On the Path Integral Representation of Supersymmetric Dirac Resolvents

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7th International Workshop on New Challenges in Quantum Mechanics: Integrability and Supersymmetry

Outline

Supersymmetric quantum mechanics

Hilbert space: $\mathcal{H}$
N self-adjoint supercharges: $Q_i = Q_i^\dagger$, $i = 1, 2, \ldots, N$
Hamiltonian: H
Superalgebra: $\{Q_i, Q_j\} = \delta_{ij} H$

$N = 1$: In general no grading operator, $H = 2Q_1^2$

$N = 2$: Grading operator always exists

$$W := \frac{2}{H} QQ^\dagger - 1 = \frac{[Q, Q\dagger]}{\{Q, Q^\dagger\}} = W^\dagger,\ W^2 = 1$$

Complex supercharges: $Q := \frac{1}{\sqrt{2}}(Q_1 + \mathrm{i}Q_2)$, $Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - \mathrm{i}Q_2)$
Superalgebra: $[H, W] = 0, \{W, Q\} = 0, H = \{Q, Q^\dagger\},$

$$Q^2 = 0 = (Q^\dagger)^2, [H, Q] = 0 = [H, Q^\dagger].$$

Grading: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $W = \pm 1$ on $\mathcal{H}_\pm$

N=2 SUSY in matrix representation

Operators:

$$W = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$

States: $\psi^+ = \begin{pmatrix} \phi^+ \\ 0 \end{pmatrix}, \psi^- = \begin{pmatrix} 0 \\ \phi^- \end{pmatrix}, \phi^\pm \in \mathcal{H}_\pm$

SUSY transformations: $H_\pm \phi_E^\pm = E\phi_E^\pm$

$$A\phi_E^- = \sqrt{E}\phi_E^+, \quad A^\dagger\phi_E^+ = \sqrt{E}\phi_E^- \quad \text{for} \quad E > 0$$

Ess. iso-spectral: $\mathrm{spec}\, H_+ \backslash \{0\} = \mathrm{spec}\, H_- \backslash \{0\}$

Unbroken SUSY: $E = 0$ eigenvalue of H_- and / or H_+

Pauli and Dirac Hamiltonian with magn. field

External magnetic field: $\vec{B} = \vec{\nabla} \times \vec{A}$

Pauli: $H_{\rm P} = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{g}{2}\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}$ on $\mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$

$N = 1$ SUSY: $H_{\rm P} = 2Q_1^2$ with $Q_1 := \frac{1}{\sqrt{4m}}\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)$ iff $g = 2$

Dirac: $H_{\rm D} = c\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2$ on $\mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$

$N = 2$ SUSY: $H_{\rm D} = Q_1 + W\mathcal{M} = \begin{pmatrix} mc^2 & A \\ A & -mc^2 \end{pmatrix}$, where

$W = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathcal{M} = \begin{pmatrix} mc^2 & 0 \\ 0 & mc^2 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$,

and $A := c\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)$, $Q_2 = -\mathrm{i}WQ_1$.

But $H_{\rm D}$ is a supercharge, not a SUSY Hamiltonian.

General SUSY Dirac Hamiltonian

Definition: $$H_{\rm D} := Q_1 + W\mathcal{M} = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix}$$

is called *generalized supersymmetric Dirac Hamiltonian* if

$$M_\pm = M_\pm^\dagger \geq 0, \qquad AM_- = M_+A, \qquad A^\dagger M_+ = M_- A^\dagger$$

Matrix representation:

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathcal{M} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, Q_1 = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix},$$

with $\{Q_1, W\} = 0$, $[\mathcal{M}, W] = 0$, $W^2 = 1$, $[Q_1, \mathcal{M}] = 0$.

Note: $H_{\rm D}^2 = Q_1^2 + \mathcal{M}^2 = \begin{pmatrix} AA^\dagger + M_+^2 & 0 \\ 0 & A^\dagger A + M_-^2 \end{pmatrix}$

has the form of a SUSY Hamiltonian

$N = 2$ SUSY structure

Let: $m > 0$ be an arbitrary mass-like parameter and

$$H := \frac{1}{2mc^2}\left(H_{\rm D}^2 - \mathcal{M}^2\right) = \frac{1}{2mc^2}\begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{2mc^2}}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \frac{1}{\sqrt{2mc^2}}\begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$

SUSY: $\boxed{H = \{Q, Q^\dagger\},\ \{Q, W\} = 0 = \{Q^\dagger, W\},\ Q^2 = 0 = \left(Q^\dagger\right)^2}$

As A is linear in $\vec{p}$, $H_\pm$ is quadratic in $\vec{p} \Longrightarrow$ non-rel. Hamiltonian

Example: $A := c\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)$, $M_\pm = mc^2 \Longrightarrow$

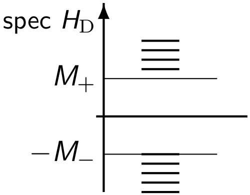
$$H_{\rm D} = c\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2,\ H_\pm = \frac{1}{2m}\left(\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\right)^2 = H_{\rm P}$$

Spectral properties

Diagonalize with $U = a_+ + a_- W \,\mathrm{sgn}\, Q_1$ and $a_\pm = \sqrt{\frac{1}{2} \pm \frac{\mathcal{M}}{2|H_\mathrm{D}|}}$.

$$\tilde{H}_\mathrm{D} := U H_\mathrm{D} U^\dagger = \begin{pmatrix} \sqrt{2mc^2 H_+ + M_+^2} & 0 \\ 0 & -\sqrt{2mc^2 H_- + M_-^2} \end{pmatrix}$$

Let: $H_\pm \phi_n^\pm = \varepsilon_n \phi_n^\pm \Longrightarrow \tilde{H}_\mathrm{D} \tilde{\psi}_n = E_n^\pm \tilde{\psi}_n$ where



$$E_n^\pm = \pm\sqrt{2mc^2 \varepsilon_n + M_\pm^2}, \qquad \psi_n^\pm = U^\dagger \tilde{\psi}_n^\pm,$$

$$\tilde{\psi}_n^+ = \begin{pmatrix} \phi_n^+ \\ 0 \end{pmatrix}, \qquad \tilde{\psi}_n^- = \begin{pmatrix} 0 \\ \phi_n^- \end{pmatrix}$$

$$Q\tilde{\psi}_n^- = \sqrt{\varepsilon_n}\tilde{\psi}_n^+, \qquad Q^\dagger \tilde{\psi}_n^+ = \sqrt{\varepsilon_n}\tilde{\psi}_n^-$$

$$E_0^+ = M_+ \text{ iff } \varepsilon_0 = 0 \in \mathrm{spec}\, H_+$$

$$E_0^- = -M_- \text{ iff } \varepsilon_0 = 0 \in \mathrm{spec}\, H_-$$

Spectral problem of rel. $H_\mathrm{D} \Longrightarrow$ Spectral problem of non-rel. $H_\pm$

Example: The Dirac oscillator

Dirac oscillator: $H_{\rm D} := \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p}-{\rm i}m\omega\vec{r}) \\ c\vec{\sigma}\cdot(\vec{p}+{\rm i}m\omega\vec{r}) & -mc^2 \end{pmatrix}$

SUSY with $A := c\vec{\sigma}\cdot(\vec{p}-{\rm i}m\omega\vec{r})$, $M_\pm := mc^2$

SUSY partners: $H_\pm = \frac{\vec{p}^2}{2m} + \frac{m}{2}\omega^2\vec{r}^2 \pm \hbar\omega\left(K+\frac{1}{2}\right)$

Spin-orbit operator: $K := 1 + \vec{\sigma}\cdot\vec{L}/\hbar, \qquad {\rm spec}\, K = \pm(j+1/2)$

Spectrum of SUSY partners:

$\varepsilon^+_{n,j,s} = \hbar\omega(2n+2+j+sj), \quad n = 0,1,2,3,\ldots,\ j = \frac{1}{2},\frac{3}{2},\ldots,\ s=\pm 1$

$\varepsilon^-_{n,j,s} = \hbar\omega(2n+j+1+s(j+1))$,

$\varepsilon^-_{n,j,s} = \varepsilon^+_{n-1,j+1,-s}$, ess. iso-spectral

$\varepsilon^-_{0,j,1} = 0$ unbroken SUSY

Resolvent and iterated resolvent

Resolvent: $G(z) := \dfrac{1}{H_{\rm D} - z}, \quad z \in \mathbb{C}\backslash \mathrm{spec}\, H_{\rm D}$

Iterated res.: $g(\zeta) := \dfrac{1}{H_{\rm D}^2 - \zeta}, \quad \zeta \in \mathbb{C}\backslash \mathrm{spec}\, H_{\rm D}^2$

Note: $g(\zeta) = \begin{pmatrix} g^+(\zeta) & 0 \\ 0 & g^-(\zeta) \end{pmatrix}, \; g^\pm(\zeta) = \dfrac{1}{2mc^2 H_\pm + M_\pm^2 - \zeta}$

$$G(z) = (H_{\rm D} + z) g(z^2) = \begin{pmatrix} (z + M_+) g^+(z^2) & A g^-(z^2) \\ A^\dagger g^+(z^2) & (z - M_-) g^-(z^2) \end{pmatrix}$$

$$g^\pm(\zeta) = \frac{\mathrm{i}}{2mc^2\hbar} \int_0^\infty \mathrm{d}t \exp\left\{ -\mathrm{i}t H_\pm^{\rm eff}(\zeta)/\hbar \right\},$$

with effective non-relativistic $H_\pm^{\rm eff}(\zeta) := H_\pm + \left(M_\pm^2 - \zeta \right) / 2mc^2$

Path integral representation

Let: $g^{\pm}(\vec{r}'',\vec{r}';\zeta) := \langle\vec{r}''|g^{\pm}(\zeta)|\vec{r}'\rangle = \frac{\mathrm{i}}{2mc^2\hbar}\int_0^\infty \mathrm{d}t\, P^{\pm}_{\zeta}(\vec{r}'',\vec{r}';t)$

Promotor: $P^{\pm}_{\zeta}(\vec{r}'',\vec{r}';t) := \langle\vec{r}''|\exp\left\{-\mathrm{i}tH^{\mathrm{eff}}_{\pm}(\zeta)/\hbar\right\}|\vec{r}'\rangle$

Path integral representation:

$$P^{\pm}_{\zeta}(\vec{r}'',\vec{r}';t) = \int_{\vec{r}(0)=\vec{r}'}^{\vec{r}(t)=\vec{r}''} \mathcal{D}\vec{r}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}s\, L^{\pm}_{\mathrm{eff}}(\dot{\vec{r}},\vec{r})\right\}$$

Here $L^{\pm}_{\mathrm{eff}}$ represents effective Lagrangian associated with $H^{\mathrm{eff}}_{\pm}(\zeta)$
If non-rel. $L^{\pm}_{\mathrm{eff}}$ is path integrable $\Longrightarrow g^{\pm}(\vec{r}'',\vec{r}';\zeta) \Longrightarrow G(\vec{r}'',\vec{r}';z)$
Also applicable to non-SUSY H_{D} iff H^2_{D} is of block-diagonal form

Example: The free particle

SUSY Dirac: $A = c\vec{\sigma}\cdot\vec{p},\ M_\pm = mc^2,\ H_\pm = \vec{p}^2/2m$

$$\Longrightarrow H_\pm^{\text{eff}}(\zeta) = \frac{\vec{p}^2}{2m} + \frac{m^2c^4-\zeta}{2mc^2} =: \frac{\vec{p}^2}{2m} - \frac{\mu(\zeta)^2}{2m}$$

$$\Longrightarrow L_{\text{eff}}^\pm(\dot{\vec{r}},\vec{r},t) = \frac{m}{2}\dot{\vec{r}}^2 + \frac{\mu(\zeta)^2}{2m}$$

$$P_\zeta^\pm(\vec{r}'',\vec{r}',t) = \left(\frac{m}{2\pi\mathrm{i}\hbar t}\right)^{3/2} \exp\left\{\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2t}(\vec{r}''-\vec{r}')^2 + \frac{t}{2m}\mu^2(\zeta)\right)\right\}$$

$$g^\pm(\vec{r}'',\vec{r}',\zeta) = \frac{1}{4\pi|\vec{x}|\hbar^2c^2}\exp\left\{\mathrm{i}\mu(\zeta)|\vec{x}|/\hbar\right\}, \quad \vec{x} := \vec{r}''-\vec{r}'$$

$$G(\vec{r}'',\vec{r}',z) = \frac{\mathrm{e}^{\mathrm{i}\mu(z^2)|\vec{x}|/\hbar}}{4\pi|\vec{x}|\hbar^2c^2}\left(\mathrm{i}\hbar c\frac{\vec{\alpha}\cdot\vec{x}}{|\vec{x}|^2} + c\mu(z^2)\frac{\vec{\alpha}\cdot\vec{x}}{|\vec{x}|} + \beta mc^2 + z\right)$$

Example: Generalised Dirac Oscillator

SUSY Dirac: $A = c\vec{\sigma}\cdot(\vec{p} - \mathrm{i}\hbar\vec{e}_r U'(r))$, $M_\pm = mc^2$, $U = U(r)$

$$\Longrightarrow H_\pm = \frac{\vec{p}^2}{2m} + \frac{\hbar^2}{2m}U'^2(r) \pm \frac{\hbar^2}{2m}U''(r) \pm \frac{\hbar^2 U'(r)}{mr}K$$

Partial wave exp.:

$$P^\pm_\zeta(\vec{r}'', \vec{r}', t) = \sum_{j,s} P^\pm_{\zeta\ell}(r'', r', t) \sum_{m_j=-j}^{j} \varphi^{(s)}_{jm_j}(\theta'', \phi'')\bar{\varphi}^{(s)}_{jm_j}(\theta', \phi')$$

Pauli spinors:

$$\varphi^{(s)}_{jm_j}(\theta, \phi) = \langle\theta, \phi|j, m_j, s\rangle = \begin{pmatrix} \sqrt{\frac{\ell+sm_j+1/2}{2\ell+1}}\, Y_\ell^{m_j-1/2}(\theta, \phi) \\ s\sqrt{\frac{\ell-sm_j+1/2}{2\ell+1}}\, Y_\ell^{m_j+1/2}(\theta, \phi) \end{pmatrix}$$

$\vec{J}^2|j, m_j, s\rangle = j(j+1)|j, m_j, s\rangle$, $\quad j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

$J_z|j, m_j, s\rangle = m_j|j, m_j, s\rangle$, $\quad m_j = -j, \ldots, j$

$K|j, m_j, s\rangle = \kappa|j, m_j, s\rangle$, $\quad \kappa = s(j+1/2)$, $\quad j = \ell + s/2$, $\quad s = \pm 1$

Example: Generalised Dirac Oscillator

Radial promotor:

$$P^{\pm}_{\zeta\ell}(r'', r', t) = \int_{r(0)=r'}^{r(t)=r''} \mathcal{D}r \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}s\, L^{\pm}_{\mathrm{eff}}(\dot r, r, s) \right\}$$

Eff. Lagrangian: $L^{\pm}_{\mathrm{eff}} = \frac{m}{2}\dot r^2 - \frac{\hbar^2\ell(\ell+1)}{2mr^2} - V_{\pm}(r) + \frac{\mu(\zeta)^2}{2m}$

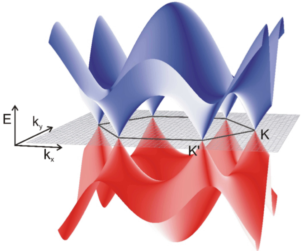
Potential: $V_{\pm}(r) = \frac{\hbar^2}{2m}U'^2(r) \pm \frac{\hbar^2}{2m}U''(r) \pm \frac{\hbar^2 U'(r)}{mr}\kappa,$

Explicit results:

$$\begin{array}{ll} U(r) = \frac{m\omega}{2\hbar} r^2 & V_{\pm}(r) = \frac{m}{2}\omega^2 r^2 \pm \hbar\omega(\kappa + 1/2) \\ U(r) = \frac{r}{a} & V_{\pm}(r) = \frac{\hbar^2}{2ma^2}(1 \pm 2\kappa\frac{a}{r}) \\ U(r) = \gamma \ln(r/a) & V(r) = \frac{\hbar^2}{2mr^2}\left[\gamma^2 \pm \gamma(\kappa - 1)\right] \end{array}$$

Graphene

Dirac cones characterize band structure of spinless electron/hole near $K(+)$ and $K'(-)$ edge of Brillouin zone.



$$H_{\rm D}^{(\pm)} = \begin{pmatrix} m_{\rm eff}\, v_{\rm F}^2 & A_{(\pm)} \\ A_{(\pm)}^\dagger & -m_{\rm eff}\, v_{\rm F}^2 \end{pmatrix}$$

$m_{\rm eff} \geq 0$ introduces band gap at K and K'

$A_{(\pm)} := v_{\rm F}\,[p_1 \mp {\rm i} p_2]$

Fermi velocity: $v_{\rm F} \sim 10^6\, m/s$

Orth. magn. field: $\vec{B} = B(x_1, x_2)\vec{e}_3$, $\quad B(x_1, x_2) = \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}$

SUSY structure:

$M_\pm = m_{\rm eff}\, v_{\rm F}^2$, $\quad A_{(\pm)} := v_{\rm F}\left[(p_1 - \frac{e}{c}a_1) \mp {\rm i}(p_2 - \frac{e}{c}a_2)\right]$

2D Pauli system

Non-rel. partner Hamiltonians: m arbitray mass parameter

$$H_+^{(\pm)} = \frac{1}{2mv_{\mathrm{F}}^2} A_{(\pm)} A_{(\pm)}^\dagger , \qquad H_-^{(\pm)} = \frac{1}{2mv_{\mathrm{F}}^2} A_{(\pm)}^\dagger A_{(\pm)}$$

2D Pauli Hamiltonian:

$$H_\pm^{(\pm)} = \frac{1}{2m} \left(\vec{p} - \frac{e}{c} \vec{a} \right)^2 \mp \frac{g^{(\pm)}}{2} \frac{e\hbar}{2mc} B(x_1, x_2)$$

where $g^{(\pm)} = \pm 2$

Two-dimensional Pauli Hamiltonian exhibits a SUSY structure for $g = 2$ as well as for $g = -2$.

1D Dirac systems

$$H_{\mathrm{D}} = \begin{pmatrix} mc^2 & A \\ A^\dagger & -mc^2 \end{pmatrix} \text{ on } L^2(\mathbb{R}) \otimes \mathbb{C}^2,\ A = cp - \mathrm{i}W(x)$$

SUSY pot.: $W : \mathbb{R} \mapsto \mathbb{R}$ cont. diff.

Witten model: $W(x) = \sqrt{2mc^2}\Phi(x)$

$$H_\pm = \frac{p^2}{2m} + \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}}\Phi'(x)$$

For all exactly solvable $H_\pm \Longrightarrow$ exact solutions for H_D

Eff. Lagrangian:

$$L^\pm_{\mathrm{eff}} = \frac{m}{2}\dot{x}^2 - \frac{1}{2mc^2}W^2(x) \mp \frac{\hbar}{2mc}W'(x) - \frac{\mu^2(\zeta)}{2m}$$

Similar, all path integrable $L^\pm_{\mathrm{eff}} \Longrightarrow$ resolvent for H_D

The free particle

SUSY potential: $W(x) = 0\,, \qquad \mu^2(\zeta) = \zeta/c^2 - m^2c^2$

Promotor:

$$P^{\pm}_{\zeta}(x'',x';t) = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\exp\left\{\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2t}|x''-x'|^2 + \frac{t}{2m}\mu^2(\zeta)\right)\right\}$$

Iterated res.: $g^{\pm}(x'',x';\zeta) = \dfrac{\mathrm{i}}{2\mu(\zeta)c^2\hbar}\exp\left\{\mathrm{i}\mu(\zeta)|x''-x'|/\hbar\right\}$

Resolvent:

$$G(x'',x';z) = \frac{\mathrm{i}}{2\mu(z^2)c^2\hbar}\exp\left\{\mathrm{i}\mu(z^2)|x''-x'|/\hbar\right\} \times\left[\mathrm{i}c\mu(z^2)\mathrm{sgn}(x''-x')\sigma_2 + mc^2\sigma_3 + z\right]$$

The harmonic oscillator

SUSY potential: $W(x) = mc\omega x\,, \qquad \mu^2(\zeta) = \zeta/c^2 - m^2c^2$

Eff. Lagrangian: $L_{\rm eff} = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 \mp \frac{\hbar\omega}{2} + \frac{\mu^2(\zeta)}{2m}$

$$P^{\pm}_{\zeta}(x'', x'; t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}}\, e^{\mp i\omega t} e^{it\mu^2(\zeta)/2m\hbar} \times \exp\left\{\frac{im\omega}{2\hbar}\left[(x''^2 + x'^2)\cot(\omega t) - \frac{2x''x'}{\sin(\omega t)}\right]\right\}$$

Iterated res.: By integration or alternatively

$$g^{\pm}(\zeta) = \frac{1}{2mc^2}\frac{1}{H_0 - \left(\frac{\mu^2(\zeta)}{2m} \mp \frac{\hbar\omega}{2}\right)} \qquad \text{with} \qquad H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2$$

$$g^{\pm}(x'', x'; \zeta) = \frac{1}{2mc^2}\, g_0\left(x'', x'; \frac{\mu^2(\zeta)}{2m\hbar\omega} \mp \frac{1}{2}\right) \qquad \text{harm. osc. resolvent}$$

Quasi-classical approximation

Non-relativistic SUSY-WKB formula: (unbroken SUSY)

$$\int_{x_L}^{x_R} \mathrm{d}x \sqrt{2m\left(\varepsilon - \Phi^2(x)\right)} = n\pi\hbar$$

results in a

Relativistic SUSY-WKB formula:

$$\int_{x_L}^{x_R} \mathrm{d}x \sqrt{E^2 - m^2c^4 - W^2(x)} = n\pi\hbar c$$

exact for all shape-invariant SUSY potentials W.
Similar for broken SUSY with $n \to n + 1/2$

Summary and outlook

- SUSY Dirac Hamiltonians appear in many physical systems
- Closely related to non-rel. Schrödinger-type Hamiltonians
- Relativistic spectral properties follow for non-rel. ones
- May result in closed expression for the Dirac resolvent
- Allow for a path-integral formalism a la Feynman
- Non-rel. methods (e.g. quasi-classical approx.) applicable to SUSY Dirac Hamiltonians

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THANKS

Congratulations Mikhail

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