Coherent Gamow states for the hyperbolic Pöschl-Teller potential.

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The Schrödinger equation

We begin with the one dimensional Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{2m}\frac{\alpha^2\left(\lambda(\lambda-1)\right)}{\cosh^2 \alpha x}\,. \tag{1}$$

Three possibilities

- $\lambda > 1$, potential barrier.
- $\frac{1}{2} \leq \lambda < 1$, low barrier.
- $\lambda = \frac{1}{2} + i\ell$, $\ell > 0$, high barrier.

The Schrödinger equation is

$$U''(x) + \left[k^2 + \frac{\lambda(\lambda-1)}{\cosh^2 x}\right]U(x) = 0\,, \quad k^2 = \frac{2mE}{\hbar^2}\,. \tag{2}$$

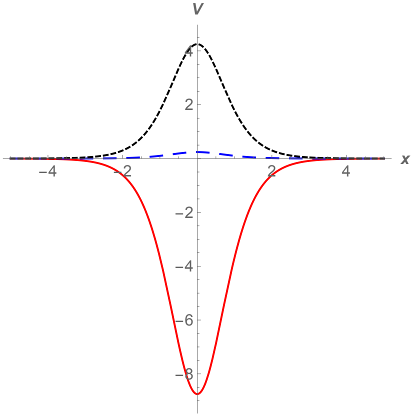

Figure: RED: Potential barrier. BLUE: Low barrier. BLACK: High barrier.

Let us introduce the new variable y

$$y(x) := \tanh x\,, \tag{3}$$

and the new function $\nu(y)$

$$U(y) = (1+y)^r(1-y)^{-r}\nu(y)\,, \qquad r := \frac{ik}{2}\,. \tag{4}$$

Then, the Schrödinger equation becomes a Jacobi equation:

$$(1-y)^2\nu''(y) + [2ik]\,\nu'(y) + \lambda(\lambda-1)\nu(y) = 0\,. \tag{5}$$

A hypergeometric equation can be obtained by using the following change of variables:

$$z := \frac{y+1}{2}\,, \tag{6}$$

so that

$$z(1-z)\nu''(z) + [ik - 2z + 1]\,\nu'(z) + [\lambda(\lambda-1)]\nu(z) = 0\,. \tag{7}$$

The general solution $U(x)$ of the original Schrödinger equation is a linear combination of products of

$$(1+\tanh x)^{ik/2}\,(1-\tanh x)^{-ik/2}$$

times hypergeometric functions on the variable $(1+\tanh x)/2$, for which the asymptotic form is well known.

$$U(x) = A\,(1+\tanh x)^{ik/2}\,(1-\tanh x)^{-ik/2}$$

$$\times {}_2F_1\left(\lambda, 1-\lambda, ik+1; \frac{1+\tanh x}{2}\right)$$

$$+B\,2^{ik}\,(1+\tanh x)^{ik/2}\,(1-\tanh x)^{-ik/2}$$

$${}_2F_1\left(\lambda-ik, 1-\lambda-ik, 1-ik; \frac{1+\tanh x}{2}\right). \tag{8}$$

The asymptotic forms are:

- For $x \longmapsto \infty$:

$$U^+(x) = A'\,e^{ikx} + B'\,e^{-ikx}$$

- For $x \longmapsto -\infty$:

$$U^-(x) = A\,e^{ikx} + B\,e^{-ikx}$$

Here,

$$A' = \frac{\Gamma(ik/\alpha+1)\Gamma(ik\alpha)}{\Gamma(ik/\alpha+1-\lambda)\Gamma(ik/\alpha+\lambda)}\,A + \frac{\Gamma(1-ik/\alpha)\Gamma(ik/\alpha)}{\Gamma(1-\lambda)\Gamma(\lambda)}\,B\,, \tag{9}$$

and

$$B' = \frac{\Gamma(ik/\alpha+1)\Gamma(-ik\alpha)}{\Gamma(1-\lambda)\Gamma(\lambda)}\,A + \frac{\Gamma(1-ik/\alpha)\Gamma(-ik/\alpha)}{\Gamma(\lambda-ik/\alpha)\Gamma(1-\lambda-ik/\alpha)}\,B\,. \tag{10}$$

S and T matrices

The S matrix is given by

$$\begin{pmatrix} B \\ A' \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ B' \end{pmatrix}, \tag{11}$$

and the T transfer matrix is

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{12}$$

Their relation is

$$S = \frac{1}{T_{22}} \begin{pmatrix} -T_{12} & 1 \\ T_{11}T_{22} - T_{12}T_{21} & T_{12} \end{pmatrix} \tag{13}$$

Purely outgoing states are characterized by the condition $A = B' = 0$.

Form of the transfer matrix

$$T = \begin{pmatrix} \dfrac{\Gamma(ik+1)\,\Gamma(ik)}{\Gamma(ik+1-\lambda)\,\Gamma(ik+\lambda)} & \dfrac{\Gamma(1-ik)\,\Gamma(ik)}{\Gamma(1-\lambda)\,\Gamma(\lambda)} \\ \dfrac{\Gamma(ik+1)\,\Gamma(-ik)}{\Gamma(1-\lambda)\,\Gamma(\lambda)} & \dfrac{\Gamma(1-ik)\,\Gamma(-ik)}{\Gamma(1-\lambda-ik)\,\Gamma(\lambda-ik)} \end{pmatrix} \tag{14}$$

High barrier Pöschl-Teller potential

Here, $\lambda = \frac{1}{2} + \ell$, $\ell > 0$.

The reflection and transmission coefficients are, respectively,

$$R = \left| \frac{T_{21}}{T_{22}} \right|^2 = \left| \frac{\Gamma(ik)\,\Gamma(\lambda - ik)\,\Gamma(1-\lambda-ik)}{\Gamma(-ik)\,\Gamma(\lambda)\,\Gamma(1-\lambda)} \right|^2 = \frac{\cosh^2(\pi\ell)}{\cosh^2(\pi k) + \sinh^2(\pi\ell)}\,, \quad (15)$$

and

$$T = \left| \frac{1}{T_{22}} \right|^2 = \left| \frac{\Gamma(\lambda - ik)\,\Gamma(1-\lambda-ik)}{\Gamma(1-ik)\,\Gamma(-ik)} \right|^2 = \frac{\sinh^2(\pi k)}{\cosh^2(\pi k) + \sinh^2(\pi\ell)}\,. \quad (16)$$

Poles of the S matrix

Purely outgoing states of the Schrödinger equation are those with asymptotic behavior consisting in outgoing waves to the right and to the left of the potential range. This means that $A = B' = 0$.

Since $B' = T_{12}\, A + T_{22}\, B$, purely outgoing states are characterized by $T_{22} = 0$.

Thus, purely outgoing states are characterized by the poles of the S-matrix, called *resonance poles*.

They are solutions of the equation

$$\cosh^2(\pi k) + \sinh^2(\pi \ell) = 0\,. \tag{17}$$

There are two series of solutions

$$k_1(n) = \ell - i\left(n + \frac{1}{2}\right) , \qquad k_2(n) = -\ell - i\left(n + \frac{1}{2}\right) , \tag{18}$$

with $n = 0, 1, 2, \ldots$.

In the energy representation, we have that

$$z(n) = \frac{\hbar^2}{2m}\, k_1^2(n) = E(n) - i\frac{\Gamma(n)}{2} , \tag{19}$$

$$z^*(n) = \frac{\hbar^2}{2m}\, k_2^2(n) = E(n) + i\frac{\Gamma(n)}{2} . \tag{20}$$

with

$$E(n) = \frac{\hbar^2}{2m}\,(\ell^2 - \gamma_n)^2 , \quad \Gamma(n) = \frac{\hbar^2}{2m}\, 4\ell\,\gamma_n , \quad \gamma_n = n + \frac{1}{2} . \tag{21}$$

Ladder operators for the hyperbolic Pöscl-Teller potential

The ladder operators for the bound states corresponding to the hyperbolic Pöschl-Teller potential with $\lambda > 1$ are given by ($\hbar^2/2m = 1$)

$$B_n^- = -\cosh x\,\partial_x - \sqrt{-E(n)}\,\sinh x\,, \tag{22}$$

$$B_{n+1}^+ = \cosh x\,\partial_x - \sqrt{-E(n)}\,\sinh x\,, \tag{23}$$

with $E(n) = -(\lambda - n - 1)^2$, $n = 0, 1, 2, \ldots, [\lambda - 1]$, where $[a]$ is the entire part of a.

Here,

$$B_n^- : \psi_n(x) \longmapsto \psi_{n-1}(x)\,, \quad B_n^+ : \psi_{n-1}(x) \longmapsto \psi_n(x)\,. \tag{24}$$

Ansatz

We define the annihilation and creation operators for the resonance states (Gamow vectors) as a straightforward generalization of the above as (for $k_1(n)$)

$$B_n^- = -\cosh x \, \partial_x + (i\ell + n + \frac{1}{2}) \sinh x \,, \tag{25}$$

$$B_n^+ = \cosh x \, \partial_x + (i\ell + n + \frac{1}{2}) \sinh x \,, \tag{26}$$

with $n = 0, 1, 2, \dots$.

For $k_2(n)$, we replace ℓ by $-\ell$. Annihilation and creation operators are now, respectively,

$$C_n^- = \cosh x\,\partial_x + (-i\ell + n + \frac{1}{2})\sinh x \tag{27}$$

$$C_n^+ = \cosh x\,\partial_x + (-i\ell + n + \frac{1}{2})\sinh x\,. \tag{28}$$

Gamow vectors

A decaying state $\psi = \varphi^D + \psi^B$ is the sum of two contributions, φ^D, which decays exponentially at all times $t \geq 0$ and the background, ψ^B, which is responsible for the deviations of the exponential decay law. These deviations for very short (Zeno) and very long times (Khalfin) are difficult to be observed. The contribution φ^D is often called the decaying Gamow vector

$$e^{-itH}\,\varphi^D = e^{-iE_Rt}\,e^{-\Gamma t}\,\varphi^D\,, \qquad t \geq 0\,. \tag{29}$$

Along the decaying Gamow vector, for each resonance also exists a growing Gamow vector which decays to the past:

$$e^{-itH}\,\varphi^G = e^{-iE_Rt}\,e^{\Gamma t}\,\varphi^G\,, \qquad t \leq 0\,. \tag{30}$$

In order to construct the respective Gamow vectors for both series of resonance poles, we set for $k_1(0)$:

$$B_0^- \, \varphi_0^D(x) = 0 \Longrightarrow \varphi_0^D(x) = (\cosh x)^{i\ell+1/2} \,. \tag{31}$$

For $k_2(0)$ we just have to replace ℓ by $-\ell$ so that

$$C_0^- \, \varphi_0^G = 0 \Longrightarrow \varphi_0^G(x) = (\cosh x)^{-i\ell+1/2} \,. \tag{32}$$

Then, we apply successively the corresponding creation operators, so as to obtain for any value of n, two series of Gamow states:

$$\varphi_n^D(x) = P_n(\sinh x)^n \, \varphi_0^D \,, \quad \varphi_n^G(x) = P_n(\sinh x)^n \, \varphi_0^G \,. \tag{33}$$

Property:

$$H\varphi_n^D(x) = \frac{\hbar^2}{2m} \, [k_1(n)]^2 \, \varphi_n^D(x) \,, \quad H\varphi_n^G(x) = \frac{\hbar^2}{2m} \, [k_2(n)]^2 \, \varphi_n^G(x) \,. \tag{34}$$

Note that

$$B_n^- \, \varphi_n^D = \varphi_{n-1}^D \,, \qquad B^+ \, \varphi_{n-1}^B = \varphi_n^D \,, \tag{35}$$

and

$$C_n^- \, \varphi_n^G = \varphi_{n-1}^G \,, \qquad C_n^+ \, \varphi_{n-1}^G = \varphi_n^G \,. \tag{36}$$

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Ladder operators

Let us define the following ladder operators

$$B^- \, \varphi_n^D := \sqrt{n} \, B_n^- \, \varphi_n^D \,, \quad B^+ \, \varphi_n^D := \sqrt{n+1} \, B_n^+ \, \varphi_n^D \tag{37}$$

and

$$C^- \, \varphi_n^G := \sqrt{n} \, C_n^- \, \varphi_n^G \,, \quad C^+ \, \varphi_n^G := \sqrt{n+1} \, C_n^+ \, \varphi_n^G \,, \tag{38}$$

for all $n = 0, 1, 2, \ldots$

Coherent states

We want to obtain two families of coherent states:

$$B^{-}\,|z^D\rangle = z\,|z^D\rangle\,, \qquad C^{-}\,|z^G\rangle = z\,|z^G\rangle\,, \qquad \forall\, z \in \mathbb{C}\,. \tag{39}$$

Let us write:

$$|z^D\rangle = \sum_{n=0}^{\infty} c_n\,\varphi_n^D\,, \qquad |z^G\rangle = \sum_{n=0}^{\infty} d_n\,\varphi_n^D\,. \tag{40}$$

Then,

$$B^{-}\,|z^D\rangle := \sum_{n=1}^{\infty} c_n\,\sqrt{n}\,\varphi_{n-1}^D = z\sum_{n=0}^{\infty} c_n\,\varphi_n^G\,, \tag{41}$$

which implies that

$$c_n = \frac{c_0}{\sqrt{n!}}\,z^n\,. \tag{42}$$

Rigged Hilbert spaces

Both $|z^D\rangle$ and $|z^G\rangle$ are continuous antilinear functionals defined on the test spaces Φ_+ and Φ_-, respectively, of rigged Hilbert spaces

$$\Phi_\pm \subset \mathcal{H} \subset \Phi_\pm^\times \,, \tag{43}$$

so that $|z^D\rangle \in \Phi_+^\times$ and $|z^G\rangle \in \Phi_-^\times$.

To construct the spaces $\Phi_\pm$, we have used Hardy functions on a half plane so that for any $\phi_+ \in \Phi_+$, the bracket $\langle\phi_+|\varphi_n^D\rangle$ is the value at $z(n)$ ($\text{Im } z(n) = -\Gamma(n)/2 < 0$) of a Hardy function on the lower half plane and for any $\phi_- \in \Phi_-$ the bracket $\langle\phi_-|\varphi_m^G\rangle$ is the value at $z^*(n)$ ($\text{Im } z^*(n) = \Gamma(n)/2 > 0$) of a Hardy function on the upper half plane.

Convergence

If $\phi(z)$ is a Hardy function on either half plane, for large values of $|z|$, one has that

$$|\phi(z)| \approx |z|^{-1/2} \,. \tag{44}$$

The resonance poles in the energy representation are located at

$$z^*(n) = \frac{\hbar^2}{2m}\,[k_1(n)]^2\,, \qquad z(n) = \frac{\hbar^2}{2m}\,[k_2(n)]^2\,, \tag{45}$$

for $n = 0, 1, 2, \ldots$.

Let us write,

$$\langle\phi_+|\varphi_n^D\rangle = \phi_+^*(z(n))\,, \qquad \langle\phi_-|\varphi_n^G\rangle = \phi_-^*(z^*(n))\,. \tag{46}$$

Then, as a consequence of the previous transparency,

$$\lim_{n\to\infty} |\phi_+(z(n))| = 0\,, \qquad \lim_{n\to\infty} |\phi_-(z^*(n))| = 0\,, \tag{47}$$

so that the sequences $|\phi_+(z(n))|$ and $|\phi_-(z^*(n))|$ are bounded by a positive constant K for all values of n. Then, for any $\phi_+ \in \Phi_+$, we have

$$|\langle\phi_+|z^D\rangle| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\,|\langle\phi_+|\varphi_n^D\rangle| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\,|\phi_+(z(n))| \leq \sum_{n=0}^{\infty} \frac{K}{\sqrt{n!}} < \infty\,. \tag{48}$$

Resolutions of the identity

Consider the following expression:

$$\int_{\mathbb{C}} |z^D\rangle\langle z^G| \, \frac{1}{\pi} \, e^{-|z|^2} \, dz \,, \tag{49}$$

with $dz = dx\,dy$, $z = x + iy$. Multiplying to the left by arbitrary $\varphi_+ \in \Phi_+$ and to the right by arbitrary $\psi_- \in \Phi_-$, we have

$$\int_{\mathbb{C}} \langle\varphi_+|z^D\rangle\langle z^G|\psi_-\rangle \, \frac{1}{\pi} \, e^{-|z|^2} \, dz \,. \tag{50}$$

Here,

$$\langle\varphi_+|z^D\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \, \langle\varphi_+|\varphi_n^D\rangle \,, \quad \langle z^G|\psi_-\rangle = \sum_{n=0}^{\infty} \frac{(z^m)^*}{\sqrt{n!}} \, \langle\varphi_n^G|\psi_-\rangle \,, \tag{51}$$

so that,

$$\int_{\mathbb{C}} \langle \varphi_+ | z^D \rangle \langle z^G | \psi_- \rangle \, \frac{1}{\pi} \, e^{-|z|^2} \, dz =$$

$$= \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!}\,\sqrt{m!}} \, \langle \phi_+ | \varphi_n^D \rangle \langle \varphi_m^G | \psi_- \rangle \, \frac{1}{\pi} \int_{\mathbb{C}} z^n (z^*)^m \, e^{-|z|^2} \, dz$$

$$= \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!}\,\sqrt{m!}} \, \langle \phi_+ | \varphi_n^D \rangle \langle \varphi_m^G | \psi_- \rangle \, n! \, \delta_{n,m} = \sum_{n=0}^{\infty} \langle \phi_+ | \varphi_n^D \rangle \langle \varphi_n^G | \psi_- \rangle \, . \quad (52)$$

Then, omitting the arbitrary $\phi_+ \in \Phi_+$ and $\psi_- \in \Phi_-$, one has the following identity:

$$\int_{\mathbb{C}} |z^D\rangle\langle z^G| \, \frac{1}{\pi} \, e^{-|z|^2} \, dz = \sum_{n=0}^{\infty} |\varphi_n^D\rangle\langle\varphi_n^G| =: I_- \,. \tag{53}$$

Here, $I_- : \Phi_- \longmapsto \Phi_+^\times$ is the canonical injection. Analogously,

$$\int_{\mathbb{C}} |z^G\rangle\langle z^D| \, \frac{1}{\pi} \, e^{-|z|^2} \, dz = \sum_{n=0}^{\infty} |\varphi_n^G\rangle\langle\varphi_n^D| =: I_+ \,, \tag{54}$$

where $I_+ : \Phi_+ \longmapsto \Phi_-^\times$.

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