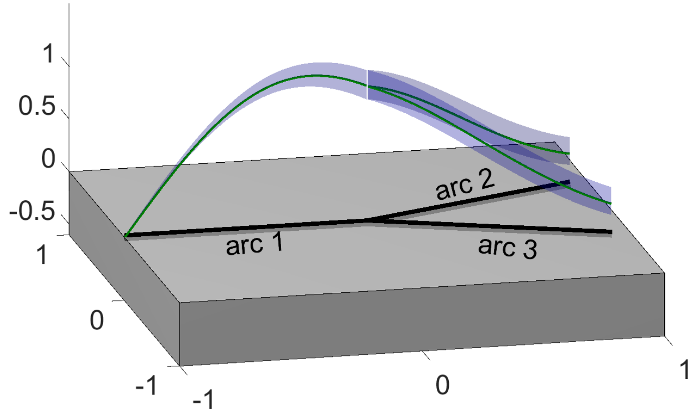
On the non–stabilizability for networks of strings

Martin Gugat, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
joint work with *Stephan Gerster (RWTH Aachen)*, see *On the limits of stabilizability for networks of strings, Systems and Control Lett. 131 (2019)*

Control and stabilization of hyperbolic systems Benasque, Wednesday, August 28, 2019



Inhalt

Application: Gas transportation through pipelines

The system dynamics in a pipe is described by

the isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta}\frac{q|q|}{\rho} - \rho\, g\, \sin(\alpha) \end{cases}$$

or a similar (linearized) model, see *A. Osiadacz, M. Chazykowski*, Comparison of isothermal and non-isothermal pipeline gas flow models, 2001.

See the results of DFG CRC 154: TRR 154 | Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks

Model for the flow in a single pipe

Ideal gas

In **ideal gas**, we have

$$p = c^2\,\rho.$$

The sound speed c is constant!

The isothermal Euler equations

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$$d_+ = c, \; d_- = -c$$

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$$\boxed{r_t + D\, r_x = F'(\bar{R})\, r.}$$

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The matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semidefinite. (**First go to RIEMANN invariants, then linearize!**) The *pressure* is given by $\exp\left(\frac{1}{2}\left(r_+ + r_- + \bar{R}_+ + \bar{R}_-\right)\right) > 0$ and the gas *velocity* is proportional to $r_+ - r_- + \bar{R}_+ - \bar{R}_-$.

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What happens if in the pde

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the matrix is *not* positive definite?
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Yes!

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Example for a closed-loop system.

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Together with the boundary conditions

$$U(t, 0) = k\, V(t, 0), \quad V(t, L) = U(t, L)$$

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(and initial data $U(0, \cdot)$, $V(0, \cdot)$))
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In our example, with a 2×2 matrix M we get

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The source term can cause **instability!**

Outline

Motivation by application: Gas transportation through pipelines

- Example for a closed-loop system.

The example by Bastin and Coron: A single interval

- A sufficient condition for non-stabilzability
- A sufficient condition for stabilizability

A tree of strings with a stabilizing and a destabilizing source term

- Sufficient conditions for non-stabilzability
- A sufficient condition for stabilizability

Inhalt

The example by Bastin and Coron: A single interval

In *Stability and Boundary Stabilization of 1-D Hyperbolic Systems* (2016), Bastin and Coron consider the diagonal system

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BASTIN & CORON construct *product solutions* of the form (separation ansatz)

$$\begin{pmatrix} U(t,\, x) \\ V(t,\, x) \end{pmatrix} = \exp(\sigma\, t) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.$$

If such a solution can be found with $\sigma > 0$,
the system is *exponentially unstable* and cannot be stabilized.

The example by Bastin and Coron

For $\sigma \in (0, c)$ define $\omega = \sqrt{c^2 - \sigma^2} > 0$. The pde and $f(0) = k\, g(0)$ imply

$$\begin{aligned} f(x) &= \quad (c + k\sigma)\sin(\omega x) \quad - k\,\omega\,\cos(\omega x), \\ g(x) &= -(\sigma + kc)\sin(\omega x) \;\; - \omega\,\cos(\omega x). \end{aligned}$$

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For $k \neq -1$ we have $f(L) = g(L)$ if $\sigma \in (0, c)$ is such that

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For $k \neq -1$, in terms of $\omega\, L$, the condition for non–stabilizability is

$$\left[c\,L + \sqrt{(c\,L)^2 - (\omega\,L)^2}\right] \frac{\tan(\omega\,L)}{\omega\,L} = \frac{k-1}{k+1}. \tag{1}$$

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So we analyze the *range* of the function on the left-hand side of (1)!

The *range* of a function

We consider

$$F(c\,L) = \sup_{s\in(0,c\,L)} \left[c\,L + \sqrt{(c\,L)^2 - s^2}\right] \frac{\tan(s)}{s} \geq 0.$$

We substitute $y = c\,L$. For $y > 0$, define $F(y) = \sup_{s\in(0,y)} \left[y + \sqrt{y^2 - s^2}\right] \frac{\tan(s)}{s}$.

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If $y > \frac{\pi}{2}$ then $G(y) \leq \lim_{s\to\frac{\pi}{2}+} \left[y + \sqrt{y^2 - s^2}\right] \frac{\tan(s)}{s} = -\infty$.

Thus for $y = c\,L \geq \pi$, the function on the left-hand side of (1) takes all values in $(-\infty, 0]$.

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Thus for $y = c\,L \geq \pi$, the function on the left-hand side of (1) takes all values in $(-\infty, 0]$.

For all $k \in (-1,\ 1]$, this implies the instability of the system

because equation (1) has a solution $s = \omega\,L \in (\pi/2, \pi)$ that corresponds by $\sigma^2 = c^2 - \omega^2$ to $\sigma \in (0,\ c)$.

Inhalt

The example by BASTIN and CORON

In fact, the following proposition is already proved by BASTIN and CORON:

Proposition:

$$\text{If} \quad c\,L \geq \pi,$$

there is *no* real value of k such that the closed loop system with the pde

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and the boundary conditions $U(t,\, 0) = k\, V(t,\, 0), \quad V(t,\, L) = U(t,\, L)$
is exponentially stable.

The example by BASTIN and CORON

In fact, the following proposition is already proved by BASTIN and CORON:

Proposition:

$$\text{If} \quad c\,L \geq \pi,$$

there is *no* real value of k such that the closed loop system with the pde

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and the boundary conditions $U(t,\,0) = k\,V(t,\,0), \quad V(t,\,L) = U(t,\,L)$
is exponentially stable.

Boundary stabilization becomes **impossible** if the length or the (negative eigenvalue of the) source term is too large!

Inhalt

A sufficient condition for stabilizability

If $\lambda > 0$, we have $\lambda L < \frac{\pi}{2}$, and for $\epsilon = \frac{\pi}{4} - \frac{\lambda L}{2}$ we have $|k| \leq \tan^2(\epsilon)$ and $c L < \lambda \tan^2(\epsilon)$

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This can be seen considering the quadratic LYAPUNOV function

$$\mathcal{L}(t) = \frac{1}{2} \int_0^L A \cot(\epsilon + \lambda x)\, U^2(t,\, x) + A^{-1}\, \tan(\epsilon + \lambda x))\, V^2(t,\, x)\, dx.$$

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  See also Charlotte's talk!

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How is the situation on *networks*?

Inhalt

A tree of strings

Now we consider a star-shaped networks of strings.

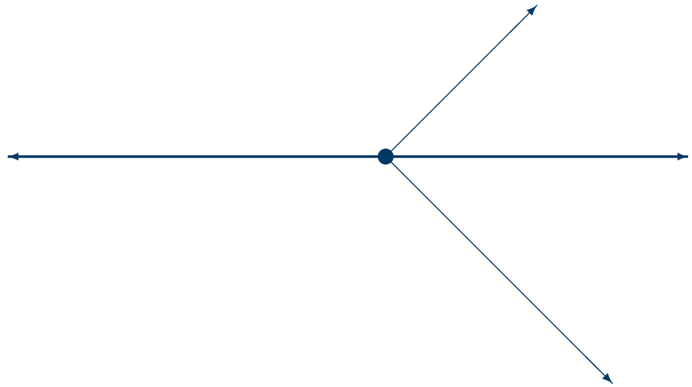

Figure: A star-shaped network with $N = 4$ edges.

A tree of strings

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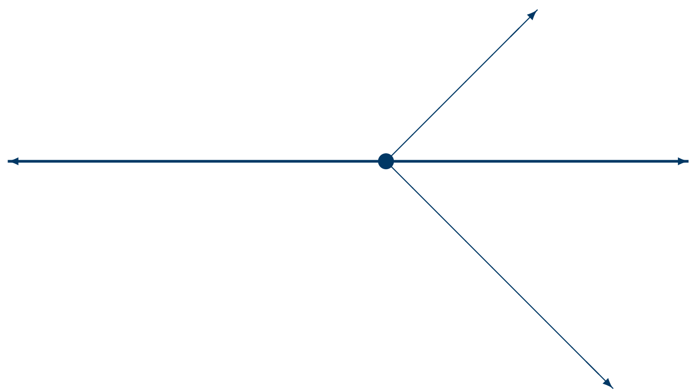

Figure: A star-shaped network with $N = 4$ edges.

We consider feedback control at all boundary nodes except one.
Let $N \in \{2, 3, 4, ...\}$ denote the number of strings.

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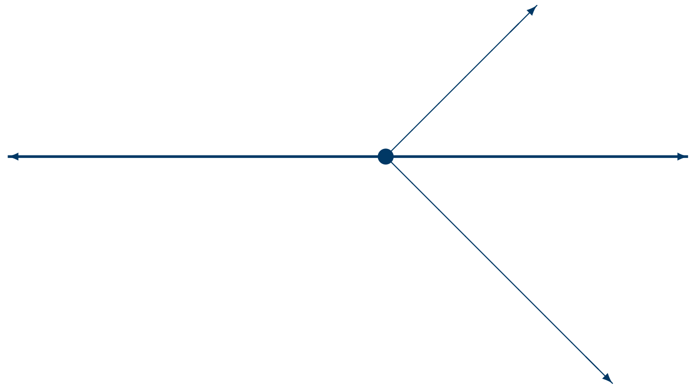

Figure: A star-shaped network with $N = 4$ edges.

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Let $N \in \{2, 3, 4, ...\}$ denote the number of strings.

For $i \in \{1, 2, ..., N\}$ let $c_i > 0$ and $\varepsilon_i \geq 0$ be given and consider the wave equation

$$U^i_{tt} = U^i_{xx} - 2\,\varepsilon_i\,U^i_t - (\varepsilon_i^2 - c_i^2)\,U^i = 0 \tag{2}$$

on the space interval $[0,\ L_i]$.

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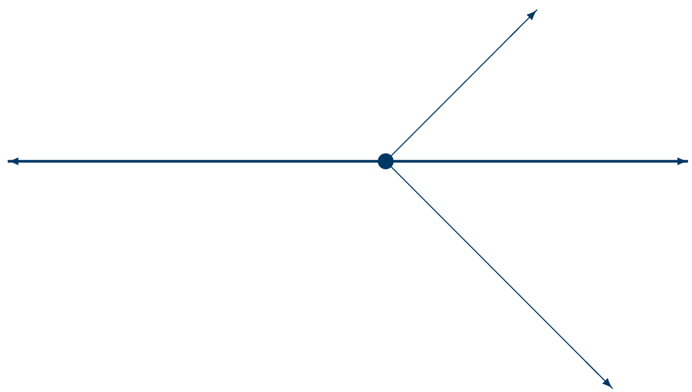

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on the space interval $[0,\ L_i]$. The edges are coupled at $x = 0$ by node conditions:

A tree of strings

At the central node:

For $i, j \in \{1, ..., N\}$

$$\begin{aligned} U^i(t, 0) - U^j(t, 0) &= 0, \\ \sum_{k=1}^{N} U_x^k(t, 0) &= 0. \end{aligned}$$

A tree of strings

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At the boundary node of edge number 1 at $x = L_1$ we have a homogeneous DIRICHLET condition

$$U_1(t, L_1) = 0$$

A tree of strings

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At the boundary node of edge number 1 at $x = L_1$ we have a homogeneous DIRICHLET condition

$$U_1(t,\;L_1) = 0$$

and **at the other boundary nodes** for $j \in \{2, ..., N\}$ at $x = L_j$ we have a NEUMANN velocity feedback

$$U_x^j(t,\;L_j) = K_j\,U_t^j(t,\;L_j)$$

with a real *feedback* parameter K_j.

We have a video!

A movie

A tree of strings

The wave equation (2) can be transformed to a 2×2 system:

A tree of strings

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A tree of strings

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Then due to the definition of V^i and (2), the function (U^i, V^i) solve

$$\begin{pmatrix} U^i \\ V^i \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix}_x + \begin{pmatrix} \varepsilon^i & c^i \\ c_i & \varepsilon^i \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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Also the node conditions and boundary conditions can be transformed similarly:
For example, at $x = L_1$, we have

$$V^1(t,\ L_1) = -\frac{1}{c_1}\left(U^1_x + U^1_t\right).$$

For $i \in \{2, ..., N\}$, at $x = L_i$, we have

$$V^i(t,\ L_i) = -\frac{1}{c_i}\left(\varepsilon_i U^i + (K_i + 1)\, U^i_t\right).$$

Aim of this talk

- Also for our star of strings, *boundary feedback stabilization* is *not always possible*!

- If one of the strings is too long, it can become *impossible* for all feedback parameters!

Inhalt

A tree of strings

Limits of stabilizability: Assume that for all $i \in \{1, ..., N\}$ we have $c_i > \varepsilon_i$.

A tree of strings

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1. **I**nstability if ALL edges are sufficiently long:
   If
   $$c_1^2 \geq \varepsilon_1^2 + \frac{\pi^2}{L_1^2}$$
   and for all $i \in \{2, ..., N\}$ we have
   $$c_i = c_1,\ \varepsilon_i = \varepsilon_1,\ L_i = L_1,\ K_i = K_2$$
   there are **no values** of $K_2 \in (-\infty, \infty)$ such that the closed loop system with the wave equation (2), the node conditions and the boundary conditions is **asymptotically stable**.
   In fact, there are solutions with **exponentially increasing norms** in $\text{X}_{i=1}^{N} L^2(0,\ L_i)$.

A tree of strings

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2. **I**nstability if ONE edge is sufficiently long:
   If
   $$c_1^2 > \varepsilon_1^2 + \frac{9}{4}\frac{\pi^2}{L_1^2} \ \text{ and } \ c_1 - \varepsilon_1 \leq \min_{i \in \{2,...,N\}} \{c_i - \varepsilon_i\}, \tag{3}$$
   there are **no values** of $K_2, ..., K_N \in (-\infty, 0]$ such that
   the closed loop system is **asymptotically stable.**

A tree of strings

Limits of stabilizability: Assume that for $i \in \{1, ..., N\}$ we have $c_i > \varepsilon_i = 0$ and one of the values of $L_i > 0$ is sufficiently large.

- Due to the POINCARÉ–inequality, if the $c_i > 0$ are sufficiently small, we can use the energy

$$E(t) = \frac{1}{2} \sum_{i=1}^{N} \int_0^{L_i} \left(U_x^i(t, x)\right)^2 + \left(U_t^i(t, x)\right)^2 - c_i^2 \left(U^i(t, x)\right)^2 \, dx.$$

A tree of strings

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- We have

$$E'(t) = \sum_{i \in I_F} K_i \left(U_t^i(t, L_i)\right)^2 .$$

Thus if $K_i \geq 0$, we have $E'(t) \geq 0$ and thus the energy does not decay.

A tree of strings

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- Thus there are no parameter vectors with components $K_i \geq 0$ such that the system is asymptotically stable.

A tree of strings

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- Thus there are no parameter vectors with components $K_i \geq 0$ such that the system is asymptotically stable.
- With the Result 2. above, this implies that there are no parameter vectors with components of *equal sign* such that the system is asymptotically stable.

A tree of strings

Limits of stabilizability - Result 3.: Instability for a large number of short edges

Assume that for $i \in \{1, ..., N\}$ we have $c_i > \varepsilon_i$ and

$$c_i = c_1,\ \varepsilon_i = \varepsilon_1,\ L_i = L_1,\ K_i = K_2.$$

A tree of strings

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If

$$\sin^2(\sqrt{c_1^2 - \varepsilon_1^2}\, L_1) = \frac{1}{N}$$

there are no real values of $K_2 \in (-\infty, \infty)$ ($j \in \{2, ..., N\}$) such that the closed loop system is asymptotically stable.

So here the total length of the strings $N\, L_1^2$ must be sufficiently large!

A tree of strings

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there are no real values of $K_2 \in (-\infty, \infty)$ ($j \in \{2, ..., N\}$) such that the closed loop system is asymptotically stable.
This implies

$$L_1^2\ (c_1^2 - \varepsilon_1^2) = \left(\arcsin(\frac{1}{\sqrt{N}})\right)^2.$$

Since we have $\lim_{N\to\infty} \arcsin(\frac{1}{\sqrt{N}}) = 0$, for N *sufficiently large* we obtain *arbitrarily small values of the lengths* $L_i > 0$, for which the system is not exponentially stable!

So here the total length of the strings $N\, L_1^2$ must be sufficiently large!

Inhalt

A sufficient condition for stabilizability

Assume that $c_i = c_1,\ \varepsilon_i = \varepsilon_1,\ L_i = L_1,\ K_i = K_2$. If $\varepsilon_1 > 0$ and

$$\varepsilon_1 \geq c_1 \geq 0$$

with $K_2 = 0$ the closed loop system is exponentially stable.

A sufficient condition for stabilizability

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- This can be shown by the analysis of the *eigenfunctions* of the system.
- For $\varepsilon_1 \geq c_1 \geq 0$, the matrix of the source term is positive semidefinite.

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What happens for

$$cL \in (0.177.., \pi)?$$

Thank you for your attention!