Long-time Behaviour of a Model of Rigid Structure Floating in a Viscous Fluid

G. Vergara-Hermosilla, M. Tucsnak, F. Sueur

Université de Bordeaux

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Outline

1. **A Model of rigid structure in a viscous fluid**
2. **Mittag-Leffler functions**
3. **Solutions unbounded case**
4. **Hurwitz and Anti-Hurwitz polynomials**
5. **Long-time behavior in viscous case**

The equilibrium problem

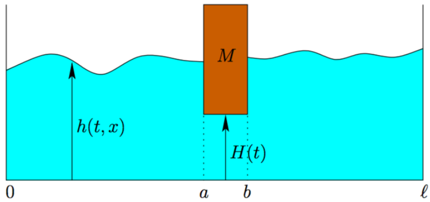

Figure 1: Configuration, Image from Maity, Tucsnak et al.

Notations

- $\mu \geq 0$ is the coefficient of viscosity of the fluid.
- $\mathcal{E} = (0, a) \cup (b, a + b)$.
- $\mathcal{I} = [a, b]$.

Equations Return to the Equilibrium

Proposition [Maity, Tucsnak et al.,2018]

If $l = a + b$, then, the return to the equilibrium problem, the position of the solid is completely determined by the integro-differential equation

$$\begin{cases} \left(1 + \frac{(b-a)^3}{12}\right)\ddot{H} = -H_0\frac{|\mathcal{I}|^2}{|\mathcal{E}|^2} - \frac{(b-a)^2}{2}F * \dot{H} - \mu(b-a)\dot{H} - (b-a)H, \\ H(0) = H_0, \quad \dot{H}(0) = 0, \end{cases} \tag{1}$$

and with F such that $\hat{F}(s) = \sqrt{1+s\mu}\tanh\left(\frac{sa}{\sqrt{1+s\mu}}\right)$.

Unbounded case

Proposition [Maity, Tucsnak et al.,2018]

If $\mathcal{E} = (-\infty, a) \cup (b, \infty)$, then the position of the solid is completely determined by the integro-differential equation

$$\begin{cases} A\ddot{H} + BF * \dot{H} + C\dot{H} + DH = 0, \\ H(0) = H_0, \quad \dot{H}(0) = \dot{H}_0, \end{cases} \tag{2}$$

where $A = 1 + \frac{(b-a)^3}{12}$, $B = (b-a)^2$, $C = (b-a)\mu$, $D = (b-a)$, and

$$F(t) = \sqrt{\mu}\left(\frac{1 - e^{-\frac{t}{\mu}}}{2\sqrt{\pi t^3}}\right) + D^{1/2}\delta_0.$$

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Goal: Understand the long-time behavior of solutions $H(t)$ for the unbounded case

Mittag-Leffler functions

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Definition

The two-parametric Mittag-Leffler function, is the complex-valued function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{with } \alpha > 0, \beta \in \mathbb{C}. \tag{3}$$

In the case when $\beta = 1$ the function is known has the classical Mittag-Leffler function and denoted by $E_\alpha(z)$.

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Examples

- $E_{1,1}(z) = e^z$,
- $E_{2,1}(z) = \cosh(\sqrt{z})$,
- $E_{\frac{1}{2},1}(z) = e^{z^2}\mathrm{erfc}(-z)$.

Mittag-Leffler functions

Theorem [Popov-Sedletskii, 2013]

For any $0 < \alpha < 1$, $\beta \in \mathbb{C}$, $p \in \mathbb{N}$, and
$z \in \mathcal{L}_{1/\alpha} = \{z \in \mathbb{C} : |\arg z| \leq \pi\alpha\}$, the following asymptotics hold

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + R_m^{[1]}(z;\alpha,\beta), \qquad (4)$$

where the remainder $R_m^{[1]}$ admits the estimate

$$\left|R_m^{[1]}(z;\alpha,\beta)\right| \leq \begin{cases} \dfrac{2^{\frac{b+2}{2}}\Gamma(b+1)e^{\left(\frac{5\pi}{4}|\Im\beta|\right)}}{\alpha\pi|z|^{p+1}}, \text{if } b = \alpha(p+1) - \Re\beta \geq 0 \\ \dfrac{(6 + \frac{2}{\alpha\pi})e^{\left(\frac{5\pi}{4}|\Im\beta|\right)}}{|z|^{p+1}}, \text{ if } b < 0. \end{cases} \qquad (5)$$

The first of estimates in (12) is valid for all $z \in \mathcal{L}_{1/\alpha}$, and the second under the additional condition $|z| \geq 2$.

Mittag-Leffler functions

Theorem [Popov-Sedletskii, 2013]

For any $0 < \alpha < 1$, $\beta \in \mathbb{C}$, $p \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathcal{L}_{1/\alpha}$, $z \neq 0$, the following asymptotics formulas hold

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + R_m^{[2]}(z;\alpha,\beta), \tag{6}$$

where the remainder $R_m^{[2]}$ admits the estimate

$$\left| R_m^{[2]}(z;\alpha,\beta) \right| \leq \begin{cases} \dfrac{2^{\frac{b+2}{2}}\Gamma(b+1)e^{\left(\frac{3\pi}{4}|\Im\beta|\right)}}{\alpha\pi|z|^{p+1}}, & \text{if } b \geq 0 \\ \dfrac{(6+\frac{2}{\alpha\pi})e^{\left(\frac{3\pi}{4}|\Im\beta|\right)}}{|z|^{p+1}}, & \text{if } b < 0. \end{cases} \tag{7}$$

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Laplace transform of Mittag Leffler functions

In this work we are interested in the case when $\alpha = \beta > 0$ and $z = \lambda t^{\alpha}$, with $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$. For simplicity we introduce the notation

$$\mathcal{E}_{\alpha}(\lambda, t) := E_{\alpha,\alpha}(\lambda t^{\alpha}).$$

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Lemma

The Laplace transform of $\mathcal{E}_\alpha(\lambda, t)$ is given by

$$\mathcal{L}[t^{\alpha-1}\mathcal{E}_\alpha(\lambda, t)](s) = (s^\alpha - \lambda)^{-1},$$

where $Re(s) > 0$ and $|\lambda s^{-\alpha}| < 1$.

Solutions unbounded case

Inviscid case

If we consider $\mu = 0$ in equation (2), the model is reduced to

$$\begin{cases} A\ddot{H} + B\dot{H} + DH = 0, \\ H(0) = H_0, \quad \dot{H}(0) = \dot{H}_0. \end{cases}$$

Applying Laplace transform to the equation above, and after simplifications, we obtain

$$\hat{H}(s)\left[As^2 + Bs + D\right] = H_0\left[As + B\right] + A\dot{H}_0.$$

Solutions of the model, inviscid case

Proposition [V-H, Sueur, Tucsnak, 2019]

The solution $H(t)$ of equation (2) with $\mu = 0$ is given by

$$e^{-\frac{B\,t}{2\,A}} \left\{ H_0 \left(\cosh\left(\frac{t\sqrt{\frac{B^2}{4} - A\,D}}{A} \right) + \frac{B \sinh\left(\frac{t\sqrt{\frac{B^2}{4} - A\,D}}{A} \right)}{2\sqrt{\frac{B^2}{4} - A\,D}} \right) \right.$$

$$\left. + \dot{H}_0 \frac{2\,A\,\sin\left(\frac{t\sqrt{4\,A\,D - B^2}}{2\,A} \right)}{\sqrt{4\,A\,D - B^2}} \right\}.$$

Viscous case

Proposition [V-H, Sueur, Tucsnak, 2019]

The solution of the equation (2) with $\mu > 0$ is given by

$$H(t) = e^{-\frac{t}{\mu}} t^{-\frac{1}{2}} \sum_{i=1}^{4} \left[H_0 r_i + \dot{H}_0 \dot{r}_i \right] \mathcal{E}_{\frac{1}{2}}(\lambda_i, t).$$

where each λ_i is a root of the polynomial

$$p(\lambda) = A\lambda^4 + \overline{B}\lambda^3 + \left(C - \frac{2A}{\mu} \right) \lambda^2 - \frac{\overline{B}}{\mu}\lambda + \frac{A}{\mu^2}, \tag{8}$$

and the parameters r_i and $\dot{r}_i$ depends of the roots.

Hurwitz and Anti-Hurwitz polynomials

Hurwitz polynomials

A polynomial $f(X)$ is said to be *Hurwitz* if the real part of all its complex roots is negative, that is $Re(u) < 0$ for all $u \in$ such that $f(u) = 0$.

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Theorem (Stodola condition)

If a polynomial with real coeficients is Hurwitz, then all its coeficients are of the same sign.

Anti-Hurwitz polynomials

A polynomial $f(X)$ is said to be *Anti-Hurwitz* if the real part of all its complex roots is negative, that is $Re(u) > 0$ for all $u \in$ such that $f(u) = 0$.

Theorem [V-H, Luca, 2019]

Let $f(X) = a_0 X^n + a_1 X^{n-1} + \cdots a_{n-1} X + a_n$ be a polynomial with coefficients in $\mathbb{R}$ and degree ≥ 3. Then, $f(X)$ is anti-Hurwitz polynomial, if and only if are satisfies the follows conditions:

1. $(-1)^i a_i > 0$ for all $i \in \{0, \ldots, n\}$.
2. $(-1)^{[i]} \Delta_i > 0$, for all $i \in \{1, \ldots, n\}$.

Long-time Behavior Viscous Case

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In this section we denote by $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ the set of roots of the polynomial $P(\lambda)$ in eq. (8).

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Remark

Since the coefficients of the polynomial $p(\lambda)$ in eq. (8) admit different signs, we conclude that

- $\Lambda \cap \mathcal{L}_2 \neq \emptyset$, and
- $\Lambda \cap \mathbb{C} \setminus \mathcal{L}_2 \neq \emptyset$.

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Idea: Combine results of Popov-Sedletskii and solution of eq. (2)

Long-time Behavior Viscous Case

Conjecture

Let $\Theta_i = (H_0 r_i + \dot{H}_0 \dot{r}_i)$ for $i = 1, 2, 3, 4$, and let $\Lambda_1 = \mathcal{L}_2 \cap \Lambda$. For all $p \in \mathbb{N}$, the following asymptotic formula hold:

$$H(t) = \sum_{\lambda_i \in \Lambda_1} \frac{\Theta_i}{2} \lambda_i \sqrt{t} e^{\left(\lambda_i \sqrt{t}\right)^2 - \frac{t}{\mu}} - \sum_{i=1}^{4} \Theta_i e^{-\frac{t}{\mu}} \left[\sum_{k=1}^{p} \frac{\left(\lambda_i \sqrt{t}\right)^{-k}}{\Gamma\left(\frac{1}{2} - \frac{k}{2}\right)} \right] + \mathcal{R}_p, \tag{9}$$

where the remainder $\mathcal{R}_p$ admits the estimate

$$|\mathcal{R}_p| \leq e^{-\frac{t}{\mu}} \cdot \frac{2^{\frac{p+8}{4}} \Gamma(1 + p/2)}{\pi |\sqrt{t}|^{p+1}} \cdot \sum_{\lambda_i \in \Lambda} \frac{1}{|\lambda_i|^{p+1}}. \tag{10}$$

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If $\lambda_i \in \Lambda$ is such that

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Perspectives

1. Solve the question

$$\mathcal{L}_2 \cap \Lambda \subset \left\{ \lambda \in \mathbb{C} : \; Re(\lambda)^2 - Im(\lambda)^2 \leq \frac{1}{\mu} \right\} ?$$

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Perspectives

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2. If 1 does not follow, look for a set

$$\left\{ (\mu, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ : \; \text{the conjecture follow} \right\}.$$

Thanks for your attention !