Controllability of systems of fourth order parabolic PDEs.

Alberto Mercado Saucedo

Universidad Técnica Federico Santa María, Chile,
Institut de Mathématiques de Toulouse.

Joint works with N. Carreño, E. Cerpa, P. Guzmán (UTFSM, Chile);
Marcos Lopez-García (UNAM, México).
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Controllability of PDE's. Introduction.

We consider the control problem of the **heat equation** in an open bounded set $\Omega \subset \mathbb{R}^n$ with a boundary control h: :

$$\begin{cases} \partial_t u - \Delta u = 0 & \Omega \times (0,T) \\ u = h 1\!\!1_{\Gamma_0} & \Gamma \times (0,T) \\ u(0) = u_0 & \Omega \end{cases}$$

Where

- The set $\Gamma_0 \subset \Gamma$ is the control zone,
- The initial condition u_0 is given.
- The function h is the control.

We intend to impose that $u(T)$ would take a prescribed value, choosing an adequate h.

Controllability of PDE's. Introduction.

Some controllability properties:

1. **Exact Controllability**. Any state $u(T,\cdot) \in X$ can be reached by a solution of the system with some control $h \in H$.
2. **Null Controllability**. Any initial condition $u_0 \in X$ can be driven to $u(T,\cdot) = 0$.

Controllability of PDE's. Introduction.

Our model: **heat equation**.

1. **Exact Controllability**. Any state $u(T, \cdot) \in X$ can be reached by a solution of the system with some control $h \in H$.

Not satisfied in $X = L^2(\Omega)$, due to **regularizating effect** .

2. **Null Controllability**. Any initial condition can be driven to $u(T, \cdot) = 0$.

Duality.

We consider the adjoint equation

$$(P^*) \begin{cases} -\partial_t \varphi - \Delta \varphi &= 0 & \Omega \times (0,T) \\ \varphi &= 0 & \Gamma \times (0,T) \\ \varphi(T) &= \varphi_T & \Omega \end{cases}$$

Then we have

$$\int_\Omega u_0(x)\varphi(0,x)dx - \int_\Omega u(T,x)\varphi_T(x)dx = \int_0^T \int_{\Gamma_0} h \frac{\partial \varphi}{\partial n} dxdt$$

$$\forall \varphi_T \in L^2(\Omega), \forall u_0 \in L^2(\Omega)$$

Duality.

For instance, null controllability is equivalent to

For any $u_0 \in L^2(\Omega)$, there exists $h \in L^2(0,T)$ such that

$$\int_\Omega u_0 \varphi(0,x)dx = \int_0^T \int_{\Gamma_0} h \frac{\partial \varphi}{\partial n} dxdt$$

$$\forall \varphi_T \in L^2(\Omega),$$

The key idea of moments method:

To use φ solutions from $\varphi_T =$ the eigenfunctions of the equation.

One dimension. moments method.

If $\Omega = (0, \pi)$ and $\Gamma_0 = \{0\}$, from the characterization of null controllability:

$$\int_\Omega u_0 \varphi(x,0)dx = \int_0^T h(t) \frac{\partial \varphi}{\partial n}(t,0)dt \qquad \forall \varphi_T \in L^2(\Omega)$$

taking $\varphi_T = \sin(nx)$ eigenfunction of $-\Delta$ with $\lambda_n = n^2$ as eigenvalue, we have

$$\varphi = e^{-n^2(T-t)} \sin(nx),$$

and then

$$\int_\Omega u_0(x) e^{-n^2 T} \sin(nx)dx = \int_0^T h(t) n \cos(0) e^{-n^2(T-t)} dt \qquad \forall n \in \mathbb{N}$$

This is, (change of var: $t \to T - t$)

$$e^{-n^2 T} \underbrace{\int_\Omega u_0(x) \sin(nx)dx}_{a_n} = n \int_0^T \widetilde{h}(t) e^{-n^2 t} dt \qquad \forall n \in \mathbb{N}$$

Moments method for heat equation.

Recall that $u_0 \in L^2(\Omega)$ if and only if $\{a_n\} \in \ell^2$, since $\{\sin(nx)\}$ is a basis for $L^2(0,\pi)$.

We look for a function $\tilde{h} \in L^2(0,T)$ (the control) such that

$$\int_0^T \tilde{h}(t) e^{-n^2 t} dt = \frac{e^{-n^2 T}}{n} a_n \qquad \forall n \in \mathbb{N}$$

This means, we have to express the space $L^2(0,T)$ decomposed by functions

$$e^{-\lambda_n t}, \quad n \in \mathbb{N}$$

The functions $e^{-n^2 t}$ are a basis for $L^2(0,T)$?

Hector Fattorini, David Russell.

Figure : Héctor Fattorini

Figure : David L. Russell

Hector Fattorini, David Russell.

Main ingredients

- If $\sum \frac{1}{\lambda_n} < \infty$ then the family $\{e^{-\lambda_n t}\}$ is minimal en $L^2(0,T)$.

$$\text{i.e.} \quad e^{-\lambda_n t} \notin \overline{\langle e^{-\lambda_k t} : k \neq n \rangle}.$$

- If $|\lambda_m - \lambda_k| \geq \rho |m-k|$ (gap condition) then

$$\text{dist}(e^{-\lambda_n t}, \overline{\langle e^{-\lambda_k t} : k \neq n \rangle}) \leq C e^{\varepsilon \lambda_n}.$$

This is the case for $\lambda_n = n^2$.

Moments method.

In Fattorini-Russell (1971), using properties of families of real exponentials in $L^2(0,T)$, it was proved the existence of a sequence $\{\theta_n\}$ which is a **biorthogonal family** to $\{e^{-\lambda_n t}\}$.

$$\int_0^T \theta_n(t) e^{-\lambda_m t} dt = \delta_{n,m},$$

In this way, the control h can be obtained by

$$\tilde{h}(t) = \sum_n \tilde{a}_n \theta_n(t)$$

Fatrorini-Russell.

Theorem (Fattorini-Russell 1971)

One-dimensional heat equation is null-controllable, for any $T > 0$.

Since then, this method has been applied to several control problems for different equations in **dimension one**.

Fourth order parabolic equation.

We consider the following Kuramoto-Sivashinshky (KS) control system

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} = 0, & x \in (0,1),\ t > 0, \\ y(t,0) = h(t), \quad y(t,1) = 0, & t > 0, \\ y_x(t,0) = 0, \quad y_x(t,1) = 0, & t > 0, \end{cases} \tag{1}$$

where the state is given by $y = y(t,x)$ and the time-dependent functions h_1, h_2 are boundary controls. This equation was derived as a model for phase turbulence and plane flame propagation.
We have some results for **linear version** of KS equation:

Theorem (Cerpa-Guzmán-M 2017)

Consider $h_2 = 0$ and $\lambda > 0$. We show that if

$$N = \{4k^2\pi^2\} \cup \{(n^2 + m^2)\pi^2\}$$

then the linear version of system (8) *is null-controllable if and only if $\lambda \in \mathbb{R}^+ \setminus N$.*

PROFF: If w_n is the eigenfunction, $w_n'(0) \neq 0$ if and only if $\lambda \in \mathbb{R}^+ \setminus N$.

Boundary control of coupled system

Consider the boundary control of coupled system:

$$\begin{cases} u_t - u_{xx} = v \\ v_t - dv_{xx} = 0 \\ u(t,0) = u(t,L) = 0, \\ v(t,0) = h(t),\ v(t,L) = 0 \end{cases} \tag{2}$$

([Ammar; Benabdallah; González-Burgos; de Teresa. *Minimal time for the null controllability ...* J. Funct. Anal. 267 (2014)])

In this case, the eigenvalues are

$$\Lambda_d = \{dk^2, m^2\}_{k,m\in\mathbb{N}}$$

The controllability of the system depends on d.

Boundary control of coupled system

Directly, we have:

Theorem

The system is not controlable if $\sqrt{d} \in \mathbb{Q}$.

PROFF: In that case $dk^2 = m^2$ for some $k, m \in \mathbb{N}$ and then $\{e^{-\lambda_n t}\}$ is not minimal.

For $\sqrt{d} \notin \mathbb{Q}$?

We need:

1. The family $\{e^{-\lambda t}\}$ to be minimal. (OK iff $\sqrt{d} \notin \mathbb{Q}$).
2. An estimate of the norm $\|\theta_n\|_{L^2}$.

The norm depends on how close are the elements of $\Lambda = \{\lambda_n\}$ one from each other.

Boundary control of coupled system

It was obtained ([Ammar et al, (2014)]) an explicit formula

$$c(\{\lambda_k\})$$

called the *condensation index*, satisfying

$$\|\theta_n\|_{L^2} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\lambda_n}$$

where

$$c(\Lambda) = \textit{ index of condensation } \text{ of the sequence } \Lambda$$

Now, recalling that

$$\tilde{h}(t) = \sum_n e^{-\lambda_n T} a_n \theta_n(t)$$

we have that, if $T \geq c(\Lambda)$, then

$$\|\theta_n\|_{L^2} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon-T)\lambda_n}$$

A minimal time for controllability for the case $T_0 = c(\Lambda) > 0$.

Boundary control of coupled system

Theorem

1. *System is null-controllable if* $T > T_0$.
2. *System is not null-controllable if* $T < T_0$.

Theorem

For the case $\Lambda_d = \{dk^2, m^2\}_{k,m\in\mathbb{N}}$*:*

1. $c(\Lambda) = 0$ *for almost all* $d \in (0, \infty)$ *(in particular for algebraic numbers* $\sqrt{d}$*).*
2. *For each* $T_0 \in [0, \infty]$*, there exists* $d > 0$ *such that* $c(\Lambda) = T_0$.

Boundary control of coupled system

Remark:

$$c(\{\lambda_k\}) := \limsup_{k\to\infty} \frac{\ln \frac{1}{|E'(\lambda_k)|}}{\lambda_k}, \tag{3}$$

where

$$E(z) = \prod_{k\in\mathbb{N}} \left(1 - \frac{z^2}{\lambda_k^2}\right), \quad z \in \mathbb{C}. \tag{4}$$

Boundary control of coupled system

Our problem: We study the boundary control of coupled system:

$$\begin{cases} u_t + u_{xxxx} = v \\ v_t - dv_{xx} = 0 \\ u(t,0) = u_{xx}(t,0) = 0, \\ u(t,L) = u_{xx}(t,L) = 0, \\ v(t,0) = h(t),\ v(t,L) = 0 \end{cases} \tag{5}$$

In this case, the eigenvalues are

$$\Lambda_d = \{dk^2, m^4\}_{k,m\in\mathbb{N}}$$

The controllability of the system depends on d.

Boundary control of coupled system

A first result:

Theorem (Cerpa, Carreño, M (preprint))

System (5) *is not (approximate) controlable if* $\sqrt{d} \in \mathbb{Q}$.

PROFF: In that case $dk^2 = m^4$ for some $k, m \in \mathbb{N}$ and then $\{e^{-\lambda_n t}\}$ is not minimal.

What about null-controllability for $\sqrt{d} \notin \mathbb{Q}$?

We need:

1. The family $\{e^{-\lambda t}\}$ to be minimal. (OK iff $\sqrt{d} \notin \mathbb{Q}$).
2. An estimate for the norm $\|\theta_n\|$.

The norm depends on how close are the elements of $\Lambda = \{\lambda_n\}$ one from each other.

Boundary control of coupled system

We have to compute $E(z) = \prod_{k\in\mathbb{N}} \left(1 - \frac{z^2}{d^2k^4}\right)\left(1 - \frac{z^2}{k^8}\right)$.
Again, the elements of $\Lambda_d = \{dk^2, m^4\}_{k,m\in\mathbb{N}}$ are close one from each other iif $\sqrt{d}$ is well approximated by rationals.

The **irrationality measure** (or **Liouville-Roth constant**) of x is the supremum of $\mu \in \mathbb{R}$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

for an infinite number of integers p, q, with $q > 0$.

It is known that:

- $\mu = 1$ for all rational x.
- $\mu \geq 2$ for all irrational x.
- $\mu = 2$ for all irrational algebraic x (Roth, 1955),
- $\mu(\phi) = 2$, $\mu(e) = 2$,
- $\mu(\pi) \leq 7.6063085$ (Salikhov 2008).
- There exist numbers with $\mu = \infty$ (**Liouville numbers**).

If $\mu < \infty$ then OK

Hence we get:

Theorem (Cerpa, Carreño, M. (preprint))

Given $\sqrt{d}$ an irrational number, then

$$\mu(\sqrt{d}) < \infty \Longrightarrow c(\{dk_2, m^4 \ : \ k, m \in \mathbb{N}\}) = 0.$$

And then

Theorem (Cerpa, Carreño, M. (preprint))

- *Suppose $\sqrt{d}$ is an irrational number with* **finite irrationality measure**. *Then system* (5) *is controlable with one boundary control for any $T > 0$.*
- *Given any $T_0 \in [0, \infty]$, there exists $d \in \mathbb{R}^+$ such that $c(\Lambda) = T_0$. Then system* (5)*:*
  1. *Is controlable if $T > T_0$,*
  2. *Is not controllable if $T < T_0$.*

Fourth order parabolic equation.

We consider the following Kuramoto-Sivashinshky (KS) control system

$$\begin{cases} y_t + \varepsilon y_{xxxx} + v y_x = 0, & x \in (0,1),\ t > 0, \\ y(t,0) = h_1(t), \quad y(t,1) = 0, & t > 0, \\ y_x(t,0) = h_2(t), \quad y_x(t,1) = 0, & t > 0, \end{cases} \tag{6}$$

We are interested in the controllability when

$$\varepsilon \to 0$$

Known related results:

- **Carreño-Guzmán (2016)**. When $\varepsilon \to 0$, the cost of the control remains bounded if

$$T \geq 40L/|v|.$$

Rmk: Hypothesis seems to be not sharp (The natural minimal time: $T_0 = L/|v|$).

Main tools of the previous results: **Carleman estimates**.

Second order, limiting case.

The corresponding second-order problem:

$$\partial_t u - \varepsilon \Delta u + v \cdot \nabla u = 0 \tag{7}$$

when $\varepsilon \to 0$.
Known related results:

- **Guerrero-M-Osses (2007).**nD Cost of the approximate (regional) controllability.
- **Guerrero-Coron (2007).** $1D$: The cost of null controllability remains bounded if $T \geq 58L/|v|$.
- **Guerrero-Lebeau (2007).** nD: The cost of the control remains bounded under geometric conditions and T large enough.
- **Glass (2010)**]. $1D$: The cost of the control remains bounded if $T \geq 6L/|v|$.
- **Lissy (2015)**]. $1D$: The cost of the control remains bounded if $T > 4,2L/v$ for $v > 0$ and $T > 6,1L/|v|$ for $v < 0$. If $T < \frac{2\sqrt{2}L}{|v|}$ then the cost of the control is not bounded as $\varepsilon \to 0$.
- **Darde, Everdoza (2017)** $1D$: The cost of the control remains bounded if $T > 3,33L/v$ for $v > 0$ and $T > 5,33L/|v|$ for $v < 0$.

Back to our problem.

Recall the Kuramoto-Sivashinshky (KS) equation

$$y_t + \varepsilon y_{xxxx} + v y_x = 0, \tag{8}$$

Recall that we look a control

$$\tilde{h}(t) = \sum_n \tilde{a}_n \theta_n(t)$$

where $\{\theta_n\}$ is such that

$$\left\langle \theta_n, e^{-\lambda_m t} \right\rangle = \delta_{n,m}$$

Difficulty: diagonalization of the differential operator. Instead we deal with

$$y_t + \varepsilon y_{xxxx} + \delta y_{xxx} + v y_x = 0, \tag{9}$$

More precisely:

$$\begin{cases} y_t + \varepsilon y_{xxxx} + \delta y_{xxx} + v y_x = 0, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0,\ y(t,L) = 0, & t \in (0,T), \\ By(t,0) = h(t),\ By(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{10}$$

where we have defined $\delta = -2\varepsilon^{2/3} M^{1/3}$ and $By = 2\varepsilon y_{xx} + \delta y_x$.

and the adjoint system is given by

$$\begin{cases} -\varphi_t + \varepsilon \varphi_{xxxx} - \delta \varphi_{xxx} - v \varphi_x = 0, & (t,x) \in (0,T) \times (0,L), \\ \varphi(t,0) = 0,\ \varphi(t,L) = 0, & t \in (0,T), \\ B^*\varphi(t,0) = 0,\ B^*\varphi(t,L) = 0, & t \in (0,T), \\ \varphi(T,x) = \varphi_0(x), & x \in (0,L), \end{cases} \tag{11}$$

Theorem

The eigenfunctions are

$$e_k(x) = e^{Ax} \sin(kx) \quad (12)$$

with $k \in \mathbb{N}\backslash\{0\}$ *and corresponding eigenvalues*

$$\lambda_k := \varepsilon(k^2 + B)^2 - C. \quad (13)$$

We follow the work:
Olivier Glass, *A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit*, Journal of Functional Analysis 258, 2010.
They study the analogous problem for

$$-\varepsilon y_{xx} + v y_x = 0.$$

We need θ_n such that

$$\int_0^T \theta_n(t) e^{-\lambda_m t} dt = \delta_{n,m} \quad \forall m, n.$$

PROOF:

1. If $J_n = \mathcal{F}(\theta_n)$, then

$$J_n(-i\lambda_k) = \delta_{kn}$$

2. We define Φ having simple zeros exactly at $\{-i\lambda_k : k \in \mathbb{N} \backslash \{0\}\}$.
3. For $n \in \mathbb{N}$ we define

$$J_n(z) := \frac{\Phi(z)}{\Phi'(-i\lambda_n)(z + i\lambda_n)}.$$

4. Then, to use the Paley-Wiener Theorem:

$$\theta \in L^2(-A, A) \Leftrightarrow |\mathcal{F}(\theta)(z)| \leq Ce^{A|z|}$$

We need θ_n such that

$$\int_0^T \theta_n(t) e^{-\lambda_m t} dt = \delta_{n,m} \quad \forall m, n.$$

PROOF:

1. If $J_n = \mathcal{F}(\theta_n)$, then

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3. For $n \in \mathbb{N}$ we define

$$J_n(z) := \frac{\Phi(z)}{\Phi'(-i\lambda_n)(z + i\lambda_n)} f(z).$$

4. Then, to use the Paley-Wiener Theorem:

$$\theta \in L^2(-A, A) \Leftrightarrow |\mathcal{F}(\theta)(z)| \leq Ce^{A|z|}$$

5. Beurling-Malliavin multiplier.

Theorem (Lopez-García, M, (preprint))

Given $L, T > 0$, *there exist* $c, C > 0$ *s.t.* $\forall\, y^0 \in L^2(0, L)$, $\varepsilon \in (0, 1)$, *there is* $u \in L^2(0, T)$ *s.t. the solution* y *satisfies*

$$y(\cdot, T) = 0 \in L^2(0, L), \quad \|u\|_{L^2(0,T)} \leq C \exp(-c/\varepsilon^{1/3}) \left\|y^0\right\|_{L^2(0,L)},$$

whenever

$$T > 4,57L/v,\ M > 0; \quad T > 6,19L/|v|,\ v < 0.$$

Future/ongoing work:

- The consequences on the cost of fast controls.
- To deal with the original equation.

Thanks!

¡Gracias!