Optimal shapes for the 2d Neumann-Kelvin problem

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2. Rankine body for the 2d Neumann-Kelvin problem
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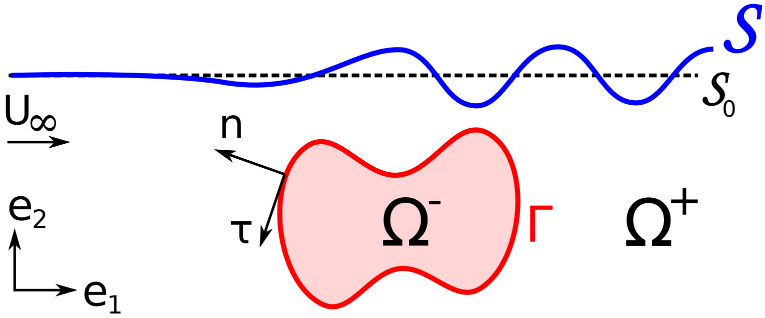
The Neumann-Kelvin problem : setting



- A cylinder obstacle (i.e. a 2d flow) under water
- The flow is inviscid, irrotational and incompressible
- A steady state has been reached
- A **linear theory** is used : **potential flow** and the condition on the free boundary is linearized (Bernoulli + kinematic condition).

The Neumann-Kelvin problem reads: find $\Phi \in C^2(\overline{\Omega^+})$ such that

$$\begin{cases} \Delta\Phi = 0\,, & \text{in } \Omega^+\,,\\ \partial^2_{11}\Phi + \nu\,\partial_2\Phi = 0\,, & \text{for } x_2 = 0\,,\\ \partial_n\Phi = -U_\infty n\cdot e_1\,, & \text{on } \Gamma\,,\\ \sup\limits_{\Omega^+}|\nabla\Phi| < \infty \ \text{ and } |\nabla\Phi| \to 0 & \text{as } x_1 \to -\infty\,, \end{cases}$$

where $\nu = g/U_\infty^2$ and $\Omega^+ = \mathbf{R}^2_- \setminus \overline{\Omega^-}$.

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where $\nu = g/U_\infty^2$ and $\Omega^+ = \mathbf{R}^2_- \setminus \overline{\Omega^-}$.

NB : $V = \nabla\tilde{\Phi} = U_\infty e_1 + \nabla\Phi$ is the velocity field (irrotational)
$\tilde{\Phi} = U_\infty x_1 + \Phi$ is the unperturbed potential
div $V = 0 = \Delta\Phi$ is the incompressibility condition
$\partial_n\tilde{\Phi} = 0$ on Γ is the no-slip condition on Γ.

The (stationary) Bernoulli equation reads

$$\frac{1}{2}|V|^2 + \frac{1}{\rho}P + gx_2 = C \text{ in the water.}$$

At the free surface, $P = P_0$ (constant) and $x_2 = h(x_1)$ is the free surface elevation. Thus,

$$\frac{1}{2}\left((U_\infty + \partial_1\Phi)^2 + (\partial_2\Phi)^2\right) + \frac{P_0}{\rho} + gh(x_1) = C.$$

When $x_1 \to -\infty$, we have $\nabla\Phi \to 0$ and $h(x_1) \to 0$ so

$$\frac{1}{2}U_\infty^2 + \frac{P_0}{\rho} = C.$$

Using that $|\nabla\Phi| << U_\infty$ and $|h(x_1)|$ is small, keeping only the first order terms yields

$$U_\infty\partial_1\Phi + gh = 0 \tag{1}$$

At the free surface, the no-slip condition reads $V \cdot N = 0$ with $N = (-\partial_1 h, 1)/\sqrt{1 + (\partial_1 h)^2}$, i.e.

$$-(U_\infty + \partial_1 \Phi)\partial_1 h + \partial_2 \Phi = 0.$$

Keeping only the first order terms yields

$$-U_\infty \partial_1 h + \partial_2 \Phi = 0 \tag{2}$$

Differentating (1) with respect to x_1 yields

$$U_\infty \partial_{11} \Phi + g \partial_1 h = 0.$$

Using (2), we obtain

$$U_\infty \partial_{11} \Phi + \frac{g}{U_\infty} \partial_2 \Phi = 0 \text{ for } x_2 = 0,$$

that is the condition on the free surface.

The Neumann-Kelvin problem reads: find $\Phi \in C^2(\overline{\Omega^+})$ such that

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where $\nu = g/U_\infty^2$ and $\Omega^+ = \mathbf{R}^2_- \setminus \overline{\Omega^-}$.

NB: Froude invariance $U_\infty^2/\mathit{length}$.

A central idea : the Green's function

The Green's function $G(x,y) = G_{y_1,y_2}(x_1,x_2)$ of the problem solves

$$\begin{cases} \Delta G_{y_1,y_2} = \delta_{y_1,y_2}\,, \text{in } \mathbb{R}^2_-, \\ \partial^2_{11} G_{y_1,y_2} + \nu \partial_2 G_{y_1,y_2} = 0\,, \quad \text{for } x_2 = 0, \\ \sup_{(x_1,x_2)\in\mathbb{R}^2_-} \left| \nabla \left[G_{y_1,y_2}(x_1,x_2) - \frac{1}{2\pi} \log(\nu |z - Z|) \right] \right| < \infty, \\ \lim_{x_1 \to -\infty} |\nabla G_{y_1,y_2}| = 0, \end{cases}$$

where $z = x_1 + ix_2$ and $Z = y_1 + iy_2$.

The Green's function for this problem is explicit:

$$\begin{aligned} G_{y_1,y_2}(x_1,x_2) \;=\; & \frac{1}{2\pi}\log(\nu|z-Z|)+\frac{1}{2\pi}\log(\nu|z-\bar{Z}|) \\ & -e^{\nu(x_2+y_2)}\sin(\nu(x_1-y_1)) \\ & +\frac{1}{\pi}\int_0^\infty \frac{\cos(k(x_1-y_1))}{k-\nu}e^{k(x_2+y_2)}dk \end{aligned}$$

Note that $\Delta G_y = 0$ in $\mathbf{R}^2_- \setminus \{y\}$.

A central idea to solve the Neumann-Kelvin problem: use the (explicit) Green function $G(x,y)$ of the problem and seek

$$\Phi(x) = \int_\Gamma G(x,y)\alpha(y)ds_y,$$

where α is a (unknown) function on Γ ("single-layer boundary potential").
For such Φ, we clearly have

$$\begin{aligned} \Delta\Phi &= 0 \text{ in } \Omega^+, \\ \partial^2_{11}\Phi + \nu\,\partial_2\Phi &= 0 \text{ for } x_2 = 0, \\ \sup_{\Omega^+}|\nabla\Phi| < \infty \text{ and } |\nabla\Phi| \to 0 \quad \text{as} \quad & x_1 \to -\infty \end{aligned}$$

Then Φ solves the NK problem if

$$\partial_n \Phi = -U_\infty n \cdot e_1 \text{ on } \Gamma,$$

that is

$$\frac{1}{2}\alpha(x) - \int_\Gamma \partial_{n(x)} G(x,y)\,\alpha(y)\,\mathrm{d}s_y = U_\infty\, n(x) \cdot e_1\,,\ \forall x \in \Gamma\,. \quad (3)$$

Theorem (Kuznetsov, Maz'ya and Vainberg'02)

The Neumann-Kelvin problem is uniquely solvable for all $\nu > 0$ with a possible exception for a finite number of values.

Main ideas :

- (3) is a Fredholm equation which depends analytically on ν
- uniquely solvable for $\nu = 0$ and $\nu = +\infty$ (Kochin'37)
- principle of isolated zeros

NB: if Ω^- is a disc, then the NK problem is uniquely solvable for all $\nu > 0$.

Not true in general (**Motygin and McIver 2010**, **McIver 1996**).

The wave-resistance

The **wave resistance** is the horizontal force exerted by the water on the obstacle in this model, i.e. the **drag**, which reads

$$R_w = -\int_\Gamma P\, n_1 \,\mathrm{d}s\,,$$

where $n_1 = n \cdot e_1$.
Using Bernoulli's formula, we have

$$R_w = \rho \int_\Gamma \frac{|V|^2}{2} n_1 \,\mathrm{d}s\,,$$

where the velocity V is obtained through $V = \nabla\Phi + U_\infty e_1$ in Ω^+.

Summing up, we have

$$R_w = \rho \int_\Gamma \left[\frac{|\nabla \Phi|^2}{2} + U_\infty \partial_1 \Phi \right] n_1 \, \mathrm{d}s \, .$$

It can be shown that

$$R_w = \frac{\rho \, \nu}{4} \left| \int_\Gamma \alpha(x) \, \mathcal{E}(x) \mathrm{d}s_x \right|^2 ,$$

where α solves the integral equation (3) and

$$\mathcal{E}(x) = e^{\nu(ix_1 + x_2)}.$$

Interpretation (1)

if Φ solves the Neumann-Kelvin problem, then

$$\Phi(x_1,x_2) = c + \Theta(x_1,x_2) + H(x_1)(\mathcal{A}\sin(\nu x_1) + \mathcal{B}\cos(\nu x_1))e^{\nu x_2},$$

as $r = (x_1^2 + x_2^2)^{1/2} \to +\infty$, where $\Theta(x_1,x_2) = O(|r|^{-1})$, $|\nabla\Theta(x_1,x_2)| = O(|r|^{-2})$, c is an arbitrary constant, H is the Heaviside function and $\mathcal{A}$, $\mathcal{B}$ are constants which depend only on ν and the values of Φ, $\partial_n\Phi$ on ∂D.
With this notation, we have

$$R_w = \frac{\rho\nu}{4}(\mathcal{A}^2 + \mathcal{B}^2).$$

Interpretation (2)

The asymptotic behaviour of Φ can be deduced from the Green's function

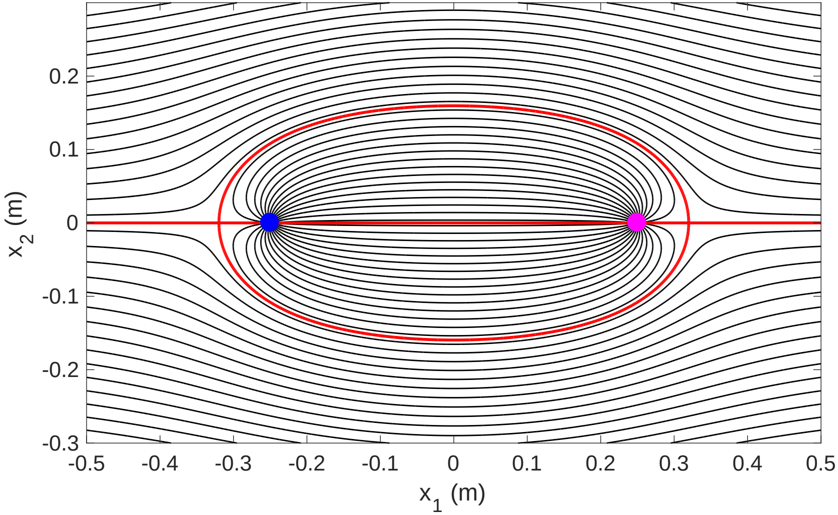
$$\begin{aligned} G_{y_1,y_2}(x_1,x_2) &= \frac{1}{2\pi}\log(\nu|z-Z|)+\frac{1}{2\pi}\log(\nu|z-\bar{Z}|) \\ &\quad -e^{\nu(x_2+y_2)}\sin(\nu(x_1-y_1)) \\ &\quad +\frac{1}{\pi}\int_0^\infty \frac{\cos(k(x_1-y_1))}{k-\nu}e^{k(x_2+y_2)}dk \end{aligned}$$

and Green's formula

$$\Phi = \int_{\Omega^+} \Phi\Delta G - G\Delta\Phi dx = \int_\Gamma \Phi\partial_n G - G\partial_n\Phi ds.$$

Optimal shape problem: is it possible to obtain "the" obstacle which minimizes the wave resistance for a given area and for a fixed center of gravity ?

NB: if there is no free surface, this problem is ill-posed: **d'Alembert's paradox** asserts that for any obstacle, the drag and the lift are zero (in 2d and 3d).
Recall that the flow is inviscid, irrotational and incompressible (Euler's equations in the irrotational case)



The classical Rankine oval (no free surface)

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The Rankine potential is the superposition of a source of strength m at $(-a, 0)$, a sink of strength m at $(a, 0)$, and a uniform stream U_∞ in the horizontal direction :

$$\tilde{\Phi}(x, y) = \frac{m}{2\pi} \ln(|z + a|) - \frac{m}{2\pi} \ln(|z - a|) + U_\infty x,$$

where $z = x + iy$.

Similarly, we define for the Neumann-Kelvin (NK) problem the potential of a source/sink couple:
For $a > 0$, $d > 0$ (depth) and $m > 0$ (strength) we introduce the potential given by

$$\Phi_{a,d,m}(x,y) = m\mathcal{G}_{(-a,-d)}(x,y) - m\mathcal{G}_{(a,-d)}(x,y) + U_\infty x,$$

where $\mathcal{G}$ is the Green function for the NK-problem.

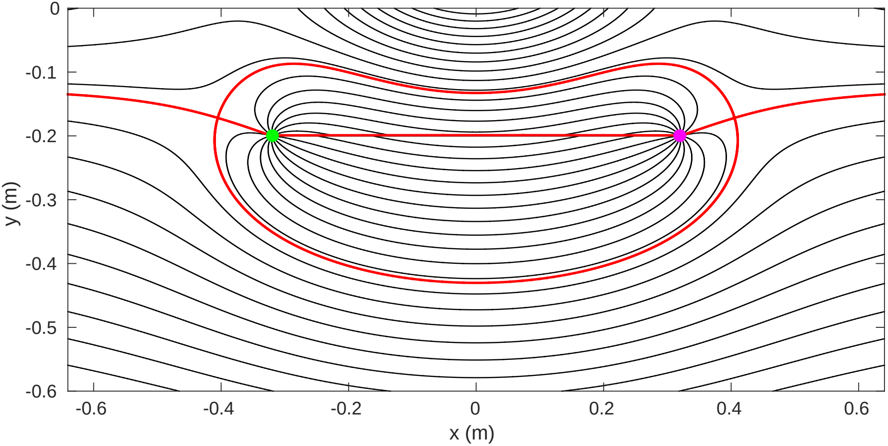
The asymptotic behaviour of $\Phi_{a,d,m}$ reads

$$\begin{aligned}\Phi_{a,d,m}(x,y) &= U_\infty x + r_{a,d,m}(x,y) \\ &\quad -2mH(x)e^{\nu(y-d)}[\sin(\nu(x+a)) - \sin(\nu(x-a))],\end{aligned}$$

where $r_{a,d,m}(x,y) = O(|z|^{-1})$ and $|\nabla r_{a,d,m}(x,y)| = O(|z|^{-2})$.
Since

$$\sin(\nu(x+a)) - \sin(\nu(x-a)) = 2\sin(\nu a)\cos(\nu x),$$

the amplitude of the waves downstream is proportional to $\sin(\nu a)$.
If $a = p\pi/\nu$ with p integer, then the potential is **waveless**.

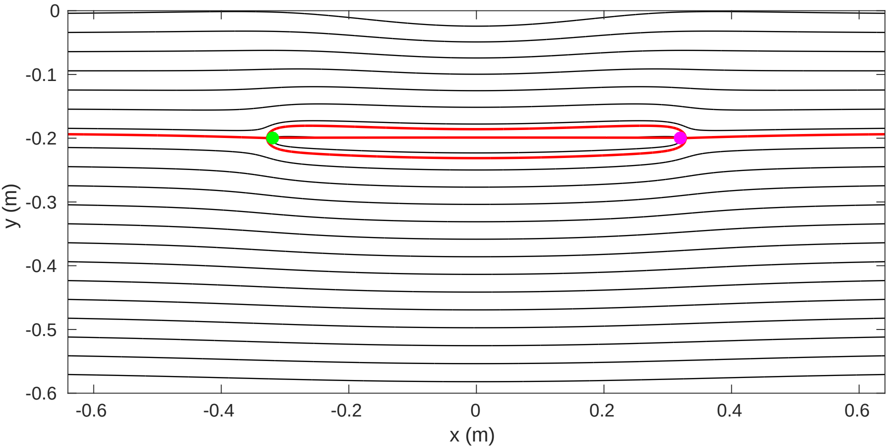


Rankine body for $d = 0.2$, $m = 0.6$

Theorem (Dambrine, Noviani, P. (in revision))

For every integer p and for every depth $d > 0$, if $b = m/(2\pi U_\infty)$ is small enough then the waveless potential $\Phi_{a,d,m}$ defines a bounded and simply connected domain $D = \mathcal{R}_{p,d,b}$ with analytic boundary for which the NK problem is solved. Moreover, this "Rankine body" $\mathcal{R}_{p,d,b}$ contains the singularities and is symmetric about the y axis.

Rk : if the potential is the *unique* solution of the NK-problem, this Rankine body has zero wave resistance (but it may not be unique and correspond to a resonance value)
Rk2 : use of the exponential integral for the numerical computations



Rankine body for $d = 0.2,\ m = 0.05$

Sketch of proof

main idea : study of the phase portrait of the ode
$(\dot{x}_1(t), \dot{x}_2(t)) = \nabla\Phi(x_1(t), x_2(t))$
We have

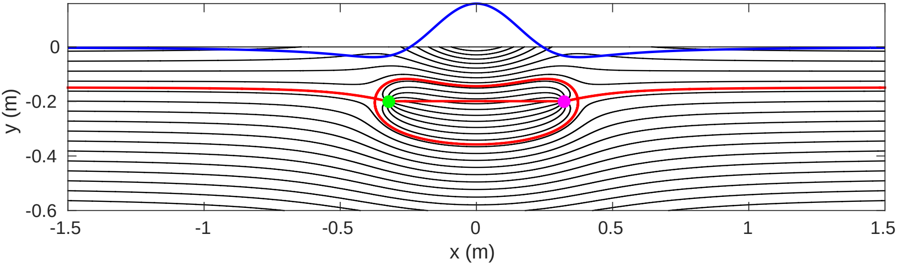
$$\begin{aligned}\Phi_{a,d,m}(x_1, x_2) \quad = \quad & U_\infty x_1 + \frac{m}{2\pi}\log(\nu|z - z_-|) + \frac{m}{2\pi}\log(\nu|z - \bar{z}_-|) \\ & -\frac{m}{2\pi}\log(\nu|z - z_+|) - \frac{m}{2\pi}\log(\nu|z - \bar{z}_+) \\ & -\frac{2m}{\pi}\int_{\mathbf{R}_+}\frac{\sin(kx_1)e^{k(-d+x_2)}\sin(kp\pi/\nu)}{k-\nu}dk.\end{aligned}$$

Sketch of proof

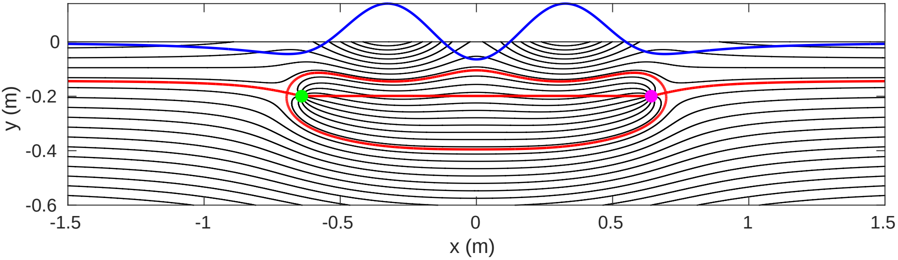
- use of the complex velocity potential $\omega(z) = \Phi(x_1, x_2) + i\Psi(x_1, x_2)$ where Ψ is the stream function
- uniqueness of a critical point
- the critical point is a non-degenerate saddle point

Consider a trajectory $(\hat{x_1}, \hat{x_2})$ arriving at z_b^+, and assume that it does not cross the x_2 axis. Then:
1. it stays in a compact set of the quadrant
2. since we have a gradient flow, $\hat{x_1}, \hat{x_2}$ originate from a critical point in the quadrant, which is necessarily distinct : a contradiction.

The Rankine body thus defined contains the singularity (otherwise, by the maximum principle applied to Ψ, it would contain a new critical point).



Rankine body with surface elevation for $a = \pi/\nu$, $d = 0.2$ and $m = 0.35$



Case $a = 2\pi/\nu$, $d = 0.2$ and $m = 0.35$

A 2d Navier-Stokes simulation

We want now to obtain **numerically** “the” obstacle which minimizes the wave resistance for a given area (and for a fixed center of gravity)

Numerical approach :

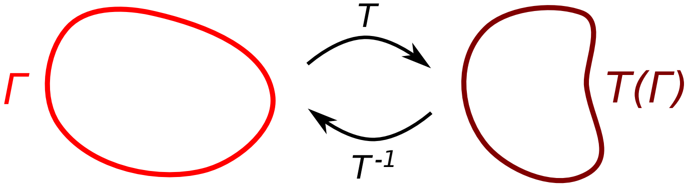
- Computation of the shape derivative for the integral problem

$$\frac{1}{2}\alpha(x) - \int_\Gamma \partial_{n(x)} G(x,y)\,\alpha(y)\,\mathrm{d}s_y = U_\infty\, n(x)\cdot e_1\,,\ \forall x \in \Gamma\,.$$

- Discretization on a regular grid
- Level-set method

A few words on shape derivative

The fundamental idea[1,2,3] is to use a local parametrization of ω with small deformations induced by a displacement field θ.



[1] J. Hadamard, *"Mémoire sur le problème d'analyse relatif à l'équilibre des*

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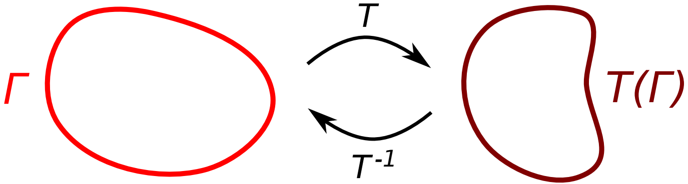

Expanding $R_w((Id + \theta)(\omega))$ to the first order with respect to θ yields

$$R_w((Id + \theta)(\omega)) = R_w(\omega) + \underbrace{R'_w(\omega)}_{\text{"shape derivative"}}(\theta) + o(||\theta||_{W^{1,\infty}})\,.$$

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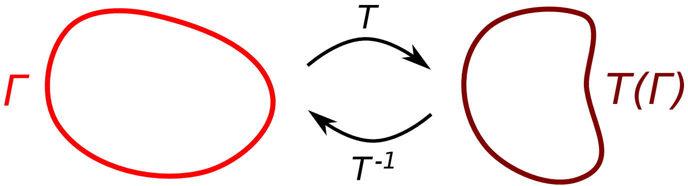

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$$R_w((Id+\theta)(\omega)) = R_w(\omega) + \underbrace{R'_w(\omega)}_{\text{"shape derivative"}}(\theta) + o(||\theta||_{W^{1,\infty}})\,.$$

Moreover, it is possible to build a "shape gradient" if $R'_w(\omega)(\theta)$ has the form :

$$R'_w(\omega)(\theta) = \int_{\partial\omega} \theta\cdot n\,\underbrace{(\ldots)}_{\text{"shape gradient"}}\,\mathrm{d}s$$

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Shape derivative of the wave-making resistance

A lengthy calcuation yields the shape gradient

$$\nabla_\Gamma R_w(\Gamma) = \Bigg[\alpha(x)\int_\Gamma \alpha(y)\,\Re\big(\partial_n\mathcal{E}(x)\overline{\mathcal{E}(y)}\big)\,\mathrm{d}s_y + \partial_\tau q(x)\,\partial_\tau \mathbb{S}(\alpha)(x)$$

$$+\,\alpha(x)\,\partial_\tau\tilde{\mathbb{S}}(\partial_\tau q)(x) + U\,\partial_\tau q(x)\,\tau(x)\cdot e_1\Bigg].$$

- $\mathbb{S}$ and $\tilde{\mathbb{S}}$ are operators with kernels G and $2G(x,y) - G(\overline{x},y)$.
- α solves the state equation

$$\frac{1}{2}\,\alpha(x) - \int_\Gamma \partial_{n(x)} G(x,y)\,\alpha(y)\,\mathrm{s}_y = U_\infty\, n_1(x)\,,\quad \forall x\in\Gamma$$

- q solves the adjoint equation

$$\frac{1}{2}\,q(y) - \int_\Gamma \partial_{n(x)} G(x,y)\,q(x)\,\mathrm{d}s_x = 2\int_\Gamma \Re\big(\mathcal{E}(x)\overline{\mathcal{E}}(y)\big)\,\alpha(x)\,\mathrm{d}s_x\,,\ \forall y\in\Gamma$$

The level-set method

We choose to represent Γ via its signed distance function:

$$d(x) = \begin{cases} -\min_{y\in\Gamma} |x-y| & \text{if } x \in \Omega^- \\ \min_{y\in\Gamma} |x-y| & \text{if } x \in \Omega^+ \\ 0 & \text{if } x \in \Gamma \end{cases}$$

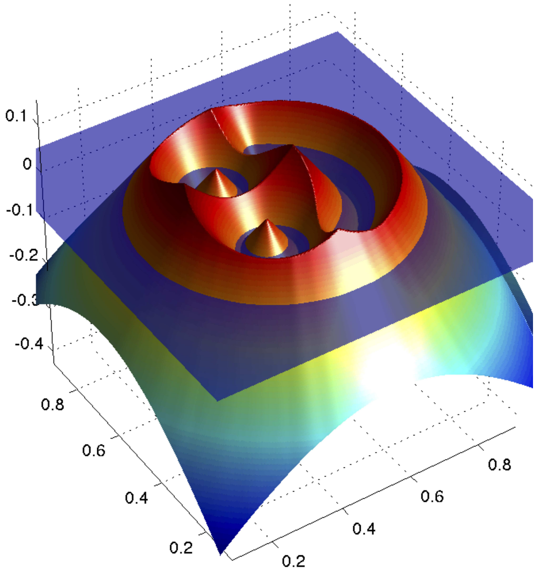
Features:

- construction solving the eikonal equation[4]:

$$|\nabla d| = 1\,, \quad d|_\Gamma = 0$$

- normal vector: $\mathrm{n} = \nabla d$
- closest point mapping: $\Pi_\Gamma = Id - d\nabla d$
- normal displacements for one iteration: $d^{n+1} = d^n + \delta_t\, \theta \circ \Pi_\Gamma$

[4] Osher, S., R.Fedkiw, "Level Set Methods and Ddynamics Implicit Surfaces. Springer", (2002)


0.1
0
-0.1
-0.2
-0.3
-0.4
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8

How about integration on Γ?

C. Kublik *et. al.* proposed[5] a method of integration that benefits from the knowledge of the closest-point mapping.

Using the co-area formula, they established:

$$\int_\Gamma f \, ds = \int_{\mathbb{R}^n} f \circ \Pi_\Gamma(x) \, J(x) \, \delta(d(x)) \, dx \,, \tag{4}$$

where $J = 1 - d\Delta d$, and δ is a function satisfying $\int_{-\varepsilon}^{\varepsilon} \delta(s) \, ds = 1$, supported in $[-\varepsilon, \varepsilon]$.

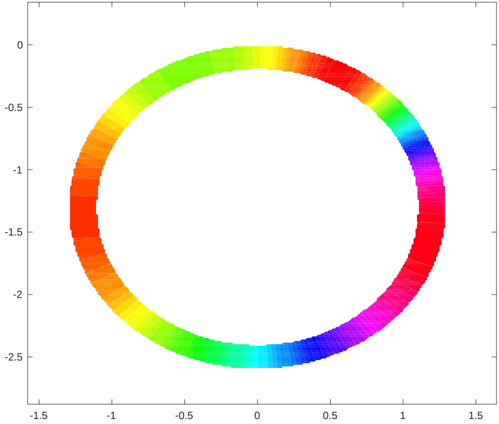
When solving boundary integral equations with (4), we recover boundary potentials that are already constant along normals!

They later exploited this fact to solve the Mullins-Sekerka problem[6]

[5] C. Kublik, N.M. Tanushev, R. Tsai (2013)

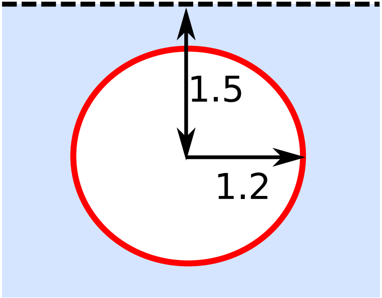
[6] C. Chen, C. Kublik, R. Tsai, 2015

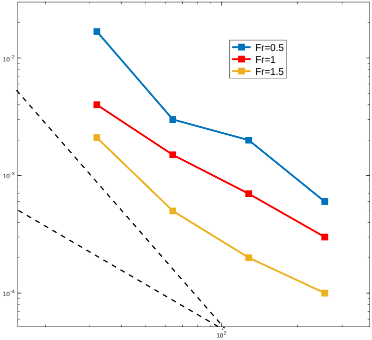
Integration on curves/surfaces



Validation on the method on Neumann-Kelvin (1)

We use Havelock's exact solution for the immersed circular cylinder, we scale ϵ with the mesh (with a weight given by $|\nabla d|_1$, as in[7]).

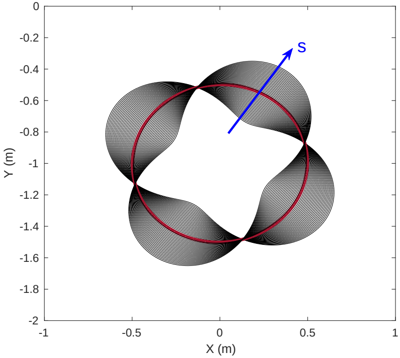


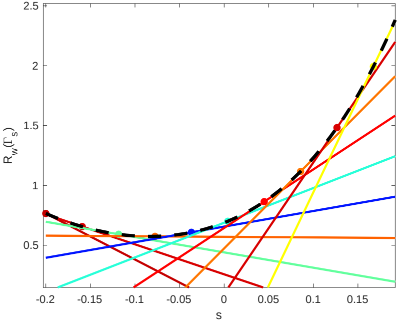


[7] B. Engquist, A.K. Tornberg, R. Tsai, "Discretization of Dirac delta functions in level set methods" Journal of Computational Physics 207 (1), 28-51

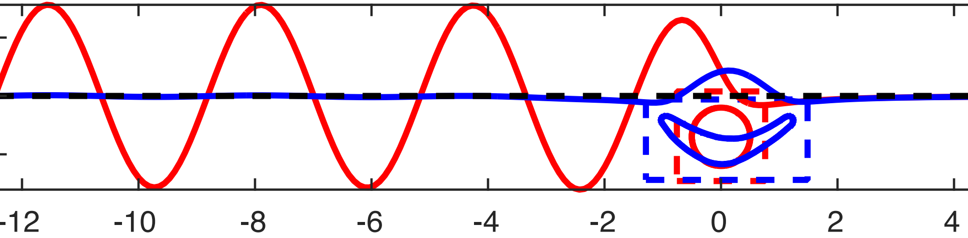
Validation on the method on Neumann-Kelvin (2)

We impose a deformation $T_s = Id + s\theta_0$ on the shape depending on a parameter s, and we compare $R_w(T_s(\Gamma_0))$ (black dotted line) with its first order expansion at several points (coloured lines).





Comparison of the wake of the initial and optimal obstacles



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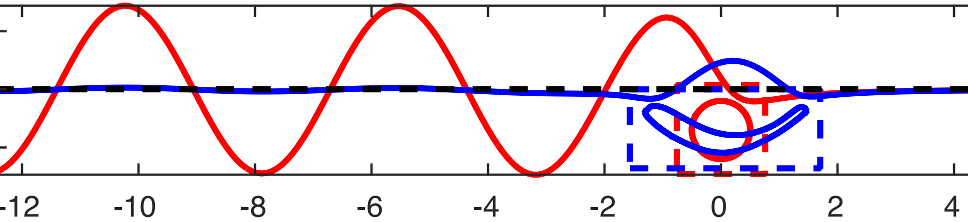

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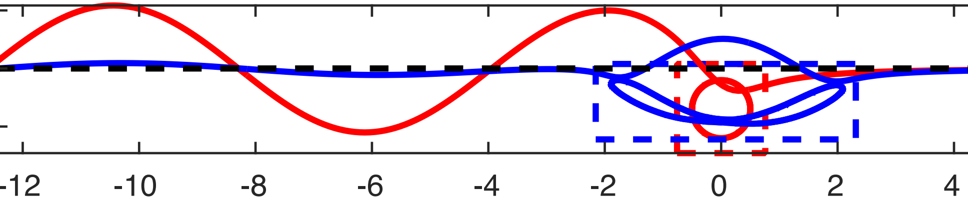
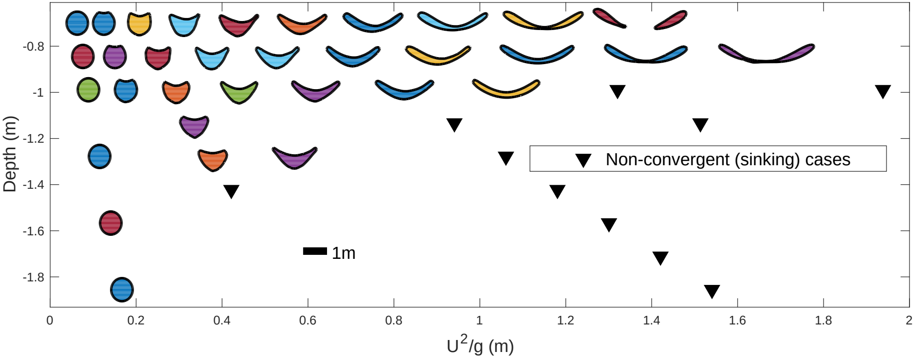
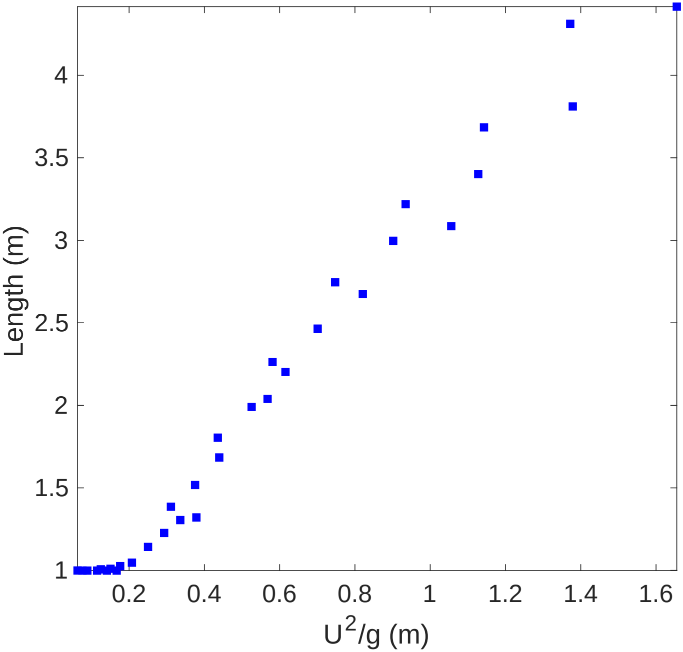
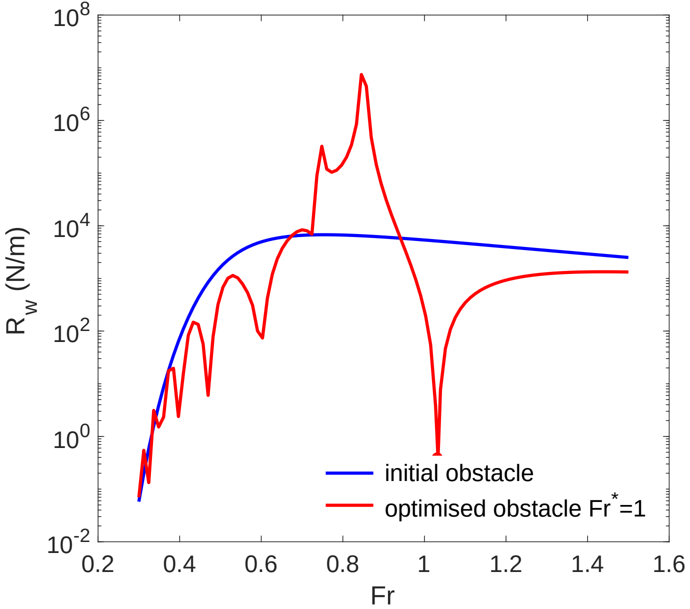




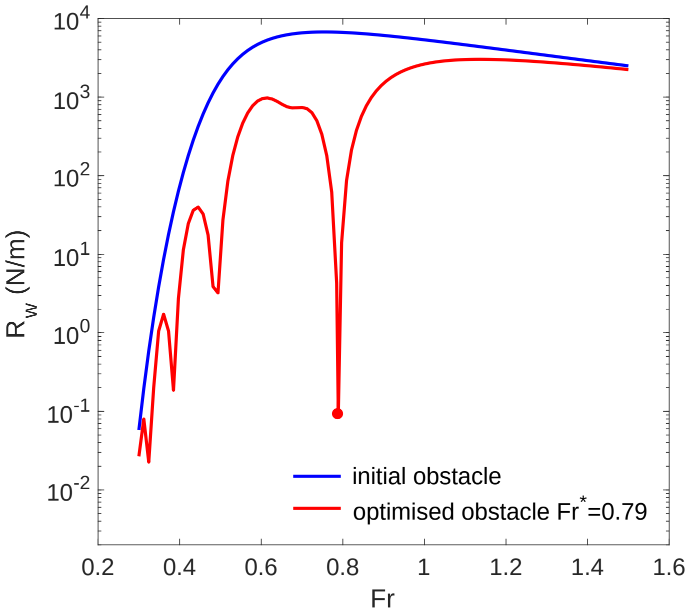
Classification of optimal shapes



Length of the optimal obstacle



Wave resistance vs Froude number



Wave resistance vs Froude number