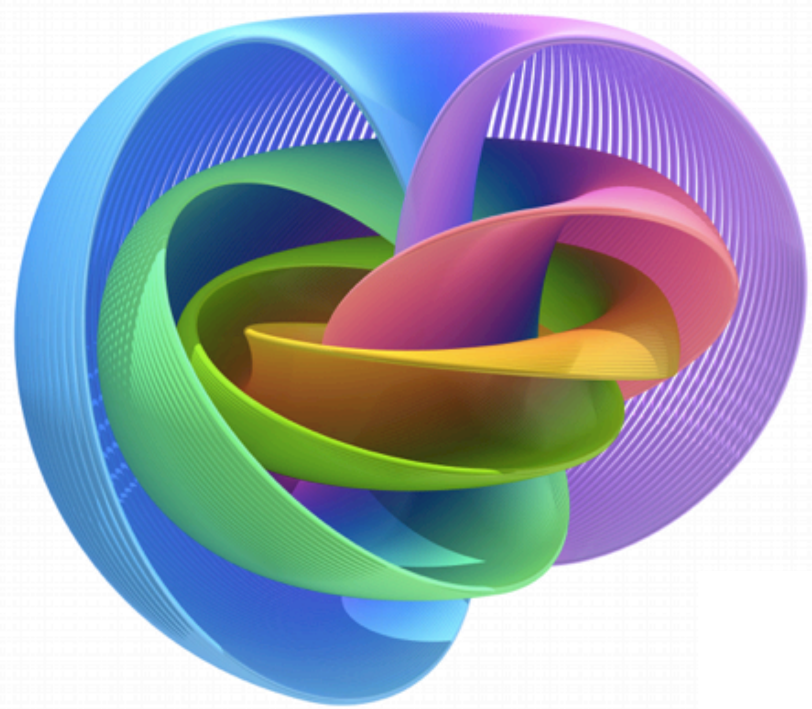
Bulk-edge dualities in topological matter

M. Asorey



Benasque 2019

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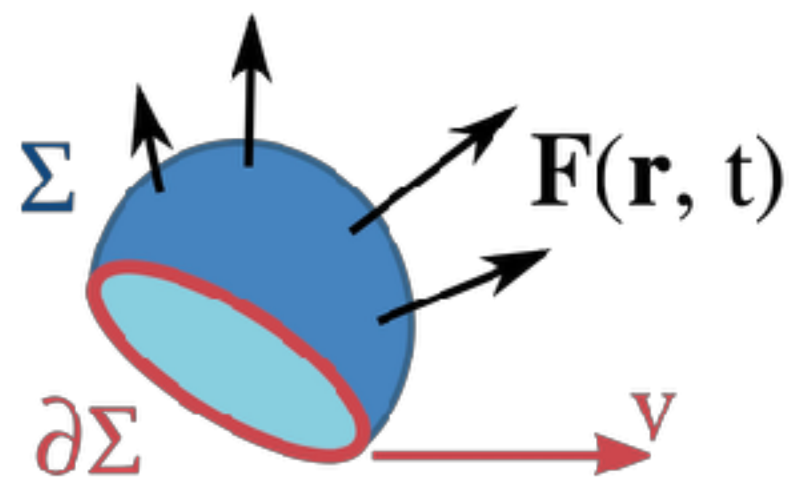
BULK-EDGE DUALITIES

Gauss-Ostrogradski theorem

$$\iint_{\Sigma} \vec{F} \cdot d\vec{\sigma} = \iiint_{V} \vec{\nabla} \cdot \vec{F}$$



Cauchy's theorem

Green's theorem

Stokes' theorem

BULK-EDGE DUALITIES

Casimir Force

Vacuum EM bulk Fluctuations

$$V(r) = \frac{\alpha}{d^7}$$

Relativistic van der Waals boundary forces

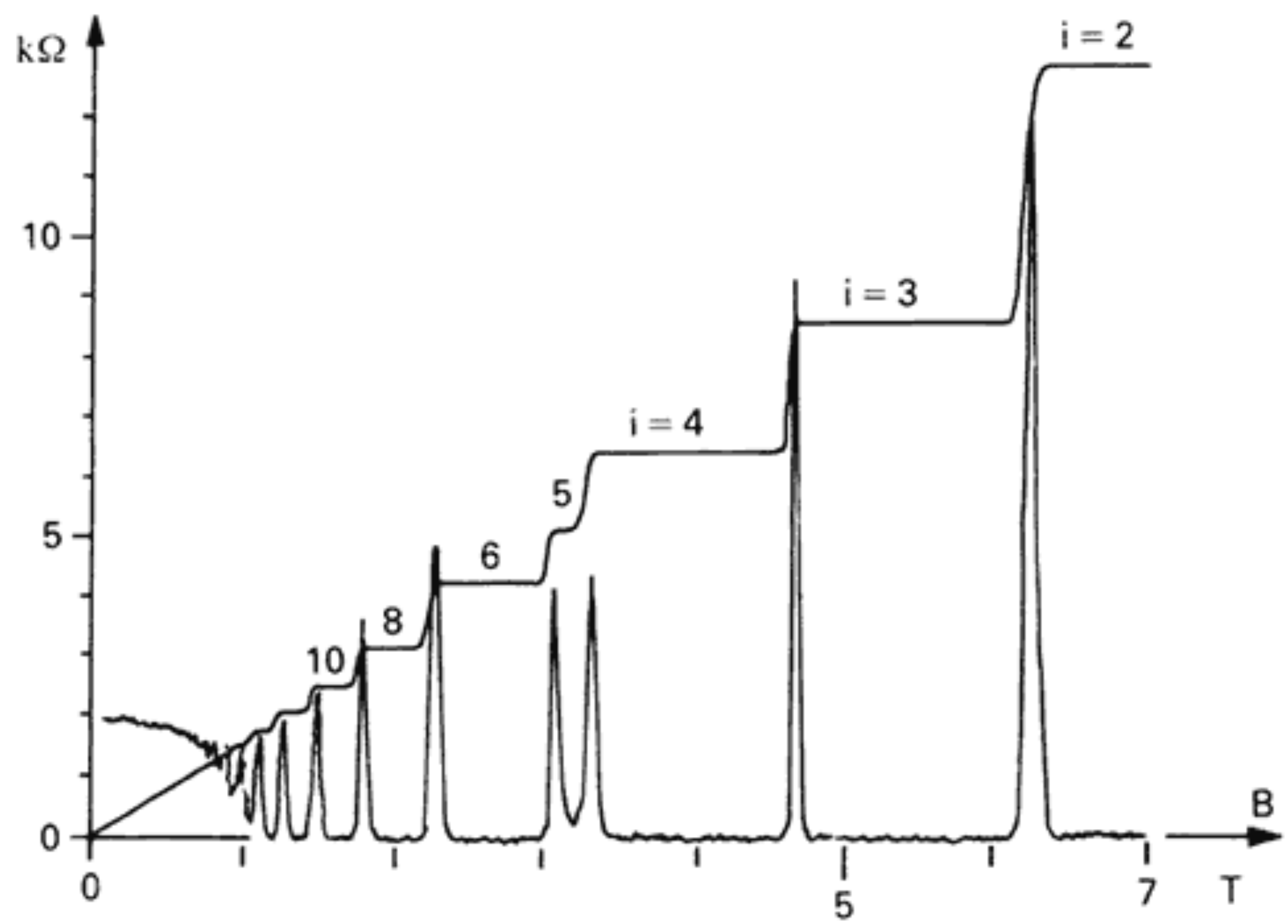
BULK-EDGE DUALITIES TOPOLOGICAL MATTER

Quantum Hall Effect [1980]

Topological Insulators [2005]

Weyl Semimetals [2015]

Integer Quantum Hall effect



[von Klitzing]

Quantum Hall effect

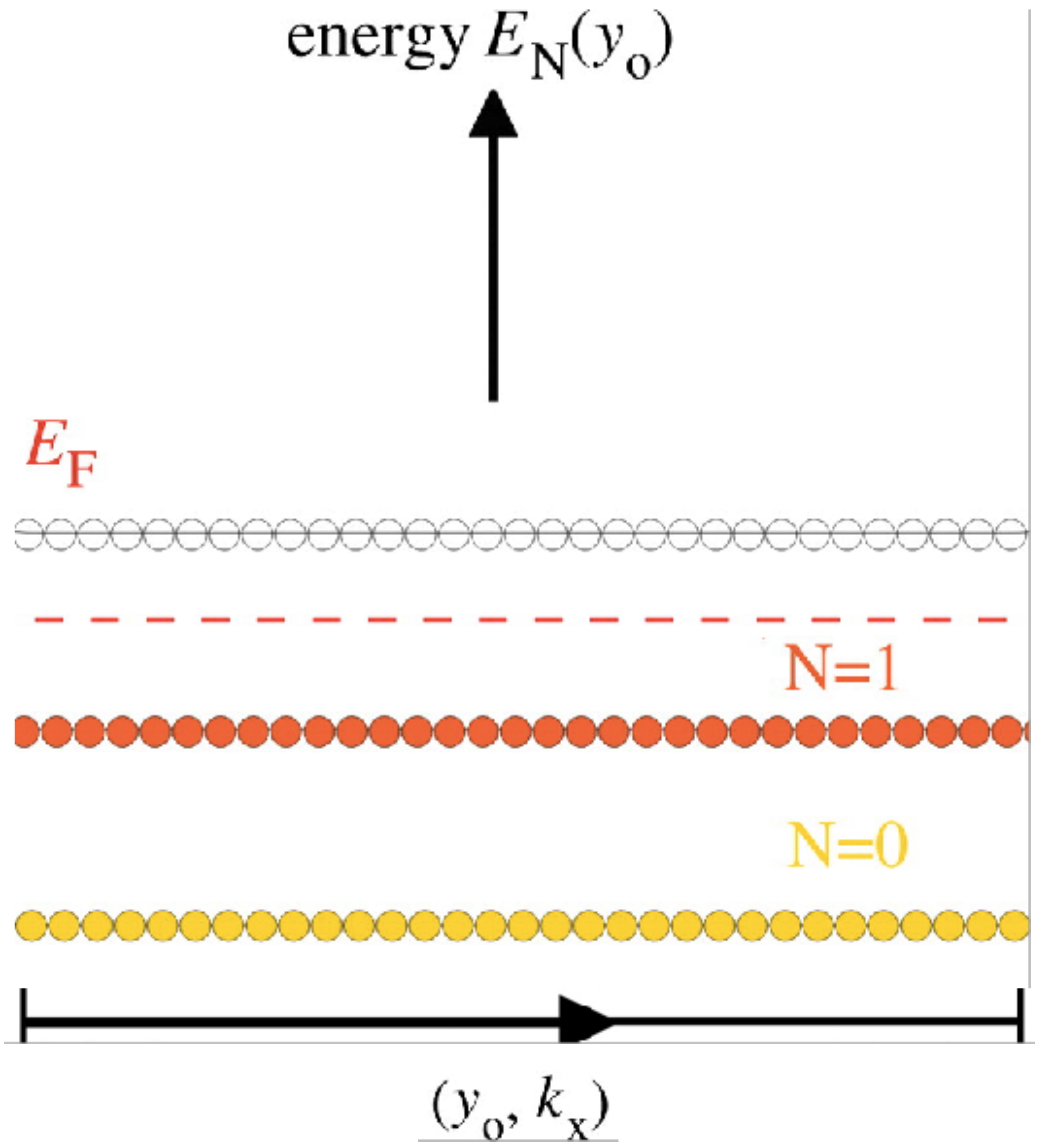
$$\mathbb{H} = -\frac{1}{2m}\left[\partial_2^2 + (\partial_1 + ieBx_2)^2\right]$$

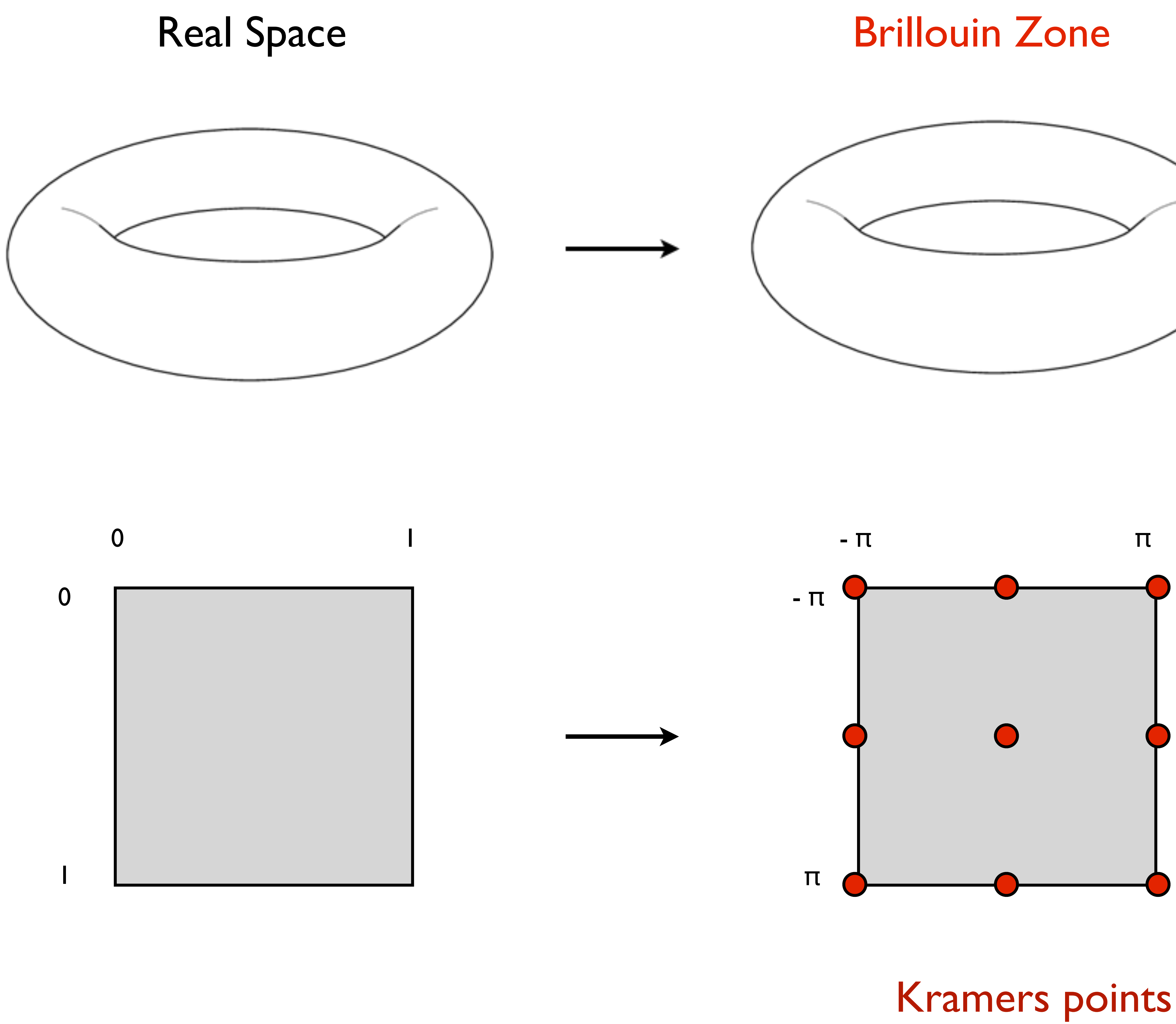
$$E_n = \frac{eB}{2m}\left(n + \frac{1}{2}\right)$$

$$\psi_{n,k_1}(x) = e^{ik_1x_1} H_n(eBx_2)\, e^{-eBx_2^2/2}$$

Landau

Integer Quantum Hall effect




Real Space
Brillouin Zone
0
l
0
l
- π
π
- π
π
Kramers points

Quantum Hall effect and Fiber bundles

Floquet-Bloch

$$L^2(\mathbb{R}^2) = \biguplus_{\lambda \in \widehat{\mathbb{T}}^2} L^2(\mathbb{T}^2)$$

The space of states adquires a bundle structure

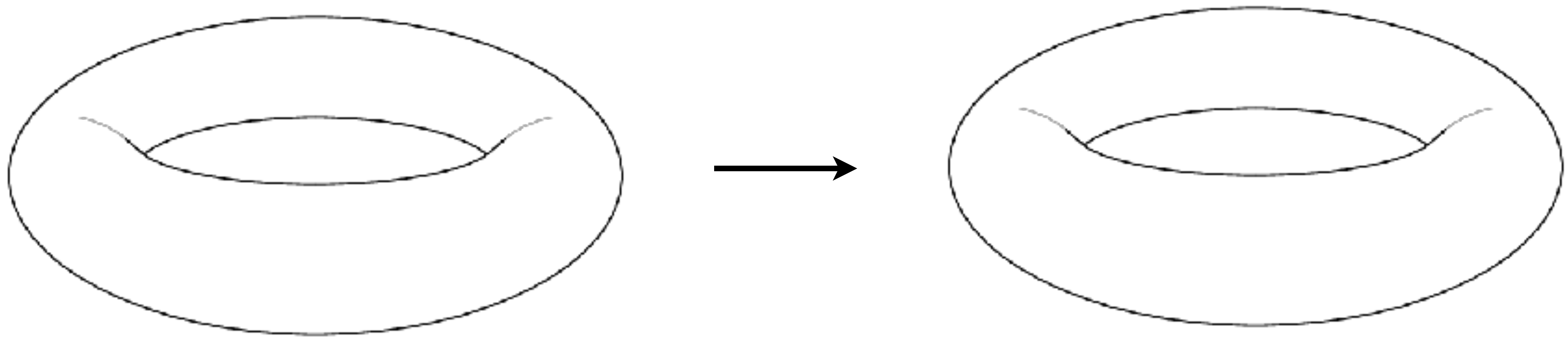
$$L^2(\mathbb{R}^2)\left(\widehat{\mathbb{T}}^2, L^2(\mathbb{T}^2)\right)$$

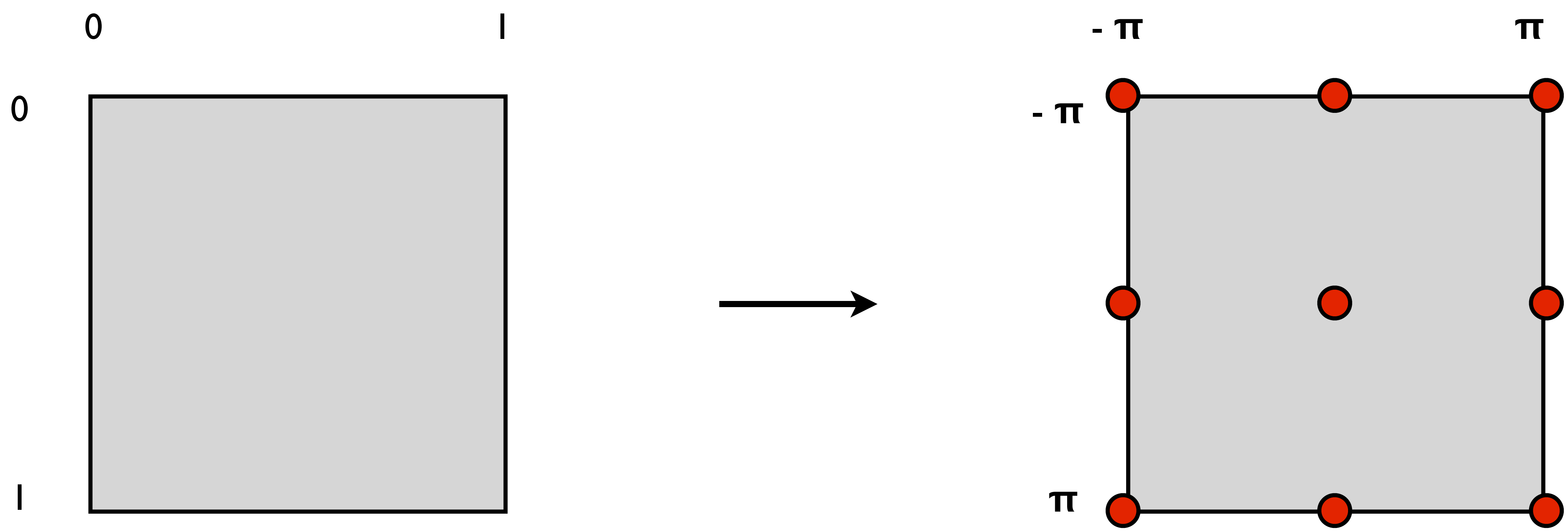
Spectral Floquet-Bloch theorem

$$\mathbb{H}_{\mathbb{R}^2} = \biguplus_{\lambda \in \widehat{\mathbb{T}}^2} \mathbb{H}^{\lambda}_{\mathbb{T}^2}$$

Real Space

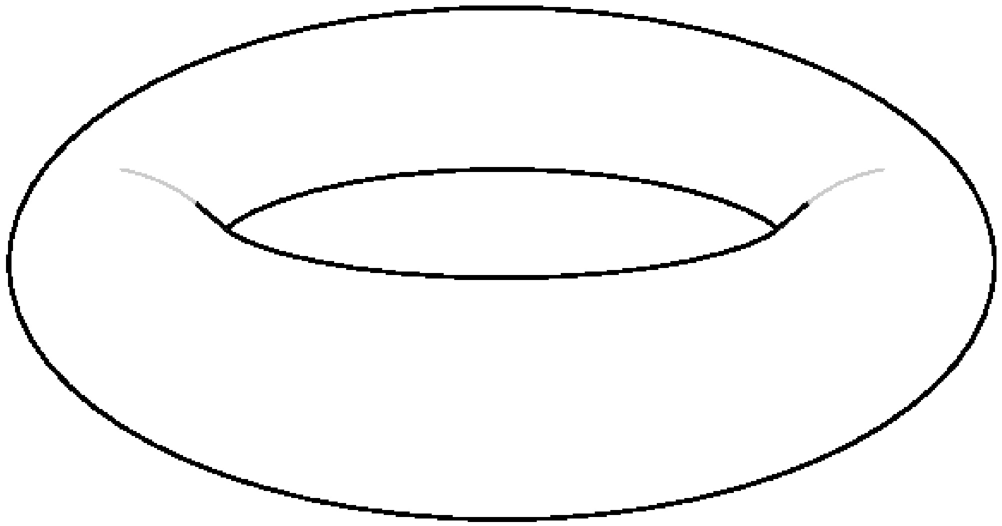
Brillouin Zone





Kramers points

Hall Effect in 2D Torus $\mathbb{T}$



$$\frac{e}{2\pi}\int_T \mathbf{F} = k \in \mathbb{Z} \qquad k = eB/2\pi$$

$$\psi(\phi_1 + 2\pi, \phi_2) = e^{i\frac{k}{2}\phi_2}\psi(\phi_1, \phi_2)$$

In complex coordinates:

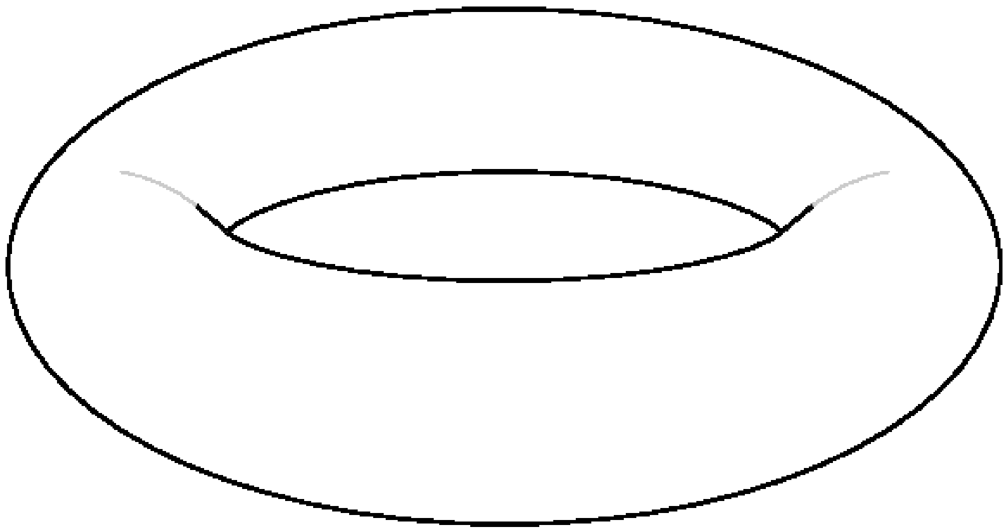
$$\psi(\phi_1, \phi_2 + 2\pi) = e^{-i\frac{k}{2}\phi_2}\psi(\phi_1, \phi_2)$$

$$\mathbb{H} = \frac{1}{2m}\left[\left(\partial_1 + i\frac{B}{2}(\phi_2 + \epsilon_2)\right)^2 + \left(\partial_2 + i\frac{B}{2}(-\phi_1 - \epsilon_1)\right)^2\right]$$

Energy levels (degeneracy: $|k|$) [Landau]

$$E_n = \frac{2\pi|k|}{m}\left(n + \frac{1}{2}\right)$$

Hall Effect in 2D Torus $\mathbb{T}$



$$\frac{e}{2\pi}\int_T \mathbf{F} = k \in \mathbb{Z} \qquad k = eB/2\pi$$

Ground State Eigenfunctions (degeneracy: $|k|$) :

Holomorphic sections of $E(T^2, \mathbb{C})$

$$\psi_0^l(\epsilon,\phi) = \frac{\mathrm{e}^{i\frac{k}{4\pi}(\phi_1+2\epsilon_1)\phi_2}}{(8\pi^4 k)^{\frac{1}{4}}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{im\left(\phi_1+\epsilon_1+2\pi\frac{l}{k}\right)-\frac{1}{4\pi k}(2\pi m+k\phi_2+k\epsilon_2)^2}$$

$$l \;=\; 0,1,2\ldots,|k|-1.$$

TKKN and the Hidden topology

The states with energies below the Fermi level define a vector bundle over the Brillouin zone torus.

$$E\left(\widehat{\mathbb{T}}^2, \mathbb{C}^N)\right)$$

In this bundle there are gauge fields defined by the Berry phases of the different states

$$\mathcal{A}_{j\,n}^{l,l'}(\epsilon) = -i\int_{\mathbb{T}^2} \psi_n^{l\;*} \partial_{\epsilon_j} \psi_n^{l'}$$

$$\mathcal{F}_n^{l,l'}(\epsilon) = \partial_{\epsilon_1} \mathcal{A}_{2\,n}^{l,l'} - \partial_{\epsilon_2} \mathcal{A}_{1\,n}^{l,l'}$$

TKKN and the Bloch bundle

First Chern class of Bloch bundle

$$C_1 = -\frac{i}{4\pi} \sum_{n=0}^{\nu} \sum_{l=0}^{|k|-1} \int_{\widehat{\mathbb{T}}^2} \mathcal{F}_n^{l,l} = \nu$$

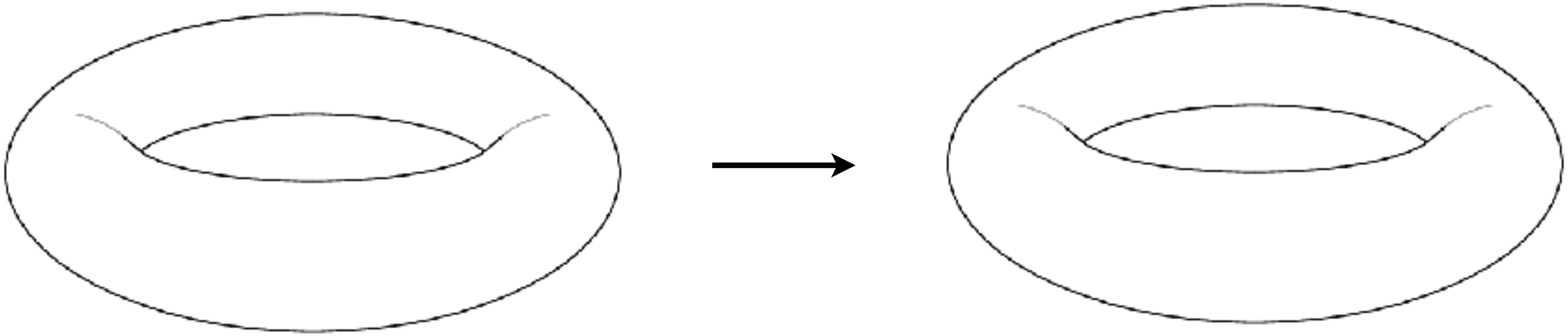
TKKN formula:

$$\sigma_{xy} = \frac{e^2}{2\pi} \nu$$

Quantization of Hall conductivity

Fourier-Mukai transform

Real Space | Brillouin Zone



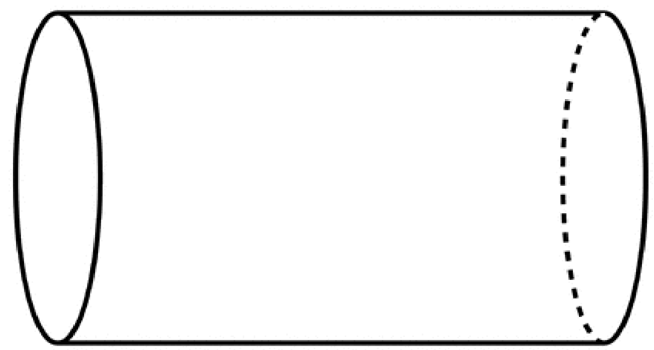
U(N) gauge field A_μ
with c_1 (A)=k

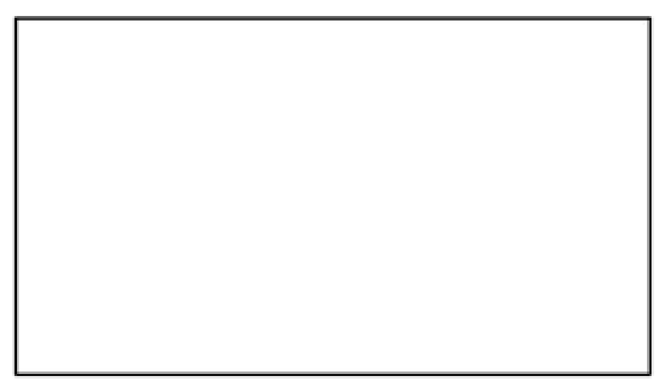
U(k) gauge field A_μ
with c_1 (A)=N

Nahm transform

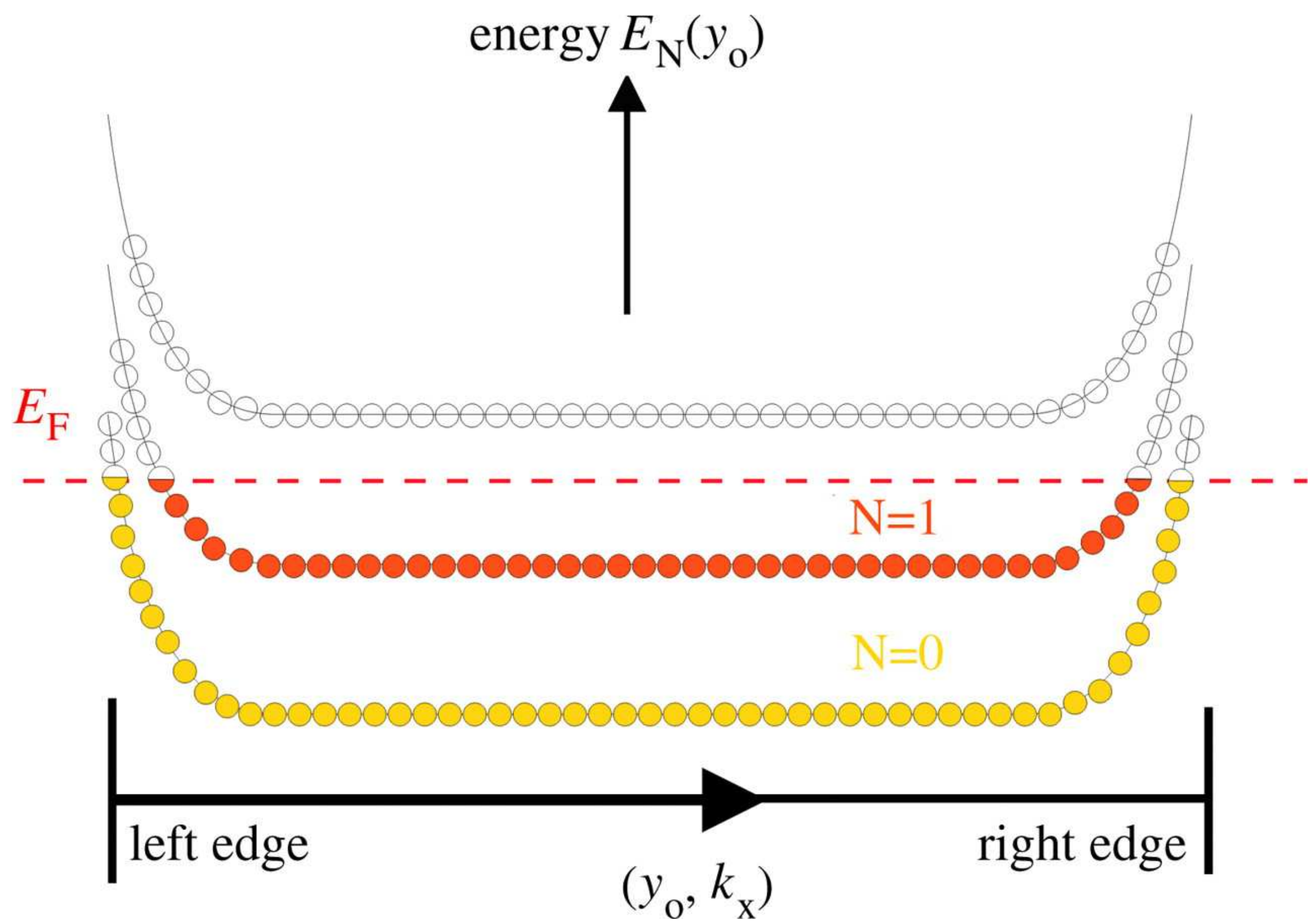
[M.A. , Nature Physics]

Hall effect with boundaries



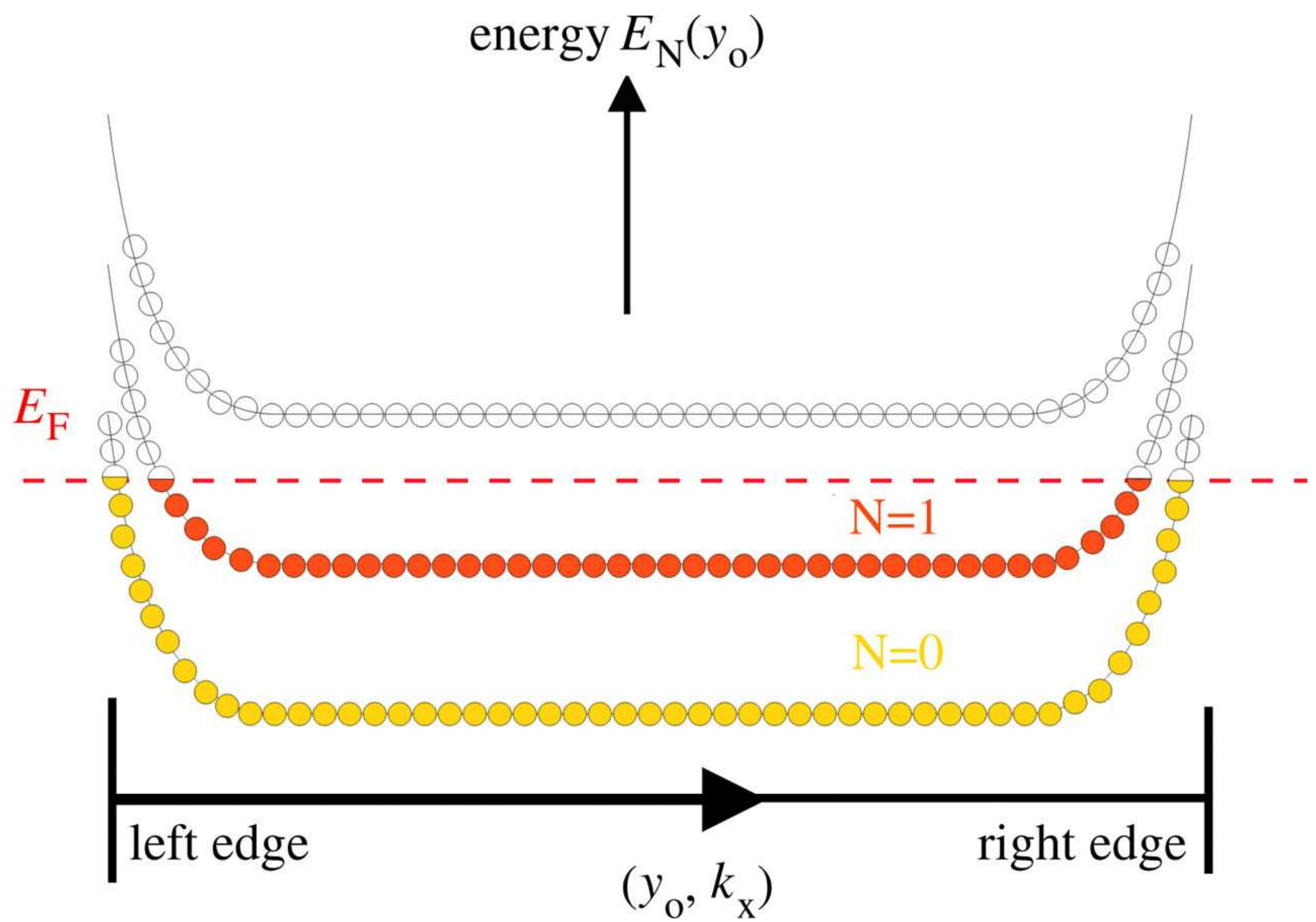


Finite size effects



Edge States

Edge view of σ_{xy} quantization

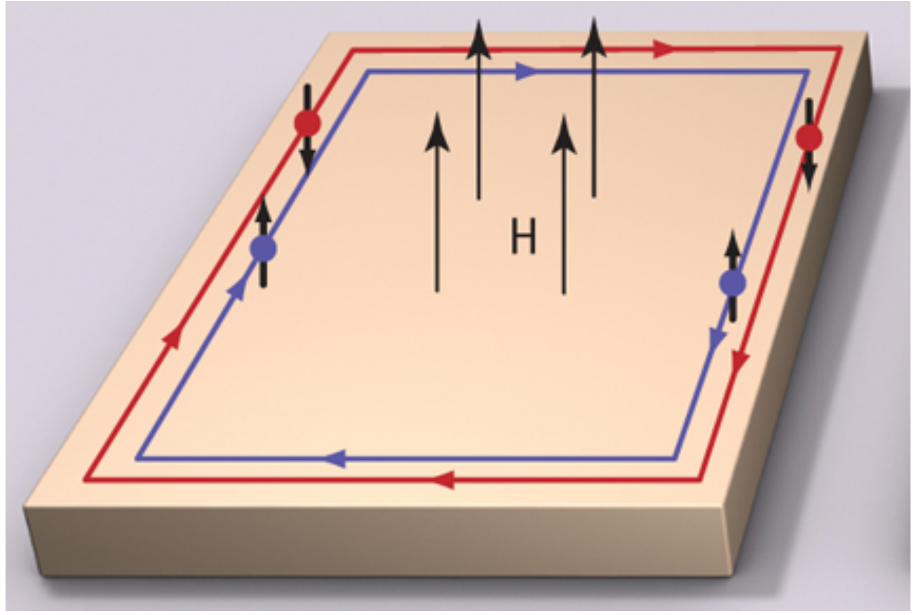


$$\sigma_{xy} = \frac{e^2}{2\pi}\nu$$

[Halperin, Laughlin]

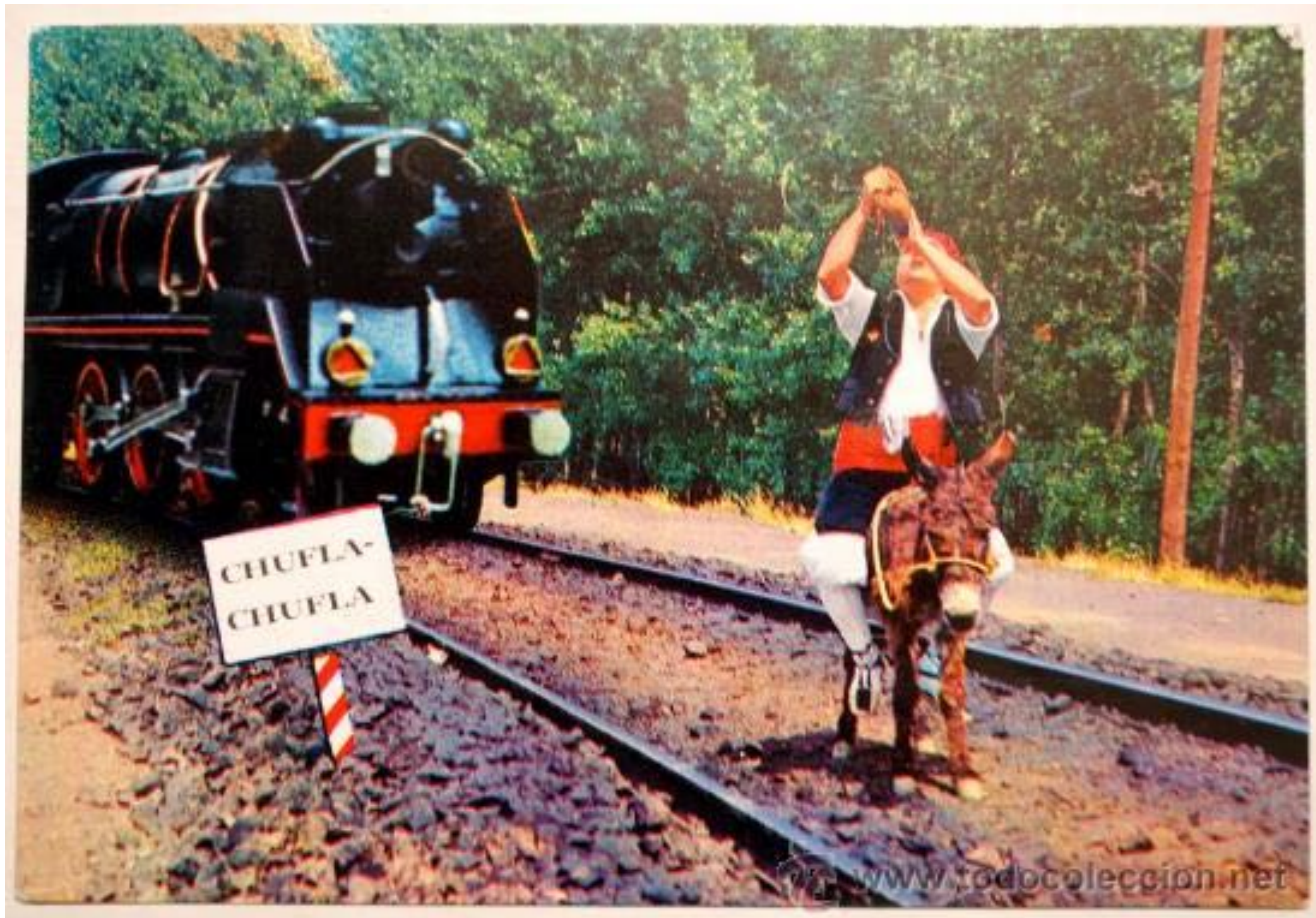
CHERN INSULATOR

Bulk insulators
Edge conductors



Integer Quantum Hall Effect

Anderson Localization

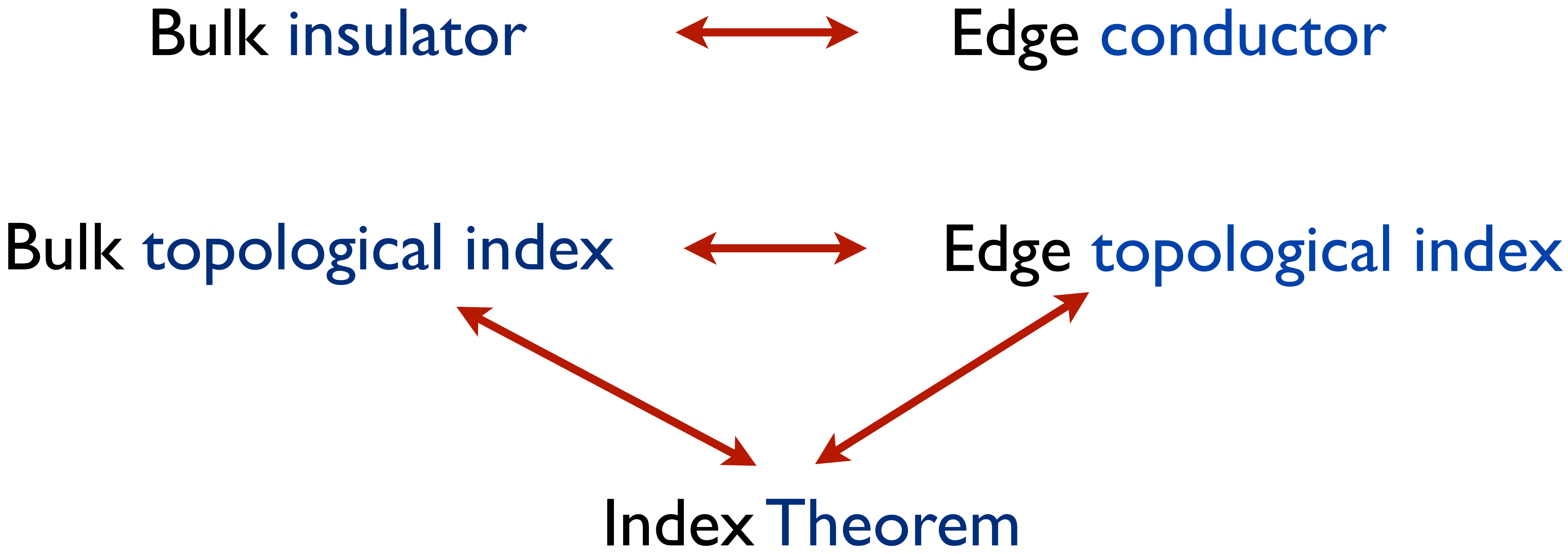


Does not affect Hall edge states

Boundary Insulators

Boundary interactions $\Rightarrow$ Anderson localization

BULK-EDGE DUALITIES
HALL EFFECT



Hall effect and Dirac operator

$$\not{D}_A = i\sigma_1(\partial_1 + ieA_1) + i\sigma_2(\partial_2 + ieA_2)$$

$$\not{D}_A^2 = -(\partial_1 + ieA_1)^2 - (\partial_2 + ieA_2)^2 + e\sigma_3 B$$

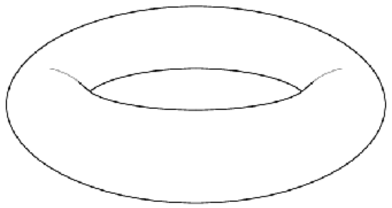
$$H = -D_A^2 = \not{D}_A^2 - eB\sigma_3$$

$E_n = eB(2n+1) \quad (n = 1, 2, \ldots)$ 2ν degeneracy

$E_0 = eB$ ν degeneracy

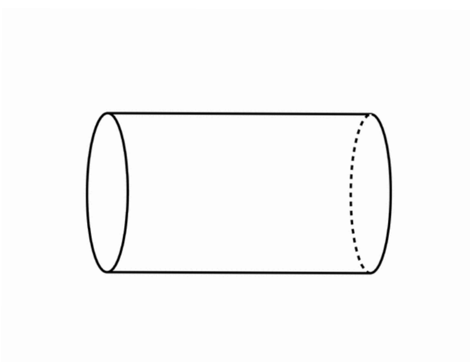
Index Theorem: $\nu = \dfrac{1}{2\pi}\displaystyle\int_{T^2} F = 2\pi B$

Atiyah-Patodi-Singer theorem



$$\nu_+ - \nu_- = \frac{1}{2\pi}\int_{T^2} F \in \mathbb{Z}$$ A-S

Edge - Bulk



$$\nu_+ - \nu_- = \frac{1}{2\pi}\int_{C} F - \frac{1}{2}\eta(A)$$ A-P-S

$$\eta(A) = 2\int_{\partial C} A - 2\,\mathrm{Int}\left[\int_{\partial C} A\right]$$

Spectral asymmetry

[M.A., López, García-Álvarez]

Edge states and Bulk-Edge correspondence

Chern-class on the cylinder C_1 is not any more an integer but

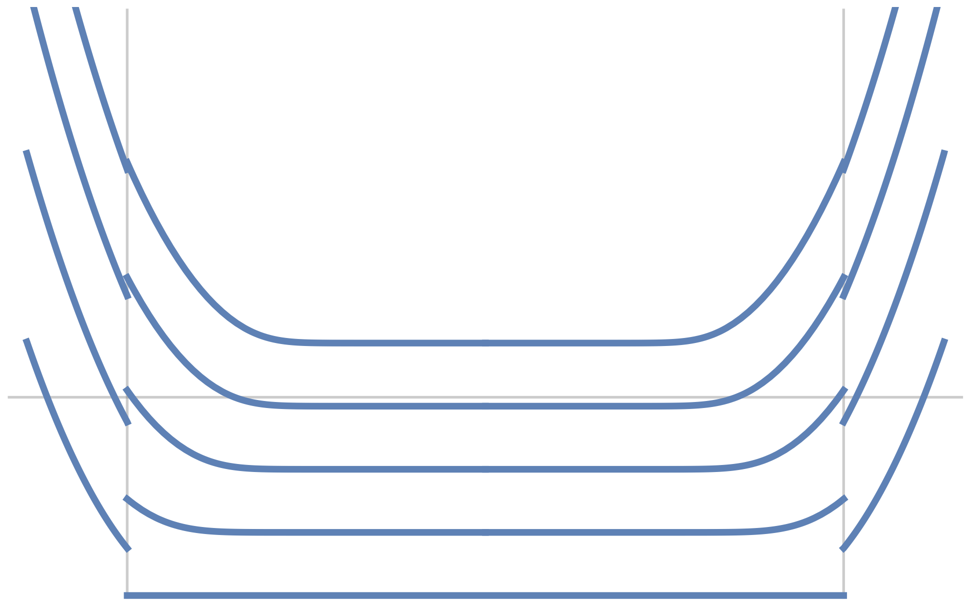
$$[C_1] = \nu_+ - \nu_- = \nu$$

is an integer quantum number

Edge states are chiral, due to the TR violation introduced by the magnetic field

Finite size Hall effects

Atiyah-Patodi-Singer boundary conditions

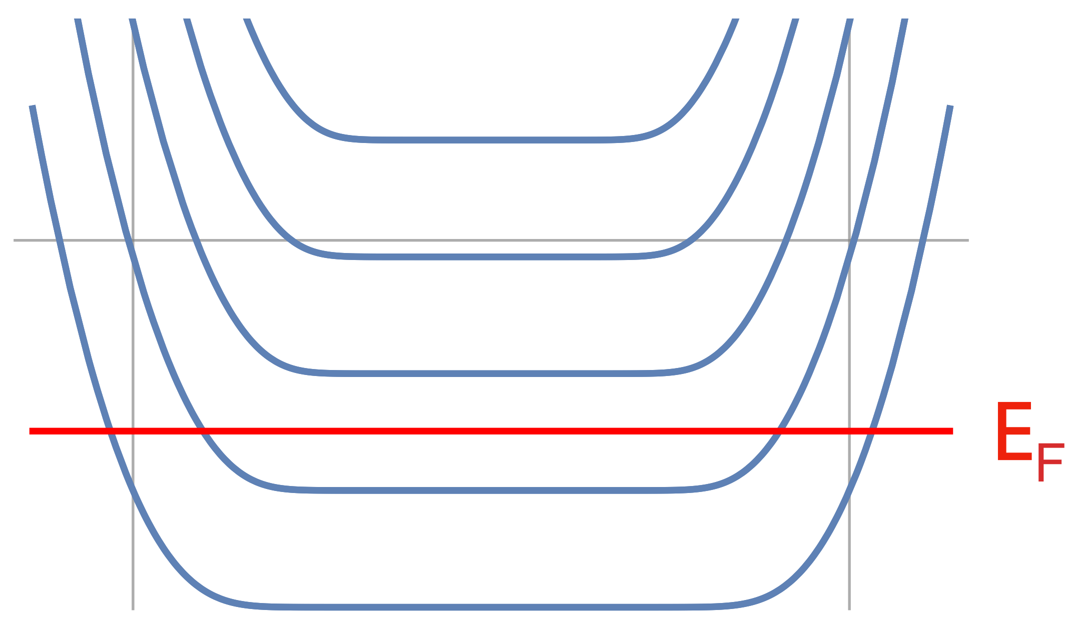


Non local and non physical

Finite size Hall effects

$$\not{D}_A^2 = -(\partial_1 + ieA_1)^2 - (\partial_2 + ieA_2)^2 + e\sigma_3 B$$

Chiral Boundary conditions: Dirichlet

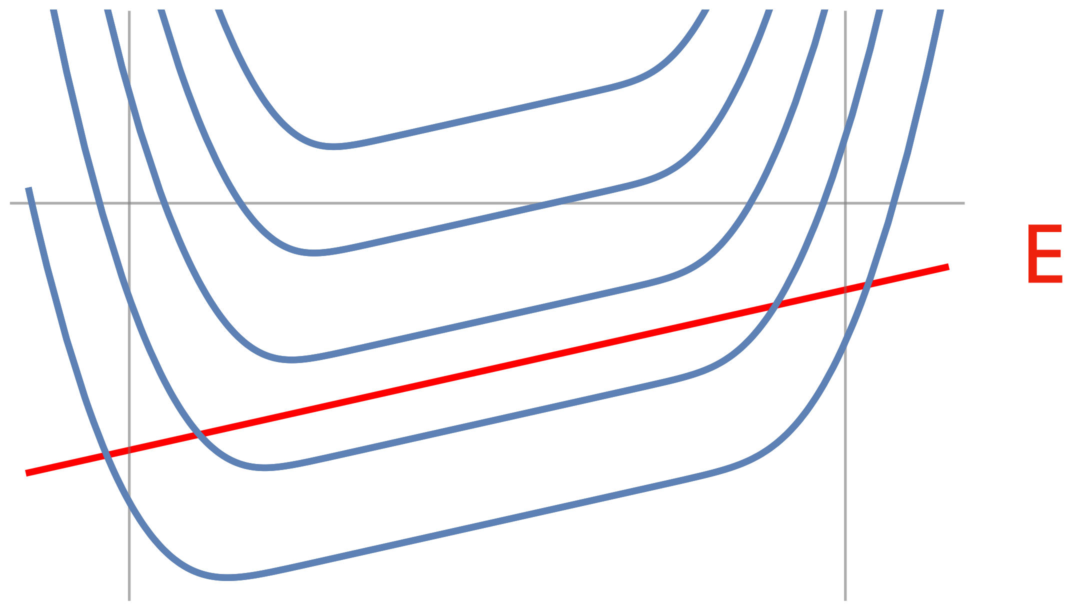


Non-chiral Edge states

[M.A.-Balachandran-Pérez-Pardo]

External Electric Field E

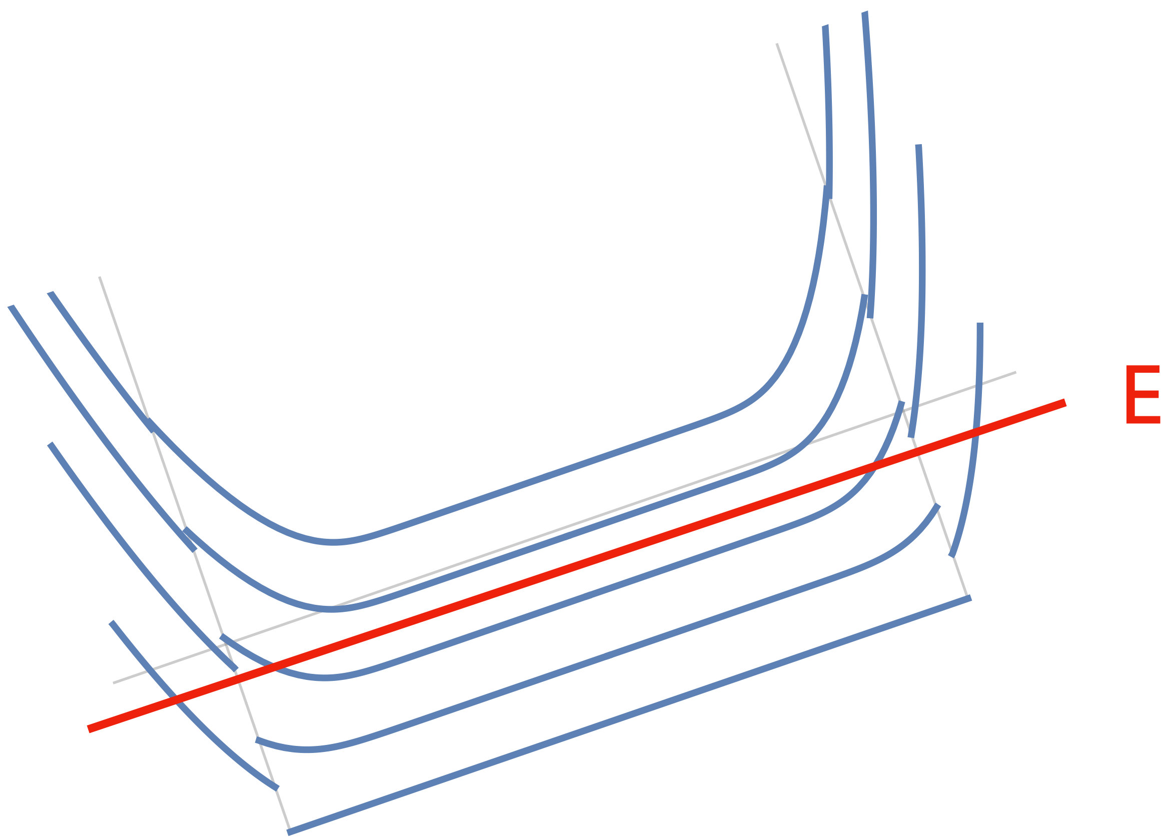
$$\not{D}_A^2 + eEx_1 = -(\partial_1 + ieA_1)^2 - (\partial_2 + ieA_2)^2 + e\sigma_3 B + eEx_1$$



$$\sigma_{xy} = \frac{I_\varphi}{2\pi E} = \sum_{n=0}^{\nu} \int_{\epsilon_n^-}^{\epsilon_n^+} d\epsilon_1 \frac{\partial_{\epsilon_1} E_n}{2\pi E} = \sum_{n=0}^{\nu} \frac{E_n(\epsilon_n^+) - E_n(\epsilon_n^-)}{2\pi E} = \frac{e^2}{2\pi}\,\nu$$

Finite size Hall effects

Atiyah-Patodi-Singer Boundary conditions



$$\sigma_{xy} \neq \frac{e^2}{2\pi}\nu$$

[M.A.]

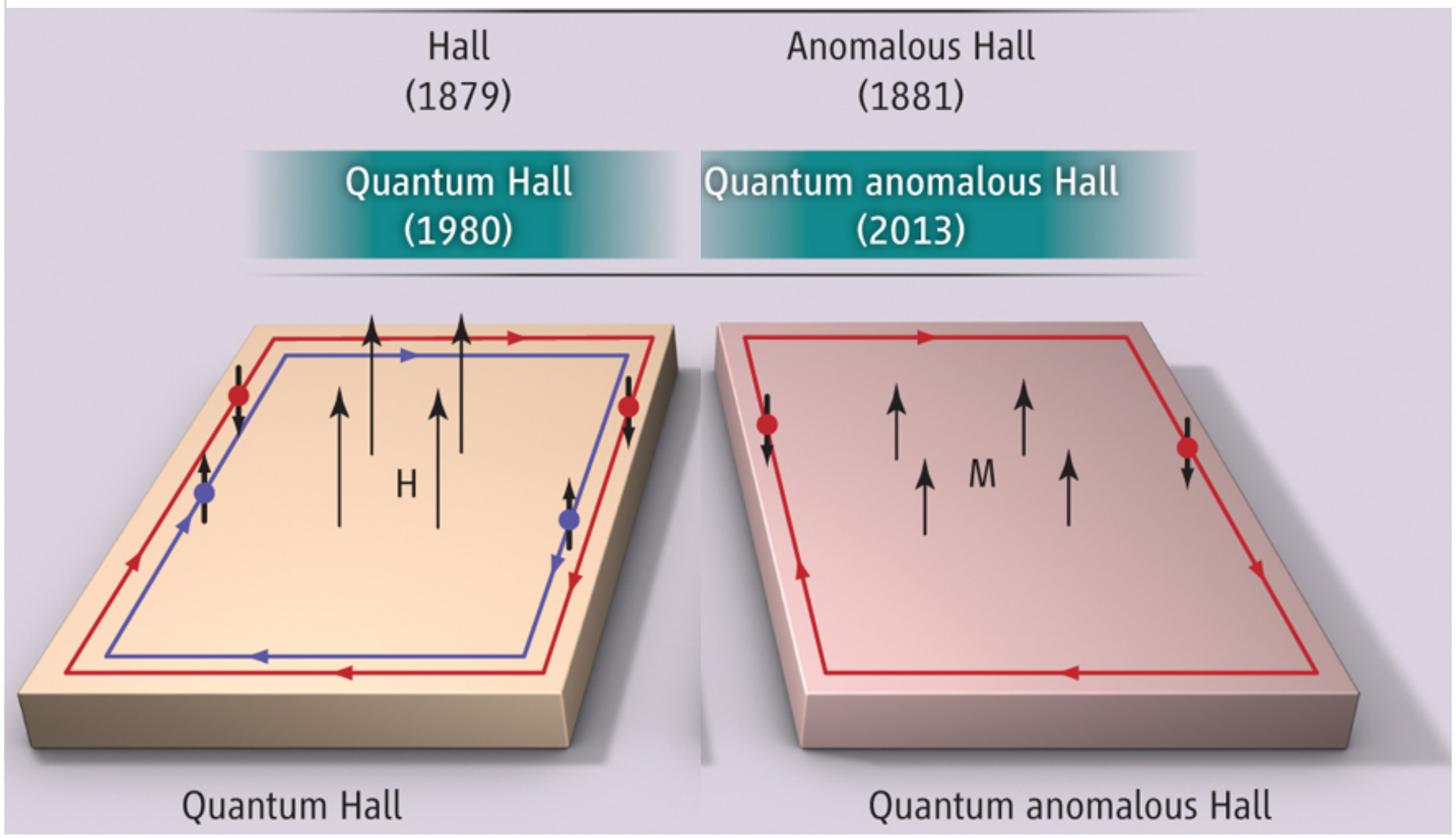
Edge states and Bulk-Edge correspondence

- The correspondence holds for local boundary conditions
  Bulk: Chern class of FM transform
  upper nearest Chern-class pbc
  ↕
  Edge: number of edge states per side below Fermi level
- Fails for APS non-local boundary conditions
- Can edge states survive without magnetic field?

YES

Anomalous Quantum Hall effect

New Quantum Hall Effect



Congratulations
Mihail

TOPOLOGICAL INSULATORS

Truncation to a finite number of bands

$$L^2(\mathbb{R}^2)\left(\widehat{\mathbb{T}}^2, L^2(\mathbb{T}^2)\right)$$

$$\downarrow$$

$$E(\mathbb{T}^2, \mathbb{C}^2)$$

$$\mathbb{H}(k) = h_\mu \sigma_\mu, \quad \sigma_0 = \mathbb{I}$$

$$E_\pm(k) = h_0(k) \pm \sqrt{\mathbf{h}(k)^2}$$

if $\mathbf{h}(k)^2 \neq 0$ there are no level crossings

Edge states and Bulk-Edge correspondence

The correspondence also holds

Bulk: Chern class of Bloch bundle= winding #
Hamiltonian spectral functions

↕

Edge: number of edge states per side below
Fermi level

- Can edge states survive without magnetic field?

YES

Anomalous Quantum Hall effect

TOPOLOGICAL INSULATORS

- Adding spin
- Two copies of Haldane model
- Spin-orbit coupling
- Time reversal symmetry
- Symmetry topological protection

Time Reversal and Kramers degeneracy

$s = \frac{1}{2}$ spin systems

$$\Theta\psi = e^{i\pi S_y}\psi^*$$

$$\Theta^2 = -I$$

Kramers theorem:

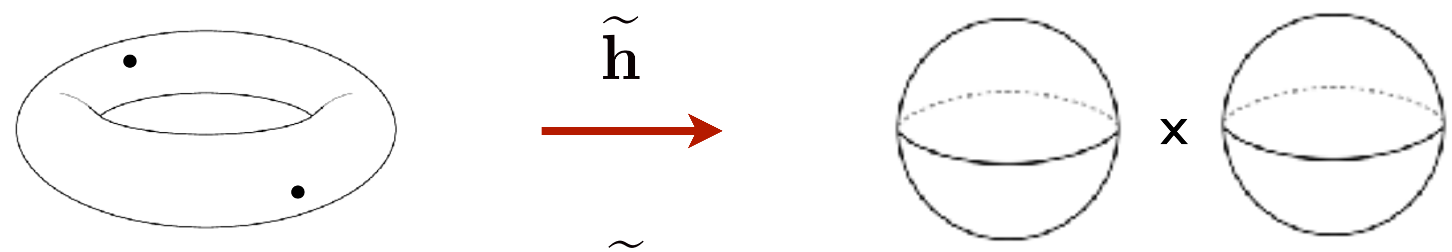
For a time reversal invariant Hamiltonian all energy levels are double degenerated at CP Kramers points

$$\Theta\psi = \lambda\psi, \qquad \Theta^2\psi = |\lambda|^2\psi = -\psi$$

For a non-degenerate energy level ψ

Nielsen-Ninomiya theorem: Lattice fermion doubling

Bulk topological index



$$\mathbb{T}^2_*/\mathbb{Z}_2 \xrightarrow{\widetilde{\mathbf{h}}} \mathbb{S}^2 \times \mathbb{S}^2$$

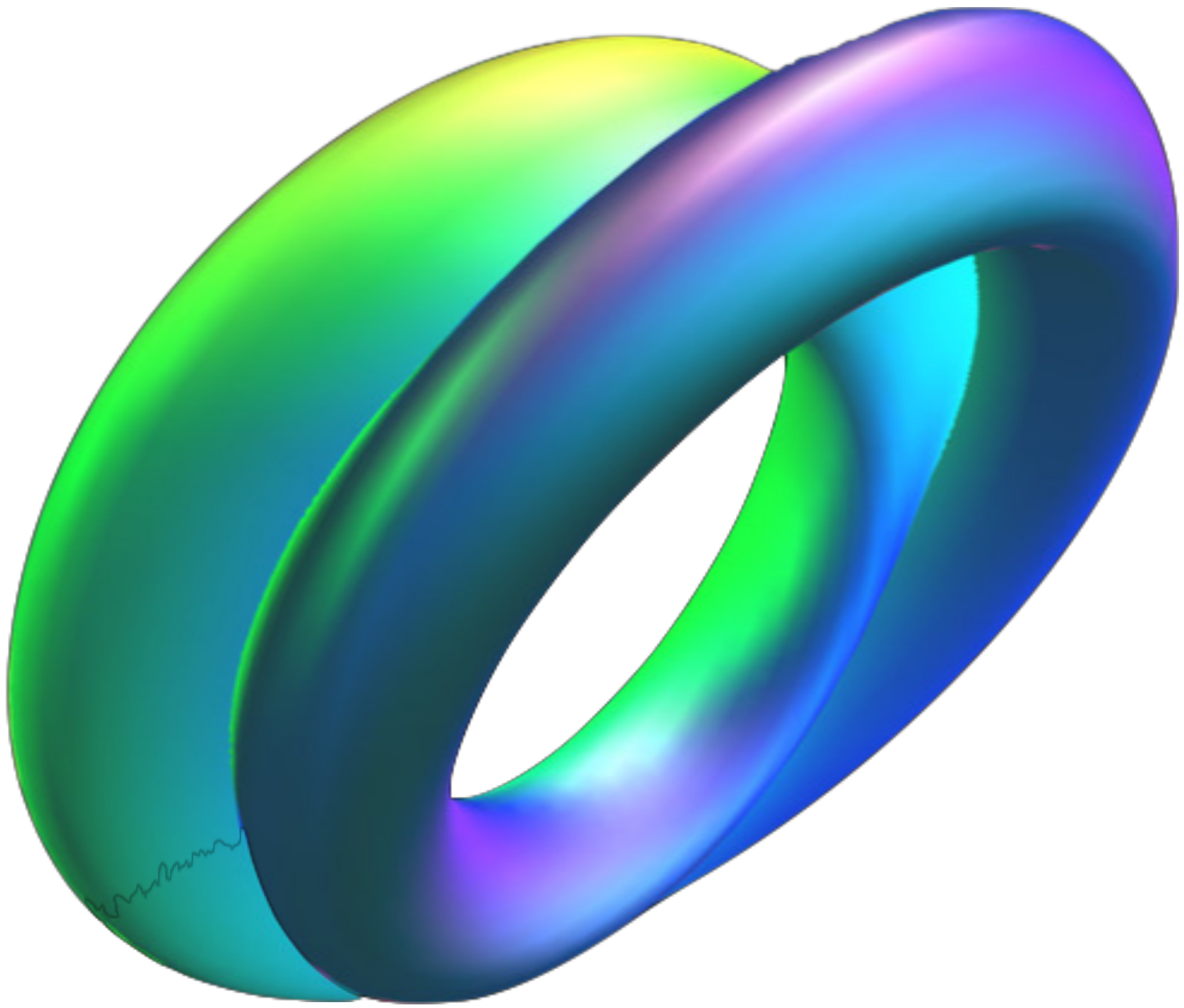
$$\mathbb{T}^2_* = \mathbb{T}^2 - \{(0,0),(0,\pi),(\pi,0),(\pi,\pi)\}$$

Vanishing Chern class

Non-trivial Bloch bundes

$$\nu \in \mathbb{Z}_2$$

Klein bottle

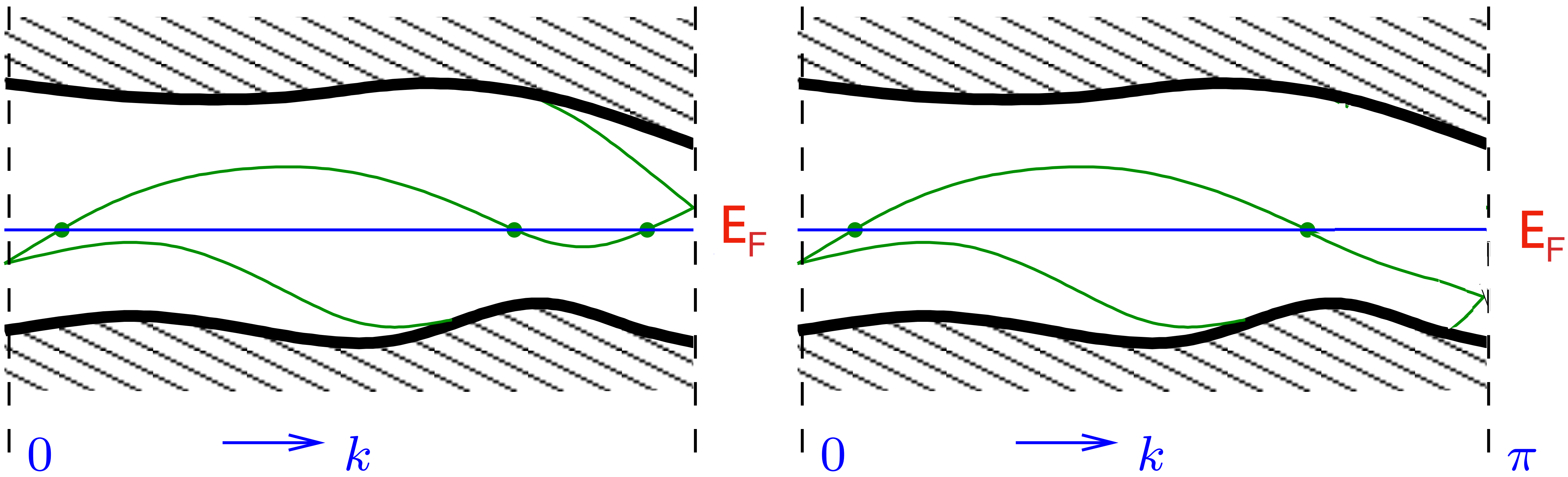


$\mathbb{T}^2_*/\mathbb{Z}_2$

Banchoff

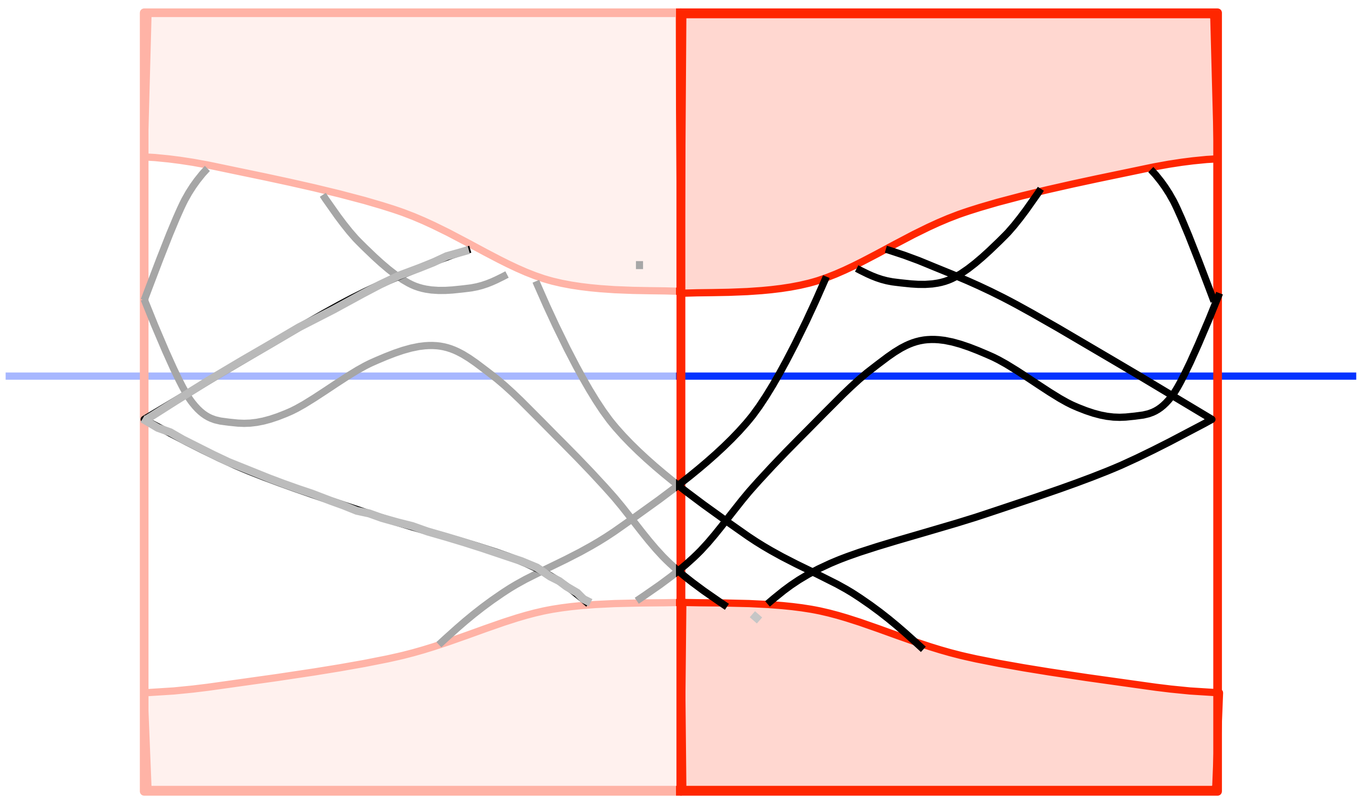
Topological Insulators

Edge Invariants [Kane-Mele]



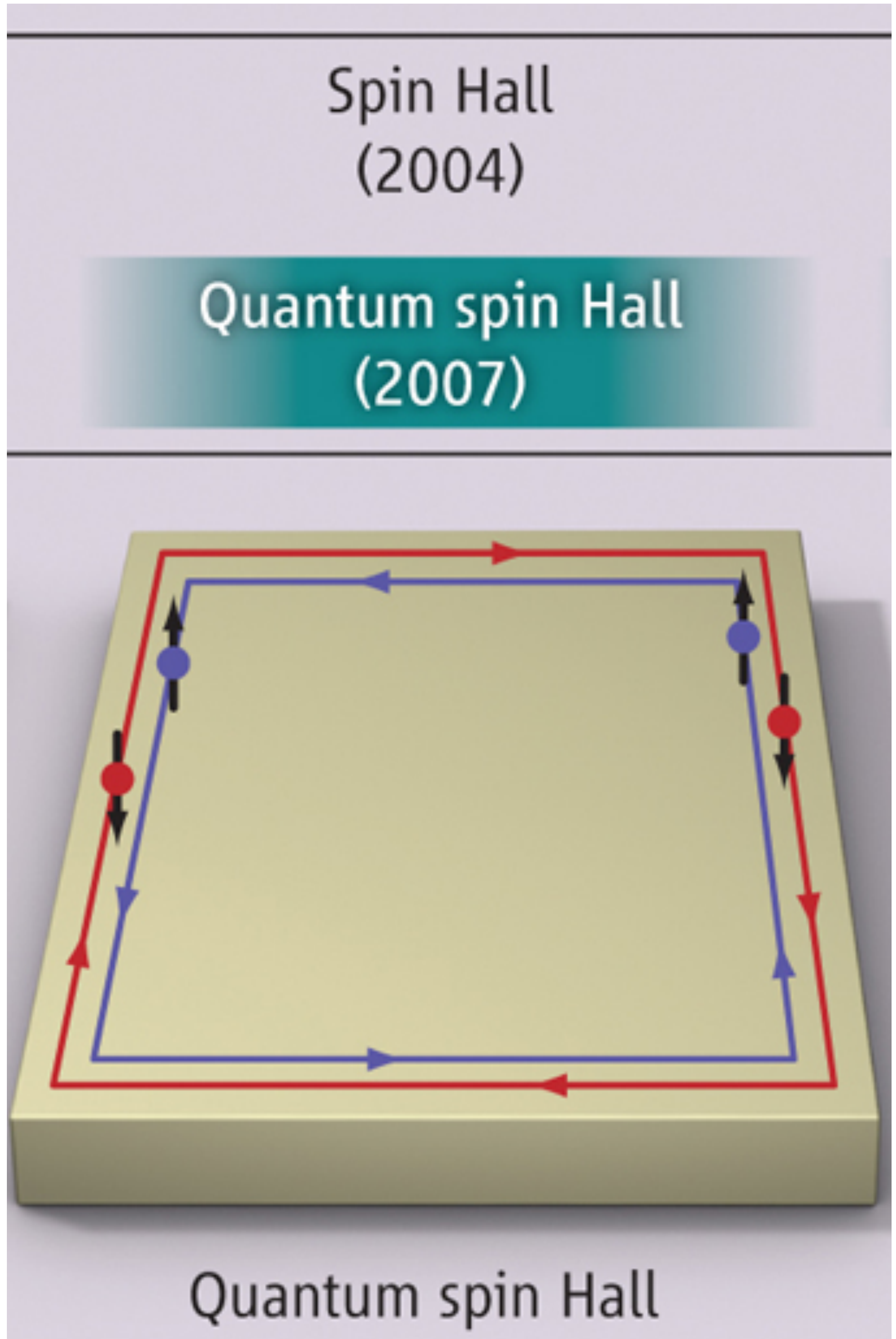
Topological Insulators **Normal Insulators**

Kane-Mele Z_2 index



$Z_2 = N_{cross} \pmod 2 = \text{Invariant}$

[Fu-Kane]


Spin Hall
(2004)
Quantum spin Hall
(2007)
Quantum spin Hall

BULK-EDGE DUALITIES
TOPOLOGICAL INSULATORS

Bulk insulators ↔ Edge conductors

Bulk topological index ↔ Edge topological index

Index Theorems

SUMMARY

- Bulk-edge correspondence in topological insulators and semimetals
- Atiyah-Patodi-Singer theorem: inspiring idea but not always working
- Bulk-edge dualities are closer to ordinary differential calculus bulk-edge theorems
- Bulk-edge dualities beyond APS and holography
- Possible applications to axion physics