Superintegrability, special functions and representations

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I will review results on classification of quantum superintegrable systems on twodimensional Euclidean space with higher order integrals. I will discuss the connection with exceptional orthogonal polynomials, Painlevé transcendents and the Chazy class of equations. I will discuss how their symmetry algebras are associated with polynomial algebras and how these algebraic structures and their Casimir operators can be used to obtain the energy spectrum algebraically.

Integrable and superintegrable, part 1

Definition

A Hamiltonian system (in n dimensions) with Hamiltonian H

$$H = \frac{1}{2} g^{ik} p_i p_k + V(\vec{x})$$

is integrable if it allows n integrals of motion that are well defined, in involution $\{H, X_a\}_p = 0$, $\{X_a, X_b\}_p = 0$, a,b=1,...,n-1 and functionally independent.

A system is superintegrable if it admits $n + k$ (with $k = 1, ..., n - 1$) functionally independent constants of the motion (well defined). Maximally superintegrable if $k = n - 1$.

QM : $\{H, X_a, Y_b\}$ are well defined quantum mechanical operators and form an algebraically independent set.

Integrable and superintegrable, part 2

- A systematic search for superintegrable systems was started some time ago.
- The best known examples are the Kepler-Coulomb system $V(r) = \frac{\alpha}{r}$ and the harmonic oscillator $V(r) = \alpha r^2$
- representations, obtained algebraic derivation, Casimir related to subalgebras
- Pauli (1926), Fock (1935), Bargmann (1936), Sudarshun, Mukunda, Raifeartaigh (1965), Barut (1965), Louck (1972), Rasmussen, Salano (1979)
- Jauch and Hill (1940), Baker (1956), Moshinsky (1962), Barut (1965), Fradkin (1965), Louck (1965), Budini (1967),Hwa (1966)

$$H = \frac{1}{2}p^2 - \frac{c_0}{r}$$

- With the following integrals

$$M_j = \frac{1}{2}\sum_{i=1}^{N}(L_{ji}p_i - p_i L_{ij}) - \frac{c_0 x_j}{r}, \quad L_{ij} = x_i p_j - x_j p_i$$

- $i, j = 1, 2, ..., N$. Moreover, $[L_{ij}, H] = [M_j, H] = 0$
- M_j, L_{ij} generate a Lie algebra to $so(N+1)/so(N,1)/e(N)$

$$[L_{ij}, L_{kl}] = i(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il})\hbar$$
$$[M_i, M_j] = -2i\hbar H L_{ij}, \quad [M_k, L_{ij}] = i\hbar(\delta_{ik}M_j - \delta_{jk}M_i)$$

Quadratically superintegrable systems : systematic approach E_2

- Winternitz, Smorodinsky, Uhlir and I.Fris, (1966,1967)

$$H = \frac{1}{2}\vec{p}^2 + V(x,y)$$

$$A_j = \sum_{i,k=1}^{2} \{f_j^{ik}(x,y), p_i p_k\} + \sum_{i=1}^{2} g_j^i(x,y) p_i + \phi_j(x,y), j = 1,2$$

- Integrability is related to separation of variables in Cartesian, Polar, Elliptic and Parabolic,
- Superintegrability, 2 such integrals

- Properties : multiseparability, exact solvability, degenerate spectrum

Results and generalization , part 1

$$V_I = \alpha(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}, \quad V_{III} = \frac{\alpha}{r}+\frac{1}{r^2}\left(\frac{\alpha}{1+cos(\phi)}+\frac{\beta}{1-cos(\phi)}\right)$$

$$V_{II} = \alpha(x^2+4y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}, \quad V_{IV} = \frac{\alpha}{r}+\frac{1}{r}\left(\beta cos(\frac{\phi}{2})+\gamma sin(\frac{\phi}{2})\right)$$

- Since 20 years many results and various generalizations, Miller, Post and Winternitz (2013)
- magnetic field, spin, families in n-dimensional curved spaces, Dunkl, Calogero type, position mass dependent
- One of the main aspect is the relation with algebraic structures, special functions/orthogonal polynomials

- Granovskii, Zhedanov and Lutzenko (1991, 1992), Vinet and Letourneau (1995), Grunbaum, Vinet, Zhedanov (2016), Sarah Post (2007,2009)

- Daskaloyannis (1993, 2001, 2006, 2007, 2011), Quesne (2007)
- Plyushchay, parafermion, deformed Heisenberg, hidden nonlinear superalgebra, reflection, (1996,2000)

- Miller, Kalnins, Kress (2005), (structure theory), Post (2010) (models, representations)

$$[A,B]=C, \quad [A,C]=\alpha A^2+\gamma\{A,B\}+\delta A+\epsilon B+\zeta$$
$$[B,C]=aA^2-\gamma B^2-\alpha\{A,B\}+dA-\delta B+z \quad .$$

- There is a cubic Casimir operator, which can be exploited to obtain algebraically the spectrum

$$K = C^2 - \alpha\{A^2, B\} - \gamma\{A, B^2\} + (\alpha\gamma - \delta)\{A, B\} + (\gamma^2 - \epsilon)B^2$$
$$+(\gamma\delta - 2\zeta)B + \frac{2a}{3}A^3 + (d + \frac{a\gamma}{3}\alpha^2)A^2 + (\frac{a\epsilon}{3} + \alpha\delta + 2z)A \quad .$$

- It allows to get algebraic derivation of the spectrum via various approaches

Quadratic algebras and superintegrable systems, part 3

- Construction of higher rank quadratic algebra
- De Bie, Genest, Lemay, Vinet, Bannai-Ito algebra (2016), De Bie, Genest, van de Vijver, Vinet, higher rank Racah algebra (2016)
- Iliev (2016,2017) generic superintegrable system on the sphere and symmetry algebra, Post (2015,2017) recoupling $QR(9)$
- Hoque, Marquette and Zhang (2014,2015,2016,2017), algebraic derivation of spectrum, Casimir operators and finite dimensional unitary representation, Liao, Marquette and Zhang (2018) , Marquette, Zhe, Zhang (2019)
- The quantum energy levels display accidental degeneracy explained by the finite dimensional unitary representations

Quadratic algebras and superintegrable systems, part 4

- The complex spaces admitting at least three 2nd order symmetries, flat space, complex 2-sphere, the four Darboux spaces, eleven 4 parameter Koenigs spaces
- There are 59 2nd order superintegrable systems in 2D, under the Stackel transform, the systems divide into 12 equivalence classes
- 6 with nondegenerate 3-parameter potentials (S9,E1,E2,E30,E8,E10), 6 with degenerate 1-parameter potentials (S3,E3,E4,E5,E6,E14)
- Contraction of the symmetry algebra of a 2D 2nd order superintegrable system and connection with the the Askey scheme, Wigner-Inonu contractions of the Lie algebras e(2,C) and o(3,C), Miller, Kalnins, Post (2013,2014)

Summary

- Many results on the classification 2nd order superintegrable systems, many properties which make them interesting from point of view of physics and mathematics
- Conserved quantities which lead to quadratic algebras, representations and spectrum can be calculated
- Higher order superintegrable systems are also very exciting and have connection to polynomial algebras, Painlevé transcendents, exceptional orthogonal polynomials and rich pattern of degeneracies
- Before some results on Painlevé transcendents

Summary

- I. Marquette, M. Sajedi, and P.Winternitz, Fourth order Superintegrable systems separating in Cartesian coordinates I. Exotic quantum potentials, J.Phys.A Theor. and Math 50 315201 (2017)

- I. Marquette, M. Sajedi, and P.Winternitz. Two-dimensional superintegrable system from operator algebras in one dimension, J. Phys. A : Math. Theor. 52 115202 (2019)

- I Marquette and P. Winternitz, A New Painlevé conjecture, Springer, Integrability, Supersymmetry and Coherent States, 103 (2019)

- I.Marquette, Higher order superintegrability, Painlevé transcendents and representations of polynomial algebras, J. Phys. : Conf. Ser. 1194 012074 (2019)

The Painlevé transcendents, part 1

- The Painlevé transcendents arise in the study of ordinary differential equations.

- Painlevé found 50 equations whose only movable singularities are poles. $\frac{d^2w}{dz^2} = F(z, w, \frac{dw}{dz})$
- The most interesting of the fifty types are those which are irreducible and serve to define new transcendents (Painlevé transcendents)

- The other 44 can be integrated in terms of classical functions and transcendents or transformed into the remaining six equations.

- Painlevé (1900, 1902, 1910), Fuchs (1905), Gambier (1910)

The Painlevé transcendents, part 2

$$P_1''(z) = 6P_1^2(z) + z$$

$$P_2''(z) = 2P_2(z)^3 + zP_2(z) + \alpha$$

$$P_3(z)'' = \frac{P_3'(z)^2}{P_3(z)} - \frac{P_3'(z)}{z} + \frac{\alpha P_3^2(z)+\beta}{z} + \gamma P_3^3(z) + \frac{\delta}{P_3(z)}$$

$$P_4''(z) = \frac{P_4'^2(z)}{2P_4(z)} + \frac{3}{2}P_4^3(z) + 4zP_4^2(z) + 2(z^2-\alpha)P_4(z) + \frac{\beta}{P_4(z)}$$

$$P_5''(z) = (\frac{1}{2P_5(z)} + \frac{1}{P_5(z)-1})P_5'(z)^2 - \frac{1}{z}P_5'(z) + \frac{(P_5(z)-1)^2}{z^2}(\frac{aP_5^2(z)+b}{P_5(z)}) + \frac{cP_5(z)}{z} + \frac{dP_5(z)(P_5(z)+1)}{P_5(z)-1}$$

$$P_6''(z) = \frac{1}{2}(\frac{1}{P_6(z)} + \frac{1}{P_6(z)-1} + \frac{1}{P_6(z)-z})P_6'(z)^2 - (\frac{1}{z} + \frac{1}{z-1} + \frac{1}{P_6(z)-z})P_6'(z) + \frac{P_6(z)(P_6(z)-1)(P_6(z)-z)}{z^2(z-1)^2}(\gamma_1 + \frac{\gamma_2 z}{P_6(z)^2} + \frac{\gamma_3(z-1)}{(P_6(z)-1)^2} + \frac{\gamma_4 z(z-1)}{(P_6(z)-z)^2})$$

The Painlevé transcendents,part 3

- Many of their properties have been studied in particular their particular solutions
- They find many applications in domain of mathematical physics
- Statistical mechanics, quantum field theory, relativity,
- Symmetry reduction of various equations (Kdv, Boussineq, Sine-Gordon, Kadomstev-Petviashvile, nonlinear Schrodinger).
- The connection with quantum models and superintegrable systems is much more recent

Integral of Nth order

- 1D, Ranada (1997), Tsiganov (2000), Hietarinta (1984,1998)
- 2D, Drach (1935), Gravel and Winternitz (2002)
- Post,Winternitz (2015), Escobar-Ruiz, Lopez Vieyra, Winternitz, Yurdusen (2018)

$$X = \frac{1}{2}\sum_{l=0}^{\lfloor \frac{N}{2} \rfloor}\sum_{j=0}^{N-2l}\{f_{j,2l}, p_1^j p_2^{N-2l-j}\}$$

$$Y = Y^{(N)} + \sum_{l=1}^{\lfloor \frac{N}{2} \rfloor}\sum_{j=0}^{N-2l} F_{j,2l} P_1^j P_2^{N-j-2l}$$

$$Y^{(N)} = \sum_{0 \leq m+n \leq N} \Lambda_{N-m-n,m,n} L_3^{N-m-n} P_1^m P_2^n$$

- Constrain, compatibility equation, other form for the integrals in polar

2D Superintegrable : 2nd and Nth, ongoing clasification

- Cartesian : N=3, Gravel (2004), Marquette (2006,2009,2009,2010), Marquette Winternitz (2007,2008)
- N=4 : Marquette, Sajedi, and Winternitz (2017) (exotic, non exotic)
- N=5 : Cartesian : Abouamal, Winternitz (2017), (doubly exotic case)

- N=3, polar : Tremblay and Winternitz (2010)
- N=4, polar : Escobar-Ruiz, Lopez Vieyra, Yurdusen, Winternitz (2017,2018) (exotic and non exotic)
- Parabolic : Popperi, Post, Winternitz (2012), Marchesiello, Post, Snobl (2015)

- Families with integrals of arbitrary order : Marquette (2011) (doubly exotic), two-dimensional anisotropic P_4 and P_5
- Classification using 1D operator algebras, (N=2,3,4,5) : Marquette, Sajedi, and Winternitz (2019)

Cartesian and $N = 3$, part 1

$$B = \sum_{i+j+k=3} A_{ijk}\{L_3^i, p_1^j p_2^k\} + \{g_1(x,y), p_1\} + \{g_2(x,y), p_2\}$$

The constants A_{ijk} and functions V, g_1 and g_2 are subject to :

$$(g_1)_x = 3f_1 V_x + f_2 V_y, \quad (g_2)_y = f_3 V_x + 3f_4 V_y,$$

$$(g_1)_y + (g_2)_x = 2(f_2 V_x + f_3 V_y)$$

$$g_1 V_x + g_2 V_y = \frac{\hbar^2}{4}(f_1 V_{xxx} + f_2 V_{xxy} + f_3 V_{xyy} + f_4 V_{yyy})$$

$$+8A_{300}(x_1 V_y - x_2 V_x) + 2(A_{210} V_x + A_{201} V_y)$$

- The functions f_i are polynomial involving the constants A_{ijk}.

Cartesian and $N = 3$, part 2

- The 10 constants and 3 functions determined from a (overdetermined) systems of 4 equations.
- The classical and quantum cases differ !
- 5 potentials are written in terms of Painlevé transcendents

$$V_a(x,y) = \hbar^2(\omega_1^2 P_1(\omega_1 x) + \omega_2^2 P_1(\omega_2 y))$$
$$V_b(x,y) = ay + \hbar^2\omega_1^2 P_1(\omega_1 x)$$

$$V_c(x,y) = bx + ay + (2\hbar b)^{\frac{2}{3}} P_2^2((\tfrac{2b}{\hbar^2})^{\frac{1}{3}} x, 0)$$
$$V_d(x,y) = ay + (2\hbar^2 b^2)^{\frac{1}{3}} (P_2'((\tfrac{-4b}{\hbar^2})^{\frac{1}{3}} x, \alpha) + P_2^2((\tfrac{-4b}{\hbar^2})^{\frac{1}{3}} x), \alpha)$$

$$V_e(x,y) = \frac{\omega^2}{2}(x^2+y^2) + \frac{\hbar^2}{2} P_4^2(\sqrt{\tfrac{\omega}{\hbar}} x, \alpha, \beta) + 2\omega\sqrt{\omega\hbar} P_4(\sqrt{\tfrac{\omega}{\hbar}} x, \alpha, \beta)$$
$$+ \frac{\epsilon\hbar\omega}{2} P_4'(\sqrt{\tfrac{\omega}{\hbar}} x, \alpha, \beta) + \frac{\hbar\omega}{3}(\epsilon - \alpha)$$

Cartesian and N=4, part 1

$$Y = \sum_{j+k+l=4} \frac{A_{jkl}}{2}\{L_3^j, p_1^k p_2^l\} + \frac{1}{2}(\{g_1(x,y), p_1^2\}$$
$$+\{g_2(x,y), p_1 p_2\} + \{g_3(x,y), p_2^2\}) + l(x,y)$$

- The quantities f_i, $i = 1, 2, .., 5$ are polynomials
- set of 6 linear PDEs for the functions g_1, g_2, g_3, and l
- If V is not known, system of 6 nonlinear PDEs for g_i, l and V.

$$g_{1,x} = 4f_1 V_x + f_2 V_y \quad g_{2,x} + g_{1,y} = 3f_2 V_x + 2f_3 V_y$$
$$g_{3,x} + g_{2,y} = 2f_3 V_x + 3f_4 V_y \quad g_{3,y} = f_4 V_x + 4f_5 V_y$$

Cartesian and N=4, part 2

$$\ell_x = 2g_1 V_x + g_2 V_y + \frac{\hbar^2}{4}\Big((f_2 + f_4)V_{xxy}$$
$$-4(f_1 - f_5)V_{xyy} - (f_2 + f_4)V_{yyy} + \ldots$$
$$\ell_y = g_2 V_x + 2g_3 V_y + \frac{\hbar^2}{4}\Big(- (f_2 + f_4)V_{xxx}$$
$$+4(f_1 - f_5)V_{xxy} + (f_2 + f_4)V_{xyy} + \ldots$$

- compatibility equation is a fourth-order linear PDE for $V(x, y)$
- 7 are written in terms of Painlevé transcendents

$$\partial_{yyy}(4f_1 V_x + f_2 V_y) - \partial_{xyy}(3f_2 V_x + 2f_3 V_y) + \partial_{xxy}(2f_3 V_x + 3f_4 V_y)$$
$$-\partial_{xxx}(f_4 V_x + 4f_5 V_y) = 0$$

Cartesian and N=4, part 3

$$V_a = -\hbar^2\delta(x^2+y^2) + \frac{a}{x^2} + \hbar^2\Big(\frac{\gamma}{P_5(y^2)-1}$$

$$+\frac{1}{y^2}(P_5(y^2)-1)(\sqrt{2\alpha}+\alpha(2P_5(y^2)-1)+\frac{\beta}{P_5(y^2)}\Big)$$

$$+y^2\Big(\frac{P_5'^2(y^2)}{2P_5(y^2)}+\delta P_5(y^2)\Big)\frac{(2P_5(y^2)-1)}{(P_5(y^2)-1)^2}+\dots$$

$$V_b = c_2(x^2+y^2) - \sqrt[4]{8c_2^3\hbar^2}\,yP_4\Big(-\sqrt[4]{\frac{2c_2}{\hbar^2}}y\Big)$$

$$+\sqrt{\frac{c_2}{2}}\hbar\Big(\epsilon P_4'\Big(-\sqrt[4]{\frac{2c_2}{\hbar^2}}y\Big)+P_4^2\Big(-\sqrt[4]{\frac{2c_2}{\hbar^2}}y\Big)\Big)$$

$$\dots$$

$$V_h = c_1 x + \frac{\hbar^2}{2}\Big(\sqrt{\alpha}P_3'(y) + \frac{3}{4}\alpha(P_3(y))^2 + \frac{\delta}{4P_3^2(y)} + \dots$$

Connection with Chazy class, part 1

- Equation of fourth order are obtained, can be integrated
- Chazy-I equation, Chazy (1911), Cosgrove (2000), (2006)

- Chazy studied the Painlevé type third order differential equations in the polynomial class and proved that they have the form

$$W''' = aWW'' + bW'^2 + cW^2W' + dW^4 + A(y)W'' + B(y)WW' + C(y)W' + D(y)W^3 + E(y)W^2 + F(y)W + G(y)$$

- where a, b, c, and d are certain rational or algebraic numbers, and the remaining coefficients are locally analytic functions of y

- Chazy classified the reduced equations into 13 classes, denoted by Chazy class I-XIII

- Canonical form for Chazy-I equation and its first integral

$$W''' = -\frac{f'(y)}{f(y)}W'' - \frac{2}{f^2(y)}\big(3k_1 y(yW' - W)^2 + \dots$$
$$+ 2k_7 W' + k_8 y + k_9\big)$$
$$(W'')^2 = -\frac{4}{f^2(y)}\big(k_1(yW' - W)^3 + k_2 W'(yW' - W)^2 + \dots$$
$$+ k_9 W' + k_{10}\big)$$

- Bureau (1964) initiated a study of ODEs of the form

$A(W', W, y)W''^2 + B(W', W, y)W'' + C(W', W, y) = 0,$
A, B and C are polynomials in W, and W'
with coefficients analytic in y

Connection with Chazy class, part 3

- Cosgrove and Scoufis (1993) give a complete classification of Painlevé type equations of second order and second degree

$W''^2 = F(W', W, y)$ where F is rational in W', and W and analytic in y

- integrating all of these equations in terms of known functions (including the six original Painlevé transcendents)
- There are six classes of them, denoted by SD-I, SD-II,...,SD-VI

- The equation SD-I equation, splits into six canonical subcases (SD-Ia, SD-Ib, SD-Ic, SD-Id, SD-Ie, and SD-If)

- N=5, 5 of doubly exotic potentials in terms Painlevé type, one confining type

Polynomial algebra integrals 2nd and Nth order

- Isaac, Marquette (2014)

$$[A,B]=C,\quad [A,C]=\sum_{i=1}^{\lfloor \frac{N}{2}+1\rfloor}\alpha_i A^i+\delta B+\epsilon+\beta\{A,B\}$$

$$[B,C]=\sum_{i=1}^{N}\lambda_i A^i+\rho B^2+\eta B+\sum_{i}^{\lfloor \frac{N}{2}\rfloor}\omega_i\{A^i,B\}+\zeta$$

- Constraints from the Jacobi equation $[A,[B,C]]=[B,[A,C]]$
- Realization as deformed oscillator algebras and Casimir

- Recover for example quartic, Marquette (2013)
- Applied to obtain spectrum of Lissajous models (EOP Jacobi), Marquette, Quesne (2016), Integral of arbitrary order

Cubic case (N=3), part 1

$$[A,B]=C$$
$$[A,C]=\alpha A^2+\beta\{A,B\}+\gamma A+\delta B+\epsilon$$
$$[B,C]=\mu A^3+\nu A^2-\beta B^2-\alpha\{A,B\}+\xi A-\gamma B+\zeta \quad .$$

- The Casimir operator

$$K=C^2-\alpha\{A^2,B\}-\beta\{A,B^2\}+(\alpha\beta-\gamma)\{A,B\}+(\beta^2-\delta)B^2$$
$$(+\beta\gamma-2\epsilon)B+\frac{\mu}{2}A^4+\frac{2}{3}(\nu+\mu\beta)A^3+(-\frac{1}{6}\mu\beta^2+\frac{\beta\nu}{3}+\frac{\delta\mu}{2}+\alpha^2+\xi)A^2$$
$$+(-\frac{1}{6}\mu\beta\delta+\frac{\delta\nu}{3}+\alpha\gamma+2\zeta)A$$

Cubic case (N=3), part 2

- Daskaloyannis (1991,1993), Realization of the cubic algebra by means of a deformed oscillator algebra $\{b^\dagger, b, N\}$

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad b^\dagger b = \Phi(N), \quad bb^\dagger = \Phi(N+1)$$

- There is a realization of the form :

$$A = A(N), \quad B = b(N) + b^\dagger \rho(N) + \rho(N) b$$

Cubic case (N=3), part 3

- Case $\beta = 0$ and $\delta \neq 0$

$$A(N) = \sqrt{\delta}(N+u), b(N) = -\alpha(N+u)^2 - \frac{\gamma}{\sqrt{\delta}}(N+u) - \frac{\epsilon}{\delta}$$

$$\rho(N) = 1, \quad \Phi(N) = \frac{\mu\delta}{8}(N+u)^4 + \ldots$$

Case $\beta \neq 0$

$$A(N) = \frac{\beta}{2}((N+u)^2 - \frac{1}{4} - \frac{\delta}{\beta^2}), b(N) = \frac{\alpha}{4}((N+u)^2) + \ldots$$

$$\Phi(N) = 384\mu\beta^{10}N^{10} + \ldots$$

N=3, Fourth Painlevé transcendent model, part 1

- The integral A and B, A related to separation of variables in Cartesian and B of order 3

$$[A,B]=C \quad [A,C]=16\omega^2\hbar^2 B$$
$$[B,C]=-2\hbar^2A^3-6\hbar^2HA^2+8\hbar^2H^3$$
$$+\frac{\omega^2\hbar^4}{3}(4\alpha^2-20-6\beta-8\epsilon\alpha)A-8\omega^2\hbar^4H+\hbar^5c(\alpha,\beta,\epsilon)$$

- The Casimir operator is given in term of the general formula, but it needs to be rewritten in terms of the central element only

$$K=-16\hbar^2H^4+\frac{4\hbar^4\omega^2}{3}(4\alpha^2-8\alpha+4-\alpha\beta)H^2+\ldots$$

N=3, Fourth Painlevé transcendent model, part 2

$$\Phi(x,u,E)=4\omega^2\hbar^4(-x-u+(\frac{E}{2\hbar\omega}+\frac{1}{2}))(x+u-(\frac{-E}{2\hbar\omega}+c_1(\alpha,\beta,\epsilon)))$$
$$(x+u-(\frac{-E}{2\hbar\omega}+c_2(\alpha,\beta,\epsilon)))(x+u-(\frac{-E}{2\hbar\omega}+c_3(\alpha,\beta,\epsilon)))$$

- We have to distinguish the two cases $\beta<0$ and $\beta>0$
- $\Phi(0,u,E)=0$, $\Phi(p+1,u,E)=0$

- solutions of the form $E_i=\hbar\omega(p+1+d_i(\alpha,\beta,\epsilon))$
- unirreps correspond to physical solutions not for all values of α, β
- Marquette and Quesne (2016) : Connection between Hermite EOP and generalized Hermite and Okamoto polynomials

Constructive approaches

- Direct approaches, lowest order integrals obtained do not necessarily allow algebraic derivation

- Many new constructive approaches have been proposed :
- Integrals in terms of building blocks, factorized form,
- Facilitates the construction of algebraic structures

- Based on recurrence relations of special functions and orthogonal polynomials, Kalnins, Miller and Kress (2011,2012,2013), Marquette (2010), Vinet and Post (2012), building block are ladder and shift operators

- classical analog Miller, Kalnins and Kress (2010), Marquette (2010,2012), Tsiganov (2008)

1D operator algebra,part 1

- Marquette, Sajedi, Winternitz : Constructive approaches to generate superintegrable systems (Cartesian)
- L_x operator of nth order
- Four types of 1D systems : Abelian (a), Heisenberg (b), Conformal (c), Ladder (d)

$$[H_x, L_x] = 0$$
$$[H_x, L_x] = \alpha_x I$$
$$[H_x, L_x] = \alpha_x H_x$$
$$[H_x, L_x] = \alpha_x L_x$$

1D operator algebra,part 2

- Case (a) Hietarinta (1989,1998), for third order operators, pure integrability (quantum)
- Case (b) : Fushchych and Nikitin (1997), Gungor, Kuru, Negro, Nieto (2015) (quantum/classical)
- Case (c) Doebner and Zhdanov 1999 (quantum)
- Case (d) Veselov and Shabat (1993), Andrianov, Cannata, Ioffe and Nishnianidze (2000), Carballo, Fernández, Negro, and Nieto (2004),Marquette (2011) (quantum), Marquette (2010,2012) (classical)
- role play these operators in context of superintegrability
- also classical analog

1D operator algebras, part3

$$H = H_x + H_y = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V_1(x) + V_2(y), \quad A = H_x - H_y$$

$$L_x = \sum f_{n_x} p_x, \quad L_y = \sum f_{n_y} p_y$$

$$(b,b) : B = \alpha_y L_x - \alpha_x L_y$$
$$(c,b) : B = \alpha_y L_x - \alpha_x H_x L_y$$
$$(d,d) : B = B_+ - B_- = (L_x^\dagger)^m (L_y)^n - (L_x)^m (L_y^\dagger)^n$$
$$(c,c) : B = \alpha_y H_y L_x - \alpha_x H_x L_y$$

- Recover : First, second, fourth, fifth Painlevé transcendents potentials (operator L up to order 5), equation of order 6 for (d_5), case with (P_3)

- Another type of reducibility, role of Painlevé property, complementarity (direct and constructive), at order three all are reducible, at order four most are directly connected to these construction

- already know that constructive approach do not provide the lowest possible order integrals, here the nonlinear differential equation can take different form (also many ways to look at ladder operators)

- Abouamal, Winternitz (2017) (another case not in term of Painlevé transcendent)

Construction type (d,d), part 1

- A 2D system with separation of variables in Cartesian :

$$H = H_x + H_y = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + V_x(x) + V_y(y)$$

- ladder operators that satisfy PHA

$$[H_x, L_x^\dagger] = \alpha_x L_x^\dagger, \quad [H_x, L_x] = -\alpha_x L_x$$
$$L_x L_x^\dagger = Q(H_x + \alpha_x), \quad L_x^\dagger L_x = Q(H_x)$$
$$[H_y, L_y^\dagger] = \alpha_y L_y^\dagger, \quad [H_y, L_y] = -\alpha_y L_y$$
$$L_y L_y^\dagger = S(H_y + \alpha_y), \quad L_y^\dagger L_y = S(H_y)$$

- α_x and α_y are constants while $Q(x)$ and $S(y)$ are polynomials
- integrals of motion ($k_1 n_1 + k_2 n_2$)
- $n_1\alpha_x = n_2\alpha_y = \alpha$, $n_1, n_2 \in \mathbb{Z}^*$

Construction type (d,d), part 2

$$A = \frac{1}{2\alpha}(H_x - H_y), \quad B_- = L_x^{n_1} L_y^{\dagger n_2}, \quad B_+ = L_x^{\dagger n_1} L_y^{n_2}$$

- ladder operators of a given order, the method allows to generate integrals of motion of an arbitrary order in a factorized form
- taking $b^\dagger = B_+$, $b = B_-$ and $N = A - u$ (where u is a representation dependent parameter that is determined using further constraints)

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b,$$
$$b^\dagger b = \Phi(N), \quad bb^\dagger = \Phi(N+1), \quad \Phi(H, u, N) = F_{n_1,n_2}(A, H).$$

Fourth Painlevé transcendent model : ladder and susy

- Andrianov, Cannata, Ioffe, and Nishnianidze (2000), Marquette (2009)
- studied using two supersymmetric quantum mechanics
- construction of a third order ladder

- At most three of the six possible states annihilated by a^- (ψ_i) and a^+ (ϕ_i) in total, only three can be square integrable

$$\psi_i = f_1(P_4, P_4')e^{\int g_1(P_4,P_4')}, \quad i = 1,2,3$$

$$\phi_i = f_2(P_4, P_4')e^{\int g_2(P_4,P_4')}, \quad i = 1,2,3$$

- For some ranges of α and β values, H_1 may admit one, two, or three infinite sequences of equidistant levels, or one infinite sequence of equidistant levels with either one additional singlet or one additional doublet.

Fifth Painlevé transcendent model : ladder and susy

- Carballo, Fernandez, Negro and Nieto (2004), Willox and Hietarinta (2003), Marquette (2011)
- studied using two supersymmetric quantum mechanics
- construction of a fourth order ladder

- At most three of the six possible states annihilated by a^- (ψ_i) and a^+ (ϕ_i) in total, only three can be square integrable

$$\psi_i = f_1(P_5, P_5')e^{\int g_1(P_5,P_5')} \quad , i = 1,2,3,4$$

$$\phi_i = f_2(P_5, P_5')e^{\int g_2(P_5,P_5')}, \quad i = 1,2,3,4$$

- For some ranges of α and β values, H_1 may admit one, two,, three, four infinite sequences of equidistant levels, or combination infinite sequences of equidistant levels with multiplet

Example, part 1

- Case $\alpha = 5$, $\beta = -8$, $f(z) = \frac{4z(2z^2-1)(2z^2+3)}{(2z^2+1)(4z^4+3)}$ and $\epsilon = 1$.

$$V(x,y) = \frac{\omega^2}{2}(x^2+y^2) - \frac{8\hbar^3\omega}{(2\omega x^2+\hbar)^2} + \frac{4\hbar^2\omega}{(2\omega x^2+\hbar)} + \frac{2\hbar\omega}{3}$$

- From the cubic algebra we get unitary representations

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)(x+3)(x+2), \quad E = \hbar\omega(p+\frac{8}{3}), p = 0, 1, \ldots$$

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)(x-3)(x-1), \quad E = \hbar\omega(p-\frac{1}{3}), p = 0$$

$$\phi(x) = 4\hbar^4\omega^2 x(p+1-x)(x+1)(x-2), \quad E = \hbar\omega(p+\frac{2}{3}), p = 0, 1$$

Example, part 2

- The eigenfunctions for the x part consist of an infinite sequence and a singlet state

$$\psi_n(x) = N_n (a^\dagger)^n e^{\frac{-\omega x^2}{2\hbar}} \frac{x(3\hbar + 2\omega x^2)}{(\hbar + 2\omega x^2)}, \chi(x) = C_0 \frac{e^{\frac{-\omega x^2}{2\hbar}}}{\hbar + 2\omega x^2}$$

$$a\psi_0(x) = 0, \quad a\chi(x) = 0, \quad a^\dagger \chi(x) = 0 \quad .$$

$$a = (\partial + W_1(x))(\partial + W_2(x))(-\partial + W_3(x))$$

Example, part 3

$$g(x) = \hbar(\hbar + 2\omega x^2)(3\hbar^2 + 4\omega^2 x^4)$$

$$W_1 = -(-\hbar + 2\omega x^2)(9\hbar^3 + 27\hbar^2\omega x^2 + 12\hbar\omega^2 x^4 + 4\omega^3 x^6)/g(x) \quad ,$$

$$W_2 = -(\hbar - 2\omega x^2)(3\hbar^2 + 3\hbar\omega x^2 + 2\omega^2 x^4)/g(x) \quad ,$$

$$W_3 = -\omega x(-9\hbar^3 + 22\hbar^2\omega x^2 + 20\hbar\omega^2 x^4 + 8\omega^3 x^6)/g(x) \quad .$$

$$\psi_{n,k} = \psi_n(x) e^{-\frac{\omega y^2}{2\hbar}} H_k\left(\sqrt{\frac{\omega}{\hbar}} y\right), \quad E = \hbar\omega\left(n + k + \frac{8}{3}\right)$$

$$\phi_m = \chi(x) e^{-\frac{\omega y^2}{2\hbar}} H_m\left(\sqrt{\frac{\omega}{\hbar}} y\right), \quad E_m = \hbar\omega\left(m - \frac{1}{3}\right)$$

- This problem occurs for states related to 1-step,2-step

- I.Marquette and C. Quesne, New families of superintegrable systems from Hermite and Laguerre exceptional orthogonal polynomials, J. Math. Phys. 54 042102 (2013).
- I.Marquette and C. Quesne, New ladder operators for a rational extension of the harmonic oscillator and superintegrability of some two-dimensional systems, J. Math. Phys. 54 102102 (2013).
- I. Marquette and C. Quesne, Combined state-adding and state-deleting approaches to type III multi-step rationally-extended potentials : applications to ladder operators and superintegrability, J. Math. Phys. 55 112103 (2014).

I.Marquette and C. Quesne, On connection betwen quantum systems involving fourth Painleve transcendent and k-step extension related to EOP, J.Math. Phys. 57 101063 (2016)

Ladder type (d) , factorization and EOP

- Case (d) is rich and ladder operator for a given system are not unique (even for the same order)
- Marquette and Quesne (2013,2014,2015,2016), Hermite and Laguerre type III
- Hoffmann, Hussin, Marquette, Zhong, (2017,2018,2019)

- Carinena, Plyushchay (2016,2017), ABC ladder operators, Inzunza, MS Plyushchay (2019,2019) extended in conformal and superconformal

- Intertwine with two nth-order differential operators $\mathcal{A}$ and $\mathcal{A}^\dagger$

$$\mathcal{A}H^{(1)} = H^{(2)}\mathcal{A}, \qquad \mathcal{A} = A^{(n)} \cdots A^{(2)}A^{(1)}$$

$$A^{(i)} = \frac{d}{dx} + W^{(i)}(x), \quad W^{(i)}(x) = -\frac{d}{dx}\log\varphi^{(i)}(x), i = 1, 2, \dots, n,$$

Exceptional orthogonal polynomials

- y_0,y_1, y_2,.... are polynomials with deg $y_n = n$

$$p(x)y_i'' + q(x)y_i' + r(x)y_i = \lambda_i y_i, \quad i = 0, 1, 2, ...$$

- p,q, r polynomials deg $p \leq 2$, deg $q \leq 1$, deg $r = 0$
- Bochner (1929), Lesky (1962) : $\{y_i\}$ are Hermite, Laguerre and Jacobi polynomials

- Odake, Sasaki, Kamran, Milson, Gomez Ulate, Quesne, (2008-2010), Post, Tsujimoto, and Vinet (2012), Gomez Ulate, Grandati, Milson (2018) : Complete set of orthogonal polynomials with gaps

- Darboux-Crum, Krein-Adler transformation of differential equation in context of quantum mechanics

Families of superintegrable models

- superintegrable systems from k-step rational extension
- They can be combination of harmonic oscillator and singular oscillator or themselves, 7 families of systems was generated
- Here only 2D "isotropic", but in fact N-dimensional generalizations exist (and "anisotropic" version)

$$H_a = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 + y^2 - 2k - 2\frac{d^2}{dx^2}\log \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \ldots, \mathcal{H}_{m_k}),$$

$$E_{i,N} = 2N, \quad N = \nu_x + \nu_y + 1$$

$$\nu_x = -m_k - 1, \ldots, -m_1 - 1, 0, 1, 2, \ldots, \qquad \nu_y = 0, 1, 2 \ldots.$$

Degeneracies

- direct calculation leads

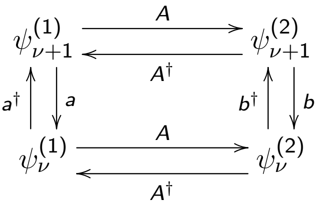
$$\deg(E_N) = \begin{cases} k - j + 1 & \text{if } N = -m_j, -m_j + 1, \ldots, -m_{j-1} - 1, \\ & \text{for } j = 2, 3, \ldots, k, \\ k & \text{if } N = -m_1, -m_1 + 1, \ldots, 0, \\ N + k & \text{if } N = 1, 2, 3, \ldots. \end{cases}$$

- pattern of degeneracies , bands of levels
- recovered from polynomial algebra and finite-dimensional unirreps
- need to combine solution for total number of degeneracies, there are as well non physical states that are obtained

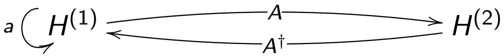
- A second 1D Hamiltonian $H^{(2)}$, related by SUSYQM with $H^{(1)}$

$$f(H^{(1)}) = A^\dagger A, \quad f(H^{(2)}) = AA^\dagger,$$
$$AH^{(1)} = H^{(2)}A, \quad A^\dagger H^{(2)} = H^{(1)}A.$$

- $A^\dagger$ and A (n-th order) $H^{(2)}$ admits a PHA ($b = AaA^\dagger$)



- we come back by the same path (also for $a^\dagger$)



Higher order SUSYQM

- We consider nth-order SUSYQM
- From seed solution $\varphi_i(x)$ of the Schrödinger equation associated with $H^{(1)}$

$$\varphi^{(1)}(x) = \varphi_1(x), \qquad \varphi^{(i)}(x) = \frac{\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_i)}{\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_{i-1})}, i = 2, 3, \ldots, n.$$

- Here $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_i)$ denotes the Wronskian of $\varphi_1(x)$, $\varphi_2(x)$, ..., $\varphi_i(x)$

$$V^{(2)}(x) = V^{(1)}(x) - 2\frac{d^2}{dx^2} \log \mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n),$$

State adding

$$V^{(1)}(x) = x^2, \qquad -\infty < x < \infty,$$

- State-Adding case : $n = k$ seed functions among the polynomial-type eigenfunctions $\phi_m(x)$ of $H^{(1)}$ below the ground-state energy $E_0^{(1)}$
- Associated with the eigenvalues $E_m = -2m - 1$

$$\phi_m(x) = \mathcal{H}_m(x) e^{\frac{1}{2}x^2}, \qquad m = 0, 1, 2, \ldots,$$

- $(\varphi_1, \varphi_2, \ldots, \varphi_n) \to (\phi_{m_1}, \phi_{m_2}, \ldots, \phi_{m_k})$
- Partner potential nonsingular if $m_1 < m_2 < \cdots < m_k$ with m_i even (resp. odd) for i odd (resp. even)

$$V^{(2)}(x) = x^2 - 2k - 2\frac{d^2}{dx^2} \log \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \ldots, \mathcal{H}_{m_k}).$$

State deleting equivalence

$$E_\nu^{(2)} = 2\nu + 1, \qquad \nu = -m_k - 1, \dots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots,$$

$$\psi_\nu^{(2)}(x) \propto \frac{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k}, \psi_\nu)}{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k})}, \qquad \nu = 0, 1, 2, \dots,$$

$$\psi_{-m_i-1}^{(2)}(x) \propto \frac{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \check{\phi}_{m_i}, \dots, \phi_{m_k})}{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k})}, \qquad i = 1, 2, \dots, k.$$

- State deleting : (at least) $n = m_k + 1 - k$ bound-state wavefunctions of $H^{(1)}$ as seed functions : $(\varphi_1, \varphi_2, \dots, \varphi_n)$ to $(\psi_1, \psi_2, \dots, \check{\psi}_{m_k - m_{k-1}}, \dots, \check{\psi}_{m_k - m_2}, \dots, \check{\psi}_{m_k - m_1}, \dots, \psi_{m_k})$

$$\bar{V}^{(1)}(x) = V^{(1)}(x)$$

$$\bar{\psi}_\nu^{(2)}(x) = \psi_\nu^{(2)}(x), \quad V^{(2)}(x) + 2m_k + 2 = \bar{V}^{(2)}(x)$$

Ladder from Krein-Adler, Darboux-Crum

- can go from $H^{(2)}$ to $H^{(2)} + 2m_k + 2$ along the following path

$$H^{(2)} \xrightarrow{\quad \mathcal{A}^\dagger \quad} H^{(1)} = \bar{H}^{(1)} \xrightarrow{\quad \bar{\mathcal{A}} \quad} \bar{H}^{(2)} = H^{(2)} + 2m_k + 2$$

$$H^{(2)} \xrightarrow{\quad c \quad} H^{(2)} + 2m_k + 2$$

- The $(m_k + 1)$th-order differential operator, $c = \bar{\mathcal{A}}\mathcal{A}^\dagger$,c† $= \mathcal{A}\bar{\mathcal{A}}^\dagger aPHAof$ m$_k$th order, $Q(H^{(2)})$ is indeed a $(m_k + 1)$th-order polynomial in $H^{(2)}$

$$[H^{(2)}, c^\dagger] = (2m_k + 2)c^\dagger, \qquad [H^{(2)}, c] = -(2m_k + 2)c,$$
$$[c, c^\dagger] = Q(H^{(2)} + 2m_k + 2) - Q(H^{(2)}),$$

$$Q_{ko}(H^{(2)}) = \prod_{i=1}^{k}(H^{(2)} + 2m_i + 1) \prod_{\substack{j=1 \\ j \neq m_k - m_{k-1}, \ldots, m_k - m_1}}^{m_k} (H^{(2)} - 2j - 1),$$

Pattern of the zero modes

- $\psi_\nu^{(2)}$ wavefunctions are related to $X_{m_1,\ldots,m_k}$ multi indexed Hermite EOP of type III
- The action of the lowering and raising operators can be calculated
- The action of c is given in both cases by

$$c\psi_\nu^{(2)} = 0, \qquad \nu = -m_k - 1, \ldots, -m_1 - 1, 1, 2, \ldots, m_k - m_{k-1} - 1,$$
$$m_k - m_{k-1} + 1, \ldots, m_k - m_1 - 1, m_k - m_1 + 1, \ldots, m_k,$$

$$c\psi_0^{(2)} \propto \psi_{-m_k-1}^{(2)},$$

$$c\psi_{m_k-m_i}^{(2)} \propto \psi_{-m_i-1}^{(2)}, \qquad i = 1, 2, \ldots, k-1,$$

$$c\psi_\nu^{(2)} \propto \psi_{\nu-m_k-1}^{(2)}, \qquad \nu = m_k+1, m_k+2, \ldots .$$

- unirreps may be characterized by (N, s) and their basis states by $|N, \tau, s, \sigma\rangle$
- $\sigma = -s, -s+1, \ldots, s$ and τ distinguishes between repeated representations specified by the same s(integer or half-integer)

$$b^{\dagger}|N, \tau, s, s\rangle = b|N, \tau, s, -s\rangle = 0.$$

- The σ is associated with each state forming this sequence
- Using notation $N = \lambda n_1 n_2 + \mu$ with appropriate values of α and μ, $|N, \nu_x\rangle = |\nu_x\rangle_1 |N - \nu_x - 1\rangle_2$

λ	μ	$2s$	$\mathcal{N}$	$\deg(E_N)$
-1	$1,\ldots,m_k-m_{k-1}$	0	1	1
-1	$m_k-m_j+1,\ldots,m_k-m_{j-1}$	0^{k-j+1}	$k-j+1$	$k-j+1$
-1	$m_k-m_1+1,\ldots,m_k$	0^k	k	k
0	0	0^k	k	k
0	$1,\ldots,m_k-m_{k-1}$	1 $0^{\mu+k-2}$	$\mu+k-1$	$N+k$
0	$m_k-m_j+1,\ldots,m_k-m_{j-1}$	1^{k-j+1} $0^{\mu-k+2j-2}$	$\mu+j-1$	$N+k$
0	$m_k-m_1+1,\ldots,m_k$	1^k $0^{\mu-k}$	μ	$N+k$

λ	μ	$2s$	$\mathcal{N}$	$\deg(E_N)$
$1,2,\ldots$	0	λ^k $(\lambda-1)^{m_k-k+1}$	m_k+1	$N+k$
$1,2,\ldots$	$1,\ldots,m_k-m_{k-1}$	$\lambda+1$ $(\lambda)^{\mu+k-2}$ $(\lambda-1)^{m_k-\mu-k+2}$	m_k+1	$N+k$
$1,2,\ldots$	$m_k-m_j+1,\ldots,m_k-m_{j-1}$	$(\lambda+1)^{k-j+1}$ $(\lambda)^{\mu-k+2j-2}$ $(\lambda-1)^{m_k-\mu-j+2}$	m_k+1	$N+k$
$1,2,\ldots$	$m_k-m_1+1,\ldots,m_k$	$(\lambda+1)^k$ $(\lambda)^{\mu-k}$ $(\lambda-1)^{m_k-\mu+1}$	m_k+1	$N+k$

Concluding remarks 1

- Higher order superintegrable systems are interesting , many very nice algebraic structures, quadratic and more generally finitely generated polynomial algebras, they share many properties of the Lie algebras.

- Classification, their Casimir would need to be obtained (PBW basis Jarvis), Calogero models, generalized Coulomb (Correa), models on surved spaces and Racah algebras (Kuru and Negro), Post and Ritter, tensor, realizations, differential operators

- Solution/Classification : special functions, exceptional orthogonal polynomials which admit holes in the sequence of polynomials, Painlevé transcendents, new transcendental functions (beyond)

Concluding remarks 2

- Many schemes of higher order analog of Painlevé, classification of superintegrable systems related to these work Chazy, Bureau, Cosgrove and Scoufis

- Post and Ritter, P6 models still unsolved, Jacobi, Laguerre EOP and P6 models

- Berntson, Miller, Marquette, A new approach to analysis of 2D higher order quantum superintegrable systems ; Notes on 3rd order superintegrable systems on the 2-sphere ; (P6 models)

- Quesne, Marquette, extend using polynomial deformation $osp(2|2)$, $osp(2m|2)$, generalization and construct their Casimir operators,