \gamma\theta_j(t) = 0, t > 0, j = 1, \ldots, n-1 \\ \dot{\theta}_j(t) + \frac{1}{\Delta^2}[\theta_{j+1}(t) - 2\theta_j(t) + \theta_{j-1}(t)] - \gamma\dot{u}_j(t) = 0, t > 0, j = 1, \ldots, n-1 \\ u_0(t) = u_n(t) = \theta_0(t) = \theta_n(t) = 0,\ t > 0 \\ u_j(0) = u_{0j}, \quad \dot{u}_j(0) = u_{1j}, \quad \theta_j(0) = \theta_{0j},\ j = 0, \ldots, n \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \ddot{\mathbf{u}}_n + B_n\mathbf{u}_n + \gamma\boldsymbol{\theta}_n = 0, \\ \dot{\boldsymbol{\theta}}_n + B_n\boldsymbol{\theta}_n - \gamma\dot{\mathbf{u}}_n = 0, \\ \mathbf{u}_n(0) = \mathbf{u}_{0n}, \quad \dot{\mathbf{u}}_n(0) = \mathbf{u}_{1n}, \quad \boldsymbol{\theta}_n(0) = \boldsymbol{\theta}_{0n}, \end{cases}$$

$$\frac{dU_n}{dt} = \mathcal{A}_n U_n, \quad U_n(0) = U_{0n},$$

$$\mathcal{A}_n = \begin{bmatrix} 0 & I_n & 0 \\ -B_n & 0 & -\gamma I_n \\ 0 & \gamma I_n & -B_n \end{bmatrix}, B_n = \frac{1}{\Delta^2} \begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ \mathbf{0} & & & -1 & 2 \end{bmatrix}$$

Finite element semi-discretization

$$\begin{cases} \ddot{\mathbf{u}}_n + B_n \mathbf{u}_n + \gamma \boldsymbol{\theta}_n = 0, \\ \dot{\boldsymbol{\theta}}_n + B_n \boldsymbol{\theta}_n - \gamma \dot{\mathbf{u}}_n = 0, \\ \mathbf{u}_n(0) = \mathbf{u}_{0n}, \quad \dot{\mathbf{u}}_n(0) = \mathbf{u}_{1n}, \quad \boldsymbol{\theta}_n(0) = \boldsymbol{\theta}_{0n}, \end{cases}$$

where $B_n = (M_n^{(2)})^{-1} M_n^{(1)}$.

$$M_n^{(1)} = \frac{1}{\Delta} \begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ \mathbf{0} & & & -1 & 2 \end{bmatrix}, M_n^{(2)} = \Delta \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & & & \mathbf{0} \\ \frac{1}{6} & \frac{2}{3} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \frac{2}{3} & \frac{1}{6} \\ \mathbf{0} & & & \frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

Uniform Polynomial Stability with finite difference et finite element method

Theorem

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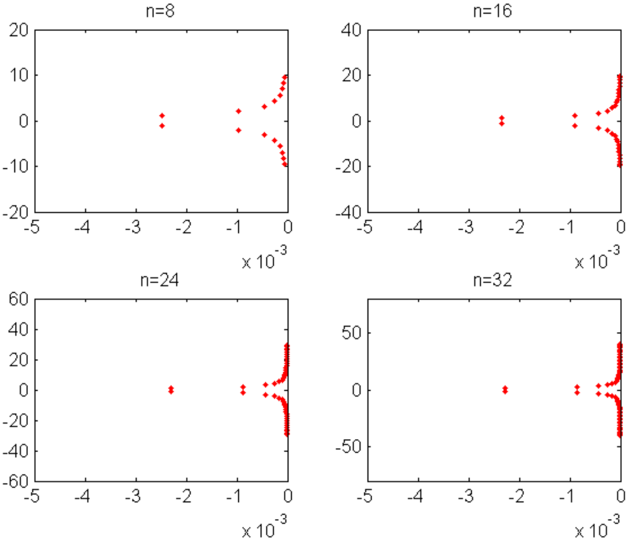
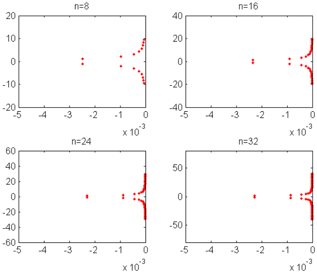
Numerical experiments





Theorem : Batkai and al 2006, Borichev and Tomilov 2010
If A is the generator of a contraction polynomially stable (of order $\alpha > 0$) C_0-semigroup on a Hilbert space X. Fix $\delta > 0$ s.t $[0, \delta] \subset \rho(A)$. Then we have for some constant C

$$|Im\lambda| \geqslant C(Re\lambda)^{-\frac{1}{\alpha}} \text{ for all } \lambda \in \sigma(A) \text{ with } Re\lambda \leqslant \delta.$$

Influence of regularity on the decay of energy !

For $T = 100$ and $\Delta t = 10^{-2}$, we consider the following initila value :

$$u(x,0) = 0, \qquad \theta(x,0) = 0, \qquad u_t(x,0) = \sqrt{\frac{2}{\pi}} sin(jx),\ j = 1, 2, 3.$$

Influence of regularity on the decay of energy !

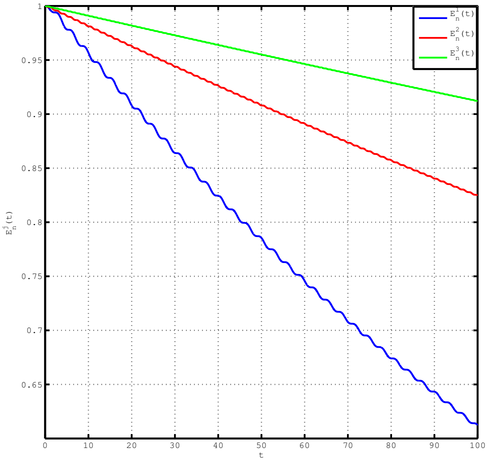


Figure – Effect of smoothness of the initial data on the rate of decay of energy.

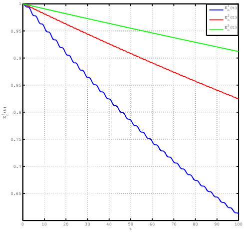


Figure – Effect of smoothness of the initial data on the rate of decay of energy.

Theorem : Batkai and al 2006

If A is the generator of a contraction C_0-semigroup on a Banach space X, with $0 \in \rho(A)$. Then we have the equivalence with $s > 0$

(a) $\|T(t)A^{-s}\| = O(t^{-r}),\ t \to +\infty$

(b) $\|T(t)A^{-s\xi}\| = O(t^{-r\xi}),\ t \to +\infty,\ \xi > 0.$

Impact of B.C on the behavior of solutions

$$(S)\begin{cases} u_{tt}(x,t)-\Delta u(x,t)+\gamma\theta(x,t) & =0, \quad \Omega\times(0,+\infty),\\ \theta_t(x,t)-\Delta\theta(x,t)-\gamma u_t(x,t) & =0, \quad \Omega\times(0,+\infty),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & \theta(x,0)=\theta_0(x)\ on\ \Omega. \end{cases}$$

(DD) $u(x,t)\mid_{\partial\Omega}=0=\theta(x,t)\mid_{\partial\Omega}$ **(Dirichlet-Dirichlet B.C)**

(DN) $u(x,t)\mid_{\partial\Omega}=0=\frac{\partial\theta}{\partial n}(x,t)\mid_{\partial\Omega}$ **(Dirichlet-Neumann B.C)**

(ND) $\frac{\partial u}{\partial n}(x,t)\mid_{\partial\Omega}=0=\theta(x,t)\mid_{\partial\Omega}$ **(Neumann-Dirichlet B.C)**

(NN) $\frac{\partial u}{\partial n}(x,t)\mid_{\partial\Omega}=0=\frac{\partial\theta}{\partial n}(x,t)\mid_{\partial\Omega}$ **(Neumann-Neumann B.C)**

Well posedness

$$\begin{cases} \frac{dU}{dt} = \mathcal{A}U, \\ U(0) = U_0 = (u_0, u_1, \theta_0)^t, \end{cases} \quad U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}$$

and

$$\mathcal{A} \in \{\mathcal{A}_{DD}, \mathcal{A}_{DN}, \mathcal{A}_{ND}, \mathcal{A}_{NN}\}$$

where

$$\mathcal{H}_O = D(A_O^{\frac{1}{2}}) \times H \times H, \qquad O \in \{D, N\}.$$

$$\mathcal{A}_{OO'} = \begin{pmatrix} 0 & I & 0 \\ A_O & 0 & -\gamma I \\ 0 & \gamma I & A_{O'} \end{pmatrix}, \quad O, O' \in \{D, N\}$$

$$D(\mathcal{A}_{OO'}) = D(A_O) \times D(A_O^{\frac{1}{2}}) \times D(A_{O'}), \quad O, O' \in \{D, N\}$$

$$D(A_D) = H^2(\Omega) \cap H_0^1(\Omega), D(A_N) = \{w \in H^2(\Omega) / \frac{\partial w}{\partial n} |_{\partial\Omega} = 0\}$$

$$D(A_D^{\frac{1}{2}}) = H_0^1(\Omega), \text{ and } D(A_N^{\frac{1}{2}}) = H^1(\Omega).$$

Well posedness

Theorem : For all $O, O' \in \{D, N\}$ the family of operators $\mathcal{A}_{OO'}$, generates a contraction semigroup $T_{OO'}(\cdot)$ on the Hilbert space $\mathcal{H}_O$.

Asymptotic behavior

$$X_{DD} = D(A_D^{\frac{1}{2}}) \times H \times H$$
$$X_{DN} = D(A_D^{\frac{1}{2}}) \times H \times H_N$$
$$X_{ND} = D(A_N^{\frac{1}{2}}) \times H_N \times H$$
$$X_{NN} = D(A_N^{\frac{1}{2}}) \times H_N \times H_N$$

where

$$H_N = \{f \in H : \langle f, 1\rangle_H = 0\}.$$

Asymptotic behavior

Theorem : For all $O, O' \in \{D, N\}$ the family of semigroups $T_{OO'}(\cdot)$ generated by $\mathcal{A}_{OO'}$:

1. is strongly stable on the Hilbert space $X_{OO'}$.
2. not exponentially stable on $X_{OO'}$.
3. polynomially stable of decay rate $\alpha = 1/2$ on $X_{OO'}$

Conclusion and open problems

- Numerical study under the class **B.C**
- Full discretization
- Numerical study when $d = 2, 3$.
- Non autonomous case $A(t),\ t > 0$
- etc...

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Thank you for your attention
Gracias