Inverse design of one-dimensional Burgers equation

Thibault Liard

Post-doctoral researcher at DeustoTech, Bilbao

Joint work with Enrique Zuazua

28th August, 2019

Scalar conservation laws

We consider the one-dimensional Burgers equation

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1}$$

where u is the state and the flux function f is a strictly convex function defined by $f(u) = \frac{u^2}{2}$. We denote by

$$(t,x) \to S_t^+(u_0)(x) \in L^\infty([0,T] \times \mathbb{R}) \cap C^0([0,T], L^1_{\text{loc}}(\mathbb{R}))$$

the weak entropy solution of (1) with initial datum $u_0 \in L^\infty(\mathbb{R})$.

The goal is to find theoretically and numerically the set of initial data u_0 such that $S_T^+(u_0)$ is close to a given target u^T as much as possible.

Motivation : minimization of the Sonic boom effects generated by supersonic aircrafts which are modeled by an augmented Burgers equation

Optimal control problem

This leads to the following optimal control problem

$$\min_{u_0 \in \mathcal{U}_{\text{ad}}^0} J_0(u_0) := \|u^T(\cdot) - S_T^+(u_0)\|_{L^2(\mathbb{R})}, \tag{2}$$

where $u^T \in BV(\mathbb{R})$ and the class of admissible initial data is defined by

$$\mathcal{U}_{\text{ad}}^0 = \{u_0 \in BV(\mathbb{R}) / \|u_0\|_{BV(\mathbb{R})} < C \text{ and } \text{Supp}(u_0) \subset K_0\}.$$

Two main difficulties arise.

- There exist multiple initial data leading to the same given target.
- The given target u^T may be unreachable along forward entropic evolution.
- Making sense of the derivative of J_0 is complex.

The backward operator S_t^-

The backward operator S_t^- associated to the Burgers dynamic is defined by

$$S_t^-(u^T)(x) = S_t^+(x \to u^T(-x))(-x),$$

for every $t \in [0, T]$ and for a.e $x \in \mathbb{R}$.

The solution $S_t^-(u^T)$ may be regarded as the zero viscosity limit of $S_T^{-,\epsilon}(u^T)$ solution of the following backward equation

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = -\epsilon \partial_{xx}^2 u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(T,\cdot) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

Using the change of variable $(t,x) \to (T-t, -x)$, we notice that the backward equation above is well-defined.

Thus, $S_T^-(u^T)$ is also called the backward entropy solution with final target u^T.

Theorem

The optimal control problem (2) *admits multiple optimal solutions. Morever, the initial datum* $u_0 \in BV(\mathbb{R})$ *is an optimal solution of* (2) *if and only if* $u_0 \in BV(\mathbb{R})$ *verifies* $S_T^+(u_0) = S_T^+(S_T^-(u^T))$.

- A full characterization of the set of initial data $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = S_T^+(S_T^-(u^T))$ is given in [**Colombo-Perrolaz, 2019**].
- If there exists an initial datum $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = u^T$ then $S_T^+(S_T^-(u^T)) = u^T$.

If u^T is a reachable target with finite number of shocks

The two following results are given in [**Colombo-Perrolaz, 2019**].

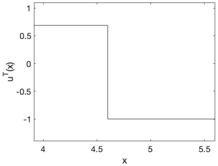
- There exists an initial datum $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = u^T$ iff u^T satisfies the Oleinik condition, means that $\partial_x u^T \leq \frac{1}{T}$ in the sense of distributions.
- A map $u_0 \in BV(\mathbb{R})$ verifies $S_T^+(u_0) = u^T$ if and only if the two following statements hold :
  - For every $x \in \mathbb{R} \setminus \cup_{i=1}^N [a_i, b_i]$, $u_0(x-) = S_T^-(u^T)(x-)$.
  - For every $x \in \cup_{i=1}^N [a_i, b_i]$

$$\int_{a_i}^{x} u_0(s)\, ds \geq \int_{a_i}^{x} S_T^-(u^T)(s)\, ds,$$

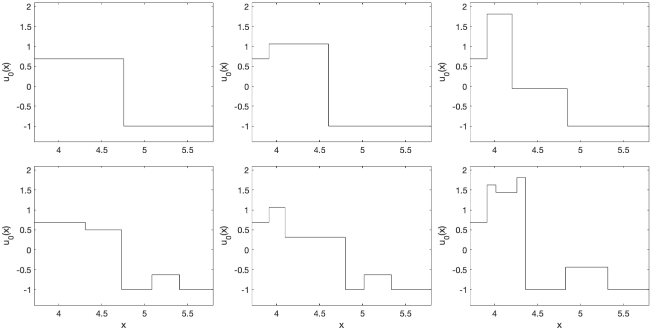
$$\int_{a_i}^{b_i} u_0(s)\, ds = \int_{a_i}^{b_i} S_T^-(u^T)(s)\, ds.$$

with $a_i := x_i^T - Tf'(u^T(x_i^T-))$ and $b_i := x_i^T - Tf'(u^T(x_i^T+))$ and $(x_i^T)_{i\in\{0,\cdots,N\}}$ the $N \in \mathbb{N} \cup \{\infty\}$ discontinuous poins of u^T such that $u^T(x_i^T+) < u^T(x_i^T-)$.

Example



The reachable target $u^T(\cdot) = 0.6875\mathbb{1}_{(-\infty,4.6)}(\cdot) - \mathbb{1}_{(4.6,\infty)}(\cdot)$

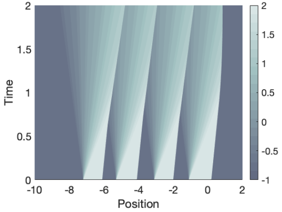


Construction of six random initial data u_0 leading to u^T using a wave-front tracking method

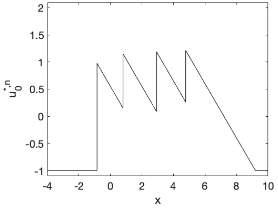
Ideas of the proof

1. Rewrite (2) as
$$\min_{q\in\mathcal{U}_{\text{ad}}^1} J_1(q) := \|u^T - q\|_{L^2(\mathbb{R})}, \tag{3}$$
where the admissible set $\mathcal{U}_{\text{ad}}^1$ is defined by
$$\mathcal{U}_{\text{ad}}^1 = \{q \in BV(\mathbb{R})/\ \partial_x q \leq \frac{1}{T} \text{ and } \|q\|_{BV(\mathbb{R})} \leq C \text{ and } \text{Supp}(q) \subset K_1\}.$$
2. Using $S_T^-(S_T^+(S_T^-(u^T)) = S_T^-(u^T)$ and a full characterization of u_0 such that $S_T^-(u_0) = S_T^-(u^T)$, we prove that $S_T^+(S_T^-(u^T))$ is a critical point of (3).

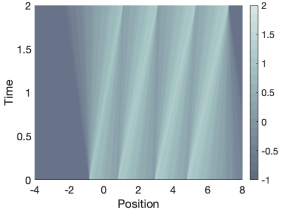
$$u^T(x) = \begin{cases} 2 & \text{if } x \in (-0.2, 1.1) \bigcup (2, 3.1) \bigcup (4.1, 5.3) \bigcup (6.1, 7.2), \\ -1 & \text{otherwise.} \end{cases}$$



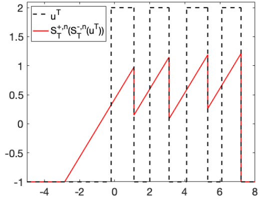
$(t, x) \quad \rightarrow \quad S_t^-(u^T)(-x)$



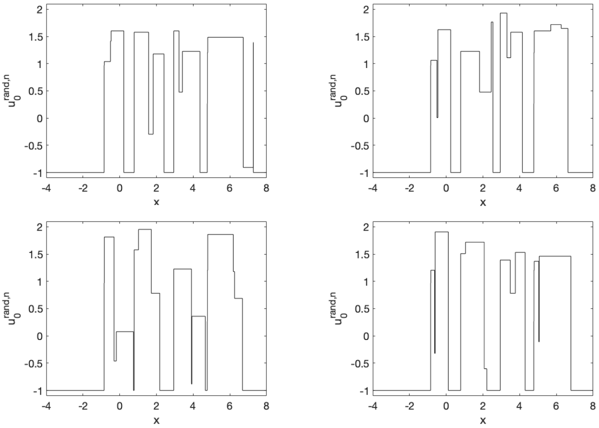
$x \quad \rightarrow \quad S_T^-(u^T)(x)$



$(t, x) \quad \rightarrow \quad S_t^+(S_T^-(u^T))(x)$

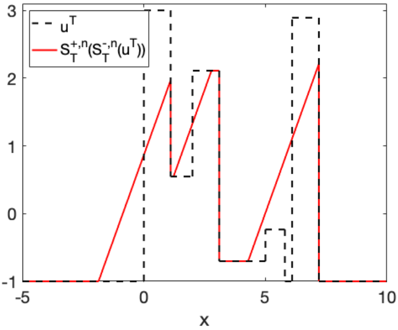


u^T and $x \rightarrow S_T^+(S_T^-(u^T))(x)$



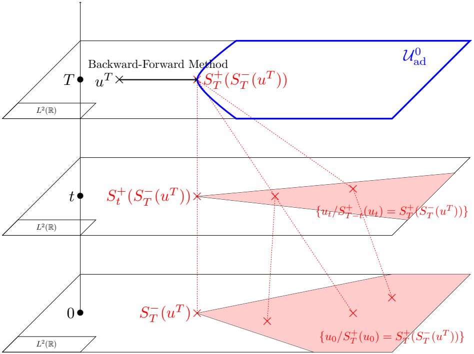
Four different $u_0^{\mathrm{rand},n}$ such that $S_T^+(u_0^{\mathrm{rand},n}) \;=\; S_T^+(S_T^-(u^T))$

$$u^T = \begin{array}{l} -\mathbb{1}_{(-\infty,0)} + 3\mathbb{1}_{(0,1.1)} + 0.55\mathbb{1}_{(1.1,2)} + 2.11\mathbb{1}_{(2,3.1)} - 0.7\mathbb{1}_{(3.1,5)} \\ -0.23\mathbb{1}_{(5,5.8)} - \mathbb{1}_{(5.8,6.1)} + 2.89\mathbb{1}_{(6.1,7.2)} - \mathbb{1}_{(7,2,\infty)}. \end{array}$$



u^T and $x \to S_T^+(S_T^-(u^T))(x)$

Conclusion



$S_T^+(S_T^-(u^T))$ is constructed using a backward-forward approach. The shaded area (in red) at time x and at time t are the set of initial data $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = S_T^+(S_T^-(u^T))$ and the set of initial data $u_t \in BV(\mathbb{R})$ such that $S_{T-t}^+(u_t) = S_T^+(S_T^-(u^T))$ respectively.

Open problems

1. It would be interesting to replace the L^2-norm by the BV-norm in (2) which seems to be more natural since we do not need the artificial constraint $\|u\|_{BV(\mathbb{R})} \leq C$ in the admissible class of initial data $\mathcal{U}_{\text{ad}}$ anymore to prove the existence of minimizers for (2).
2. We may also consider a convex-concave function as a flux function in (1) which is for instance a more realistic choice to describe the flow of pedestrian.
3. A source term may be added to the Burgers equation. In this case, the backward-forward method described in this paper may not be well-defined.
4. To finish we can also investigate systems of conservation laws in one dimension or in multi-dimension (Euler equations, Shallow water equations).

Thank you for your attention

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No694126 – DYCON).

European Research Council
Established by the European Commission

DYNAMIC CONTROL

CHAIR OF
COMPUTATIONAL
MATHEMATICS