H^1-EXPONENTIAL STABILIZATION FOR THE INTRINSIC GEOMETRICALLY EXACT BEAM MODEL

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Abstract. The *geometrically exact beam* model (or GEB) gives the position in $\mathbb{R}^3$ of a slender elastic beam that may undergo large displacements of its centerline and large rotations of its cross sections. The *intrinsic formulation* of the GEB model is a first order semilinear hyperbolic system of $d = 12$ equations, that arises when considering as states the translational and rotational velocities and strains of the beam. Here, applying a boundary feedback control a one end of the beam, we show that the steady state $v = 0$ of the *intrinsic formulation* of GEB is locally H^1 - exponential stable (when the applied external forces and moments are set to zero), in the sense that if the initial datum is sufficiently small then this model has a unique global solution in $C^0([0,\infty); H^1(0,L;\mathbb{R}^d))$ whose H^1 - norm decreases exponentially with time. The strategy relies on the study of the energy of the beam, as well as on [BC17, Th. 10.2] which amounts to finding a quadratic Lyapunov function.

References

[BC17] G. Bastin and J.-M. Coron. Exponential stability of semi-linear one-dimensional balance laws. In *Feedback Stabilization of Controlled Dynamical Systems*, pages 265–278. Springer, 2017.

H^1 - exponential stabilization for the intrinsic geometrically exact beam model

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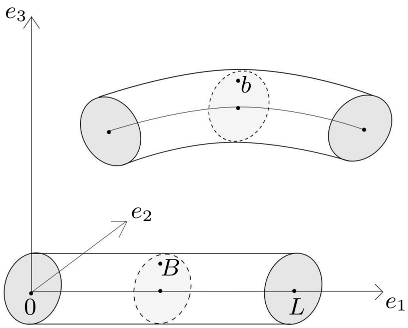
Contents

Reference straight configuration $B = (x, X_2, X_3)^\intercal$

➟ At time $t \geq 0$, $\quad b = \mathbf{p}(x,t) + \mathbf{R}(x,t)(X_2 e_2 + X_3 e_3)$.



Unknowns:

- position of centerline $\mathbf{p} = \mathbf{p}(x,t) \in \mathbb{R}^3$
- rotation of cross sections $\mathbf{R} = \mathbf{R}(x,t) \in \mathbb{R}^{3\times 3}$

Geometrically exact: any magnitude of displacement and rotation.

Small strains; isotropic material (Saint-Venant Kirchhoff); cross sections plane, no change of shape, rotate independently from $\mathbf{p}$; thin beam; lateral contraction neglected.

Governing equations in $(0,L)\times(0,T)$:

$$\begin{cases} \rho a \partial_t^2 \mathbf{p} & = \partial_x[\mathbf{R} M_1(\mathbf{R}^\intercal \partial_x \mathbf{p} - R_c^\intercal p_c')] + \bar{f}_1, \\ \rho \partial_t [\mathbf{R} J \mathrm{vec}(\mathbf{R}^\intercal \partial_t \mathbf{R})] & = \partial_x[\mathbf{R} M_2 \mathrm{vec}(\mathbf{R}^\intercal \partial_x \mathbf{R} - R_c^\intercal R_c')] \\ & \quad + (\partial_x \mathbf{p}) \times (\mathbf{R} M_1(\mathbf{R}^\intercal \partial_x \mathbf{p} - R_c^\intercal p_c')) + \bar{f}_2, \end{cases}$$

+ Dirichlet B.C. at $x = L$: $\mathbf{p} = h^{\mathbf{p}}, \quad \mathbf{R} = h^{\mathbf{R}}$,

+ Neumann B.C. at $x = 0$: $\begin{cases} -\mathbf{R} M_1(\mathbf{R}^\intercal \partial_x \mathbf{p} - R_c^\intercal p_c') = h_1 \\ -\mathbf{R} M_2 \mathrm{vec}(\mathbf{R}^\intercal \partial_x \mathbf{R} - R_c^\intercal R_c') = h_2, \end{cases}$

+ initial conditions.

Notation: $M = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \Leftrightarrow \mathrm{vec}(M) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ i.e. $Mz = \mathrm{vec}(M) \times z$.

[1] Reissner '81, Simo '85, Kapania & Li '03, Strohmeyer '18

Transformation: GEB problem $\xrightarrow{\text{Transformation}}$ IB problem

$$y = \begin{bmatrix} \mathbf{R}^\intercal \partial_t \mathbf{p} & \text{velocity of centerline } V \\ \text{vec}(\mathbf{R}^\intercal \partial_t \mathbf{R}) & \text{angular velocity } W \\ \mathbf{R}^\intercal \partial_x \mathbf{p} - R_c^\intercal p_c' & \text{translational strain } \Gamma \\ \text{vec}(\mathbf{R}^\intercal \partial_x \mathbf{R} - R_c^\intercal R_c') & \text{curvature } \Upsilon \end{bmatrix} \in \mathbb{R}^{12}$$

➥ semilinear hyperbolic system:

$$\partial_t y + \mathbf{A} \partial_x y + \widetilde{B}(x) y = \widetilde{g}(y) + \widetilde{q},$$

…of characteristic form ($v = \mathbf{L}y$):

$$\partial_t v + \mathbf{D} \partial_x v + B(x) v = g(v) + q.$$

- $d = 12$
- B indefinite
- $g_k(\varphi) := \varphi^\intercal G^k \varphi$ with $G^k \in \mathbb{R}^{12\times 12}$.

[2] Hodges 2003

More precisely,

The coefficients $\mathbf{D}, B, g$ are explicitly known.

Parameters: density ρ, cross section area a, shear modulus G, Young modulus E, area moments of inertia contained in $J \in \mathbb{R}^{3\times 3}$, correction factors k_2, k_3. Strains before deformation: $\Gamma_c, \Upsilon_c \in C^1([0,L];\mathbb{R}^3)$.

About $\mathbf{D} \in \mathbb{R}^{d\times d}$: for D_+ pos. definite diagonal matrix

$$\mathbf{D} = \mathrm{diag}(-D_+, D_+).$$

➥ Notation: for $v \in \mathbb{R}^d$, $v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix}$, where $v_-, v_+ \in \mathbb{R}^6$.

Existence and uniqueness:

- $C^1_{x,t}$ solutions to 1-d quasilinear hyperbolic systems: Wang '06 (extension of Li '10 to nonautonomous systems). Solution local and semi-global in time.
- $C^0([0,T];H^1)$ solutions to 1-d semilinear hyperbolic systems: Bastin, Coron '17 and '16.

Boundary feedback exponential stabilization: boundary condition $\mathcal{B}(y(0,t),y(L,t),u(t)) = 0$ with feedback control $u(t) = u(y(0,t),y(L,t))$. See Bastin, Coron '16.

Notation: $H^1 = H^1(0,L;\mathbb{R}^d)$ and $C^k_{x,t} = C^k([0,L]\times[0,T];\mathbb{R}^d)$.

Local H^1-exp. stabilization of IB

Assumption 1:

Let $\mu_1, \mu_2 > 0$. Assume $\bar{f}_1 = \bar{f}_2 = 0$, and the boundary conditions are

$$v_-(L,t) = -v_+(L,t), \qquad v_+(0,t) = \kappa\, v_-(0,t),$$

where κ diagonal matrix depending on μ_1, μ_2 and s.t. $\kappa_i \in (-1,1)$ for $1 \leq i \leq 6$.

(1)
$$\begin{cases} \partial_t v + \mathbf{D}\partial_x v + B(x)v = g(v) & \text{in } (0,L)\times(0,T) \\ v_-(L,t) = -v_+(L,t) & \text{for } t \in (0,T) \\ v_+(0,t) = \kappa\, v_-(0,t) & \text{for } t \in (0,T) \\ v(x,0) = v^0(x) & \text{for } x \in (0,L) \end{cases}$$

Theorem:

The steady state $v = 0$ of (1) system is H^1 - exponentially stable,

... in the sense that $\exists \varepsilon > 0$, $\alpha > 0$ and $c > 0$ s.t., for any $v^0 \in H^1(0, L; \mathbb{R}^d)$ satisfying

$$\|v^0\|_{H^1(0,L;\mathbb{R}^d)} \leq \varepsilon$$

and the C^0-compatibility conditions at $(x, t) = (0, 0)$ and $(x, t) = (L, 0)$, the solution v to (1) belongs to $C^0([0, +\infty); H^1(0, L; \mathbb{R}^d))$ and satisfies

$$\|v(t)\|_{H^1(0,L;\mathbb{R}^d)} \leq ce^{-\alpha t}\|v^0\|_{H^1(0,L;\mathbb{R}^d)}, \quad \forall t \in [0, +\infty).$$

About the boundary conditions:

The boundary condition are chosen as a result of the analysis of the beam energy $\mathcal{E}$ (which is the sum of the kinetic and strain energy).

About the proof:

The proof of the main theorem involves the general result for 1-d semilinear hyperbolic system in Bastin, coron '17, as well as a study of the structure of $\mathcal{E}$.

Future work

- Networks of beams: Write the boundary conditions for a network of IB. Stability study.

- Add source terms $\bar{f}_1 \neq 0$ and $\bar{f}_2 \neq 0$: nontrivial steady state.

Thank you for your attention!

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