Long-time Behaviour of a Model of Rigid Structure Floating in a Viscous Fluid

G. Vergara-Hermosilla, M. Tucsnak, F. Sueur

Université de Bordeaux

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Outline

The equilibrium problem

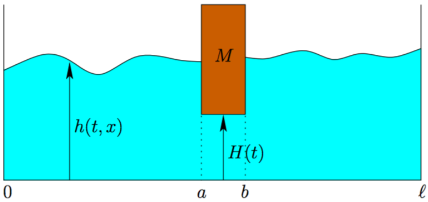

Figure 1: Con guration, Image from Maity, Tucsnak et al.

Notations

0 is the coe cient of viscosity of the uid.

$E = (0; a) [(b; a + b)$.

$I = [a; b]$.

Equations Return to the Equilibrium

Proposition [Maity, Tucsnak et al.,2018]

If $l = a + b$, then, the return to the equilibrium problem, the position of the solid is completely determined by the integro-differential equation

$$\begin{cases} \left(1 + \frac{(b-a)^3}{12}\right)\ddot{H} = -H_0\frac{|I|^2}{|E|^2} - \frac{(b-a)^2}{2}F * \dot{H} - (b-a)\dot{H} - (b-a)H; \\ H(0) = H_0; \quad \dot{H}(0) = 0; \end{cases} \tag{1}$$

and with F such that $\hat{F}(s) = \sqrt{1+s}\,\tanh\left(\frac{sa}{\sqrt{1+s}}\right).$

Unbounded case

Proposition [Maity, Tucsnak et al.,2018]

If $E = (-\infty; a) \cup (b; \infty)$, then the position of the solid is completely determined by the integro-differential equation

$$\begin{aligned} &A\ddot{H} + BF * \dot{H} + C\dot{H} + DH = 0; \\ &H(0) = H_0; \quad \dot{H}(0) = \dot{H}_0; \end{aligned} \qquad (2)$$

where $A = 1 + \frac{(b-a)^3}{12}$, $B = (b-a)^2$, $C = (b-a)$, $D = (b-a)$, and

$$F(t) = \sqrt{\ } \left(\frac{1}{2\sqrt{\ }} \frac{e^{\ t}}{t^3} \right) + D^{1/2}\ {}_0.$$

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where $A = 1 + \frac{(b-a)^3}{12}$, $B = (b-a)^2$, $C = (b-a)\ $, $D = (b-a)$, and

$$F(t) = \sqrt{\pi}\left(\frac{1}{2\sqrt{\pi}}\frac{e^{-t}}{\sqrt{t^3}}\right) + D^{1/2}\delta_0.$$

Goal: Understand the long-time behavior of solutions $H(t)$ for the unbounded case

Mittag-Leffler functions

Mittag-Leffler functions

Definition

The two-parametric Mittag-Leffler function, is the complex-valued function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{with} \quad \alpha > 0, \; \beta \in \mathbb{C}. \tag{3}$$

In the case when $\beta = 1$ the function is known has the classical Mittag-Leffler function and denoted by $E_{\alpha}(z)$.

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Examples

$$E_{1,1}(z) = e^z,$$
$$E_{2,1}(z) = \cosh(\sqrt{z}),$$
$$E_{\frac{1}{2},1}(z) = e^{z^2}\mathrm{erfc}(-z).$$

Mittag-Le er functions

Theorem [Popov-Sedletskii, 2013]

For any 0 < < 1; 2 C, p 2 N, and
$z\ 2\ L_{1=}\ = fz\ 2$ C : j arg $zj\quad g$, the following asymptotics hold

$$E_{\ ;}(z) = \frac{1}{\ } z^{(1\quad)=}\ e^{z^{1=}}\quad \sum_{k=1}^{p} \frac{z^{\ k}}{(\quad k\ )} + R_m^{[1]}(z;\ ;\ ); \qquad (4)$$

where the remainder $R_m^{[1]}$ admits the estimate

$$R_m^{[1]}(z;\ ;\ )\quad \frac{2^{\frac{b+2}{2}}\ (b+1)e^{(\frac{5}{4} j = \ j)}}{jzj^{p+1}};\ \text{if } b = \ (p+1)\ <\ 0$$

$$\frac{(6 + \frac{2}{\ })e^{(\frac{5}{4} j = \ j)}}{jzj^{p+1}};\ \text{if } b < 0:$$

(5)

The rst of estimates in (12) is valid for all $z\ 2\ L_{1=}$, and the second under the additional condition $jzj\quad 2$.

Mittag-Leffler functions

Theorem [Popov-Sedletskii, 2013]

For any $0 < \alpha < 1$, $\beta \in \mathbb{C}$, $p \in \mathbb{N}$ and $z \in \mathbb{C} \setminus L_{1/\alpha}$, $z \neq 0$, the following asymptotics formulas hold

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + R_m^{[2]}(z;\alpha,\beta), \qquad (6)$$

where the remainder $R_m^{[2]}$ admits the estimate

$$|R_m^{[2]}(z;\alpha,\beta)| \le \begin{cases} \dfrac{2^{\frac{b+2}{2}}\,\Gamma(b+1)\,e^{\left(\frac{3}{4}|\beta/\alpha|\right)}}{|z|^{p+1}}, & \text{if } b \ge 0 \\[2ex] \dfrac{\left(6 + \frac{2}{\;}\right)e^{\left(\frac{3}{4}|\beta/\alpha|\right)}}{|z|^{p+1}}, & \text{if } b < 0. \end{cases} \qquad (7)$$

The first of estimates in (7) is valid for all $z \in L_{1/\alpha}$, and the second under the additional condition $|z| \ge 2$.

Laplace transform of Mittag Le er functions

In this work we are interested in the case when $=$ > 0 and $z =$ t , with 2 C and t 2 R. For simplicity we introduce the notation

$$E \;(\;;t) := E_{\;;} \;(\; t \;):$$

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Lemma

The Laplace transform of E (; t) is given by

$$L[t\ \ ^{1}E\ (\ ; t)](s) = (s\ \ \ \ )\ ^{1};$$

where $Re(s) > 0$ and j s $j < 1$.

Solutions unbounded case

Inviscid case

If we consider = 0 in equation (2), the model is reduced to

$$
\begin{aligned}
&A\ddot{H} + B\dot{H} + DH = 0\,;\\
&H(0) = H_0; \quad \dot{H}(0) = \dot{H}_0:
\end{aligned}
$$

Applying Laplace transform to the equation above, and after simpli cations, we obtain

$$\hat{H}(s)\left(As^2 + Bs + D\right) = H_0\,[As + B] + A\dot{H}_0:$$

Solutions of the model, inviscid case

Proposition [V-H, Sueur, Tucsnak, 2019]

The solution $H(t)$ of equation (2) with = 0 is given by

$$
e^{-\frac{Bt}{2A}} \begin{cases} H_0 \left(\cosh\left(\dfrac{t\sqrt{\frac{B^2}{4} - AD}}{A} \right) + \dfrac{B \sinh\left(\dfrac{t\sqrt{\frac{B^2}{4} - AD}}{A} \right)}{2\sqrt{\frac{B^2}{4} - AD}} \right) \\[2ex] + H_0 \dfrac{2A \sin\left(\dfrac{t\sqrt{4AD - B^2}}{2A} \right)}{\sqrt{4AD - B^2}} \end{cases} .
$$

Viscous case

Proposition [V-H, Sueur, Tucsnak, 2019]

The solution of the equation (2) with > 0 is given by

$$H(t) = e^{\frac{t}{}} t^{\frac{1}{2}} \sum_{i=1}^{4} \left[H_0 r_i + H_0 r_i^{\perp} \right] E_{\frac{1}{2}}(\ _i; t):$$

where each $_i$ is a root of the polynomial

$$p(\ ) = A^{\ 4} + \overline{B}^{\ 3} + \left(C \quad \frac{2A}{} \right)^{2} \quad \frac{\overline{B}}{} + \frac{A}{^{2}}; \qquad (8)$$

and the parameters r_i and $r_i^{\perp}$ depends of the roots.

Hurwitz and Anti-Hurwitz polynomials

Hurwitz polynomials

A polynomial $f(X)$ is said to be Hurwitz if the real part of all its complex roots is negative, that is $\mathrm{Re}(u) < 0$ for all $u \in$ such that $f(u) = 0$.

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Theorem (Stodola condition)

If a polynomial with real coefficients is Hurwitz, then all its coefficients are of the same sign.

Anti-Hurwitz polynomials

A polynomial $f(X)$ is said to be Anti-Hurwitz if the real part of all its complex roots is negative, that is $\mathrm{Re}(u) > 0$ for all $u \in \mathbb{C}$ such that $f(u) = 0$.

Theorem [V-H, Luca, 2019]

Let $f(X) = a_0 X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n$ be a polynomial with coefficients in $\mathbb{R}$ and degree ≥ 3. Then, $f(X)$ is anti-Hurwitz polynomial, if and only if are satisfies the follows conditions:

1. $(-1)^i a_i > 0$ for all $i \in \{0, \ldots, n\}$.
2. $(-1)^{[i]} \Delta_i > 0$, for all $i \in \{1, \ldots, n\}$.

Long-time Behavior Viscous Case

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In this section we denote by $\ =\{\ _1;\ _2;\ _3;\ _4\}$ the set of roots of the polynomial $P(\ )$ in eq. (8).

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Remark

Since the coefficients of the polynomial $p(\lambda)$ in eq. (8) admit different signs, we conclude that

- $\Lambda \cap L_2 \neq \emptyset$, and
- $\Lambda \cap \mathbb{C} \setminus L_2 \neq \emptyset$.

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Idea: Combine results of Popov-Sedletskii and solution of eq. (2)

Long-time Behavior Viscous Case

Conjecture

Let $\;_i = (H_0 r_i + H_0 r_i^{\perp})$ for $i = 1; 2; 3; 4;$ and let $\;_1 = L_2 \cap \;$. For all $p \in \mathbb{N}$, the following asymptotic formula hold:

$$H(t) = \sum_{\in\;_1} -\frac{}{2} - \sqrt{\;t}\, e^{(-\sqrt{\;t})^2\;t} - \sum_{i=1}^{4} -e^{\;t}\left[\sum_{k=1}^{p} \frac{\left(-\sqrt{\;t}\right)^k}{\left(\frac{1}{2}\;\frac{k}{2}\right)}\right] + R_p; \quad (9)$$

where the remainder R_p admits the estimate

$$|R_p| \le e^{\;t}\,\frac{2^{\frac{p+8}{4}}\left(1 + \;p/2\right)}{|\sqrt{\;t}|^{p+1}} \sum_{\in\;2} \frac{1}{|\;-\;|^{p+1}}: \quad (10)$$

Long-time Behavior Viscous Case

Conjecture

Let $_i = (H_0 r_i + H_0 r_i)$ for $i = 1; 2; 3; 4$; and let $_1 = L_2 \cap$. For all $p \in N$, the following asymptotic formula hold:

$$H(t) = \sum_{i \in 1} \frac{i}{2} \frac{1}{i \sqrt{t}} e^{(i \sqrt{t})^2 t} - \sum_{i=1}^{4} i \, e^{t} \left[\sum_{k=1}^{p} \frac{\left(i \sqrt{t} \right)^{k}}{\left(\frac{1}{2}\right)_{\frac{k}{2}}} \right] + R_p; \qquad (11)$$

where the remainder R_p admits the estimate

$$|R_p| \le e^{t} \frac{2^{\frac{p+8}{4}} (1 + p/2)}{|\sqrt{t}|^{p+1}} \sum_{i \in 2} \frac{1}{|i|^{p+1}}: \qquad (12)$$

Remark

If $_i$ 2 is such that

$$\mathrm{Re}(\ _i)^2 \quad \mathrm{Im}(\ _i)^2 \quad \frac{1}{\ };$$

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Perspectives

1. Solve the question

$$L_2 \setminus \qquad 2\,\mathbb{C} : \ Re(\ )^2 \quad Im(\ )^2 \quad \frac{1}{\ } \quad ?$$

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the conjecture follow.

Perspectives

1. Solve the question

$$L_2 \setminus \qquad 2\,\mathbb{C} : \ Re(\ )^2 \quad Im(\ )^2 \quad \frac{1}{\ } \quad ?$$

2. If 1 does not follow, look for a set

$$(\ ;\ !)\ 2\ \mathbb{R}^+ \quad \mathbb{R}^+ :\ \text{the conjecture follow} \quad :$$

Thanks for your attention !