Spectral asymmetry/the η invariant in Dirac Hamiltonians
A novel look at Fractional Fermi numbers in QFT

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7th International Workshop on New Challenges in Quantum Mechanics- Benasque, SPAIN, 2019

Talk elaborated a' l'honneur de Mikhail Plyushchay

Outline

1. Spectral asymmetry/the eta invariant from heat kernel methods
2. Fermi charge of 1D solitons: the Niemi-Semenoff formula from heat kernel methods
3. The non-Abelian Goldstone-Wilczek model
4. Fermi fractional charge of solitonic domain walls and Hall conductivity

Fermi charge on static backgrounds: spectral asymmetry

- The Dirac Hamitonian governing Fermionic fluctuations on static backgrounds

$$
\begin{aligned}
H &= -i\sum_{j=1}^{n}\alpha^j D_j + \beta\varphi_1 + i\beta\gamma^{n+2}\varphi_2 \\
D_j &= \nabla_j - ieA_j(x^1,\cdots,x^n) \quad , \quad \varphi_1(x^1,\cdots,x^n) \;, \varphi_2(x^1,\cdots,x^n) \\
\beta^2 &= 1 \;,\; (\alpha^j)^2 = 1 \;,\; \beta\alpha^j + \alpha^j\beta = 0 \;,\; \alpha^j\alpha^l + \alpha^l\alpha^j = 0 \;,\; j,l = 1,2,\cdots,n \\
\gamma^0 &= \beta \;,\; \gamma^j = \beta\alpha^j \;,\; \{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu} \;,\; \mu,\nu = 0,1,\cdots,n \\
\gamma^{n+2} &= i\gamma^0\gamma^1\cdots\gamma^n \quad , \quad \text{focus} \quad n = 1,3,5,\cdots
\end{aligned}
$$

- Dirac spinor quantized fields:

$$H\psi_\lambda(x^1,\cdots,x^n) = \lambda\psi_\lambda(x^1,\cdots,x^n) \quad , \quad \lambda\in\mathbb{R}$$

$$\hat{\Psi}(x^0,x^1,\cdots,x^n) = \sum\!\!\!\!\!\!\int_{\lambda>0} [d\lambda]\,\hat{a}_\lambda e^{-i\lambda x^0}\psi_\lambda(x^1,\cdots,x^n) + \sum\!\!\!\!\!\!\int_{\lambda<0} [d\lambda]\,\hat{b}^\dagger_\lambda e^{i\lambda x^0}\psi_\lambda(x^1,\cdots,x^n)$$

$$\{\hat{a}_\lambda,\hat{a}^\dagger_\mu\} = \text{"}\delta(\lambda-\mu)\text{"} = \{\hat{b}_\lambda,\hat{b}^\dagger_\mu\}$$

- Fermi number of the static background ground state

$$N = \frac{1}{2}\int dx^1\cdots dx^n\,\langle 0|[\hat{\Psi}(x^1,\cdots,x^n),\hat{\Psi}^\dagger(x^1,\cdots,x^n)]|0\rangle = -\frac{1}{2}\Big(\sum\!\!\!\!\!\!\int_{\lambda>0} 1 - \sum\!\!\!\!\!\!\int_{\lambda<0} 1\Big)$$

Spectral asymmetry and the spectral eta function

- If the spectrum is discrete the spectral η function is defined by analytic continuation in the s-complex plane to a meromorphic function from the series

$$\eta(s,H) = \sum_{\lambda>0} \lambda^{-s} - \sum_{\lambda<0} (-\lambda)^{-s},$$

which is convergent if Re s is large enough.Thus,

$$N = -\frac{1}{2}\eta(0,H)$$

- Mellin transform:From the spectral heat trace to the spectral eta function

$$\eta(s,H;\rho) = \mathrm{T}r\left(\rho\cdot(H^2)^{-s/2}H/|H|\right) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty dt\, t^{\frac{s-1}{2}}\,\mathrm{Tr}\left(\rho H e^{-tH^2}\right)$$

$\rho(x^1,\cdots,x^n)$: auxiliary function of compact support. From the density j^0, we obtain the global Fermi number N after integration

$$\eta(0,H;\rho) = -\frac{1}{2}\int d^n x\, j^0(x)\rho(x), \qquad \int d^n x\, j^0(x) = N$$

Eta function and the spectral heat trace expansion

- $\eta(s,H;\rho)$ may be expressed as

$$\eta(s,H;\rho) = -\frac{1}{2\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty dt\, t^{\frac{s-3}{2}}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\,\mathrm{Tr}\left(e^{-tH_\rho^2}\right) \quad \text{where} \quad H_\rho = H+\varepsilon\rho$$

Let L be the Laplace type operator

$$L(\rho,M^2) = H_\rho^2 - M^2. = -(\nabla^2+E)$$

$$E, \text{matrix} - \text{valued potential} \quad , \quad \nabla = \partial+\omega, \text{covariant derivative} \quad , \quad M, \text{auxiliary mass}$$

$$E = -\frac{ie}{4}F_{jk}[\gamma^j,\gamma^k] - i\gamma^j\partial_j\varphi_1 + \gamma^j\gamma^{n+2}\partial_j\varphi_2 + (M^2-\varphi_1^2-\varphi_2^2) - 2\beta\varepsilon\rho(\varphi_1+i\gamma^{n+2}\varphi_2)$$

$$\omega_j = -ieA_j + i\alpha_j\varepsilon\rho \quad , \quad \Omega_{jk} \equiv [\nabla_j,\nabla_k] = -ieF_{jk} + \alpha_k\varepsilon\partial_j\rho - \alpha_j\varepsilon\partial_k\rho$$

- Asymptotic expansion of the heat trace

$$\mathrm{Tr}\left(Qe^{-tL}\right) \simeq \sum_{k=0}^\infty t^{\frac{k-n}{2}} a_k(L,Q), \qquad t\to +0$$

$$\text{convergence at } t=0^+ \Rightarrow \frac{da_k(L)}{d\varepsilon}|_{\varepsilon=0} = 0 \text{ if } k\leq n+1$$

Basic facts about the heat trace coefficients a_k: (i) they all are integrals of traces of local polynomials constructed from E, Ω and their covariant derivatives ∇ (ii) terms depending on E only have the following simple form:

$$a_{2l}(L,Q) \sim \frac{1}{(4\pi)^{\frac{n}{2}}\, l!}\int d^nx\,\mathrm{tr}\left(QE^l\right) \quad , \quad a_2(L,Q) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int d^nx\,\mathrm{tr}\,(QE)$$

The eta invariant: High mass expansion

- Behaviour of η under small localized variations:δA and $\delta\varphi_{1,2}$

$$\delta H = -\alpha^j \delta A_j + \beta\delta\varphi_1 + i\beta\gamma^{n+2}\delta\varphi_2 \ \Rightarrow\ \delta\eta(0,H) = -\frac{2}{\sqrt{\pi}} a_{n-1}(H^2, \delta H) \ \Rightarrow\ \delta\eta(0,H) = 0$$

Therefore, $\eta(0,H)$ is a topological/homotopy invariant.

- Large mass/small derivatives expansion with respect to the parameter M^2 of the Fermi number

$$\eta(0,H;\rho) = -\frac{1}{2\sqrt{\pi}} \sum_k \Gamma\left(\frac{k-1-n}{2}\right) |M|^{n+1-k} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\, a_k\Big(L(\rho, M^2)\Big)$$

The Dirac Hamiltonian for 1D solitons

- The 1D Dirac Hamiltonian: $\varphi_1(x) = \Phi(x), \ \ \varphi_2(x) = \mu$

$$\alpha^1 = \sigma^2 \ , \ \beta = \sigma^1 \, ; \quad H = \begin{pmatrix} \mu & D \\ D^\dagger & -\mu \end{pmatrix} \quad , \quad D = -\frac{d}{dx} + \Phi(x) \quad , \quad \lim_{x\to\pm\infty} \Phi(x) = \pm\nu$$

$$H_\rho^2 = \begin{pmatrix} -\frac{d^2}{dx^2} - \frac{d\Phi(x)}{dx} + \Phi^2(x) + (\mu + \varepsilon\rho(x))^2 & -2\varepsilon\rho(x)\frac{d}{dx} - \epsilon\frac{d\rho(x)}{dx} + 2\epsilon\rho(x)\Phi(x) \\ 2\varepsilon\rho(x)\frac{d}{dx} + \epsilon\frac{d\rho(x)}{dx} + 2\epsilon\rho(x)\Phi(x) & -\frac{d^2}{dx^2} + \frac{d\Phi(x)}{dx} + \Phi^2(x) + (-\mu + \varepsilon\rho(x))^2 \end{pmatrix}$$

$$H_\rho^2 = \Delta + Q(x)\frac{d}{dx} + V(x) \quad , \quad \Delta = \begin{pmatrix} -\frac{d^2}{dx^2} + \mu^2 + \nu^2 & 0 \\ 0 & -\frac{d^2}{dx^2} + \mu^2 + \nu^2 \end{pmatrix}$$

$$V(x) = \begin{pmatrix} \Phi^2 - \nu^2 - \frac{d\Phi}{dx} + 2\mu\varepsilon\rho + \varepsilon^2\rho^2 & +\varepsilon\frac{d\rho}{dx} + 2\varepsilon\rho\Phi \\ -\varepsilon\frac{d\rho}{dx} + 2\varepsilon\rho\Phi & \Phi^2 - \nu^2 + \frac{d\Phi}{dx} - 2\mu\varepsilon\rho + \varepsilon^2\rho^2 \end{pmatrix} , \ Q(x) = \begin{pmatrix} 0 & 2\varepsilon\rho \\ 2\varepsilon\rho & 0 \end{pmatrix}$$

- The heat trace expansion

$$\mathrm{Tr}\left(e^{-\tau H_\rho^2}\right) = \sum_{k=0}^{\infty}\sum_{i=1}^{2} [c_k(H_\rho^2)]_{ii}\, e^{-\tau(\nu^2+\mu^2)}\, \frac{1}{\sqrt{4\pi}}\, \tau^{k-\frac{1}{2}} \quad , \quad c_k(H_\rho^2) = a_{2k}(H_\rho^2)$$

- Fermi number: integrate over τ, derivate with respect to ε, keep the ε independent terms and take the limit $s = 0$

$$N(H) = \frac{1}{8\pi}\sum_{k=0}^{\infty}\sum_{i=1}^{2} [\bar{c}_k(H_\rho^2)]_{ii}\,(\nu^2+\mu^2)^{1-k}\,\Gamma[k-1] \quad , \quad [\bar{c}_k(H_\rho^2)] = \lim_{\rho(x)\to 1} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[c_k(H_\rho^2)] \quad .$$

Recurrence relations and the heat trace coefficients

- The kernel of the H^2_ρ-heat equation

$$\left(\frac{\partial}{\partial\tau}+H^2_\rho\right)K_{H^2_\rho}(x,y;\tau)=0 \qquad , \qquad K_{H^2_\rho}(x,y;0)=\delta(x-y)$$

- From the ansatz $K_{H^2_\rho}(x,y;\tau)=K_\Delta(x,y;\tau)\sum_{k=0}^\infty c_k(x,y)\tau^k$ the following recurrence relations between the c-densities and their derivatives are derived
-

$${}^{(l)}C_k(x)=\lim_{y\to x}\frac{\partial^l}{\partial x^l}c_k(x,y) \quad , \quad {}^{(l)}C_0(x)=\delta_{0k}\,\mathbb{I} \ ; \ K_\Delta(x,y;\tau)=\frac{e^{-(\mu^2+\nu^2)\tau}}{\sqrt{4\pi\tau}}\cdot\exp(-\frac{|x-y|^2}{4\tau})$$

$${}^{(l)}C_k(x)=\frac{1}{k+l}\Big[{}^{(l+2)}C_{k-1}(x)-\sum_{j=0}^{l}\binom{l}{j}\frac{d^jV(x)}{dx^j}\,{}^{(l-j)}C_{k-1}(x)-[v^2,{}^{(l)}C_{k-1}(x)]-$$

$$-\sum_{j=0}^{l}\binom{l}{j}\frac{d^jQ(x)}{dx^j}\,{}^{(l-j+1)}C_{k-1}(x)+\frac{l}{2}\sum_{j=0}^{l-1}\binom{l-1}{j}\frac{d^jQ(x)}{dx^j}\,{}^{(l-j-1)}C_k(x)\Big]$$

- Heat trace coefficients and soliton Fermi number

$$c_k(H^2_\rho)=\int_{-\infty}^{\infty}dx\,{}^{(0)}C_k(x)\ ,\ \ N^{(k_{\max})}=\frac{1}{8\pi}\sum_{k=0}^{k_{\max}}\sum_{i=1}^{2}[\bar{c}_k(H^2_\rho)]_{ii}\,(v^2+\mu^2)^{1-k}\,\Gamma[k-1]$$

$k_{\max}\to\infty$ limit of the partial sums.

Partial sums and the Fermi charge of solitons

- Mathematica calculations:

$$\mathrm{tr}[\overline{c}_0(H^2_\rho)]=0 \quad , \quad \mathrm{tr}[\overline{c}_1(H^2_\rho)]=0 \quad , \quad \mathrm{tr}[\overline{c}_2(H^2_\rho)]=-8\,\mu\,\nu \quad , \quad \mathrm{tr}[\overline{c}_3(H^2_\rho)]=-\frac{16}{3}\,\mu\,\nu^3,$$

$$\mathrm{tr}[\overline{c}_4(H^2_\rho)]=-\frac{32}{15}\,\mu\,\nu^5 \quad , \quad \mathrm{tr}[\overline{c}_5(H^2_\rho)]=-\frac{64}{105}\,\mu\,\nu^7 \quad , \quad \mathrm{tr}[\overline{c}_6(H^2_\rho)]=-\frac{128}{945}\,\mu\,\nu^9 \quad , \quad \dots$$

- Integration of the heat kernel densities

$$\mathrm{tr}[\overline{c}_2(H^2_\rho)]=\int_{-\infty}^{\infty}dx(-4\mu\Phi'(x))=-4\mu\Big[\Phi(+\infty)-\Phi(-\infty)\Big]=-8\,\mu\,\nu \quad ,$$

$$\mathrm{tr}[\overline{c}_3(H^2_\rho)]=-\frac{2}{3}\mu\int_{-\infty}^{\infty}dx\Big[6(\nu^2-\Phi^2(x))\Phi'(x)+\Phi'''(x)\Big]=$$

$$=-\frac{2}{3}\mu\Big[6\Big(\nu^2\Phi(x)-\frac{1}{3}\Phi^3(x)\Big)+\Phi''(x)\Big]\Big|_{-\infty}^{\infty}=-\frac{16}{3}\,\mu\,\nu^3,$$

$$\mathrm{tr}[\overline{c}_4(H^2_\rho)]=\mu\int_{-\infty}^{\infty}dx\Big[-2(\nu^2-\Phi^2(x))^2\Phi'(x)+\frac{2}{3}\Phi'(x)^3+$$

$$+\frac{8}{3}\Phi(x)\Phi'(x)\Phi''(x)-\frac{2}{3}\Big(\nu^2-\Phi^2(x)\Big)\Phi'''(x)-\frac{1}{15}\Phi^{(5)}(x)\Big]=$$

$$=-\frac{32}{15}\,\mu\,\nu^5+\mu\int_{-\infty}^{\infty}dx\Big[\frac{2}{3}\Phi'(x)^3-\frac{2}{3}(\Phi'(x))^3\Big]=-\frac{32}{15}\,\mu\,\nu^5$$

- Soliton Fermi number as a series

$$N(H)=-\frac{1}{8\pi}\sum_{k=2}^{\infty}\frac{2^{k+1}(k-2)!}{(2k-3)!!}\,\frac{\mu\,\nu^{2k-3}}{(\nu^2+\mu^2)^{k-1}}$$

Convergence of the Fermi number series

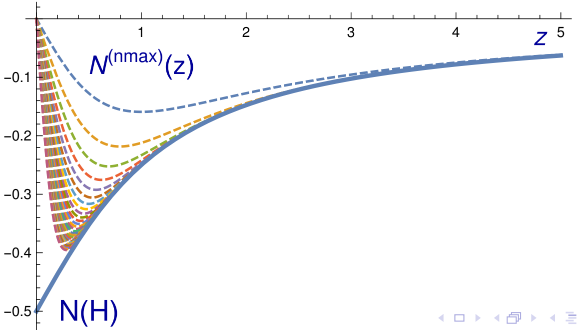
- Partial sum

$$N^{(k_{\max})} = \frac{1}{2}\sum_{k=2}^{k_{\max}} \frac{2^{k+1}(k-2)!}{(2k-3)!!}\,\frac{\mu\,\nu^{2k-3}}{(\nu^2+\mu^2)^{k-1}}$$

- In terms of special functions. Let us define $z = \frac{\mu}{\nu}$

$$N^{(k_{\max})} = -\frac{\mu}{2\sqrt{\mu^2}} + \frac{1}{\pi}\arctan\frac{\mu}{\nu} + \frac{\mu\,\nu^{2k_{\max}-1}\Gamma(k_{\max})\,{}_2F_1[1,k_{\max},\frac{1}{2}+k_{\max},\frac{\nu^2}{\nu^2+\mu^2}]}{2\sqrt{\pi}\,(\nu^2+\mu^2)^{k_{\max}}\,\Gamma[\frac{1}{2}+k_{\max}]}$$

$$N^{(k_{\max})}(z) = -\frac{z}{2\sqrt{z^2}} + \frac{1}{\pi}\arctan z + \frac{\Gamma(k_{\max})}{2\sqrt{\pi}\,\Gamma[\frac{1}{2}+k_{\max}]}\,\frac{z\,{}_2F_1[1,k_{\max},\frac{1}{2}+k_{\max},\frac{1}{1+z^2}]}{(1+z^2)^{k_{\max}}}$$



1D Dirac Hamiltonian for fermions with N flavors

- The Goldstone-Wilczek non Abelian action in 1 + 1 dimensions

$$S = \int dt\, dx\, \bar{\psi} \not{D}\, \psi, \qquad \not{D} = i\gamma^\mu \partial_\mu - \varphi_1 - i\gamma_* \varphi_2\,, \;\; \gamma_* = \gamma^0\gamma^1 = \sigma^2 \;\;,\;\; \psi(x) = \begin{pmatrix} \psi_1(x,t) \\ \vdots \\ \psi_N(x,t) \end{pmatrix}$$

$$\varphi_1(x,t) = \begin{pmatrix} \varphi_1^{11}(x,t) & \cdots & \varphi_1^{1N}(x,t) \\ \vdots & \ldots & \vdots \\ \varphi_1^{N1}(x,t) & \cdots & \varphi_1^{NN}(x,t) \end{pmatrix} \;,\; \varphi_2(x,t) = \begin{pmatrix} \varphi_2^{11}(x,t) & \cdots & \varphi_2^{1N}(x,t) \\ \vdots & \ldots & \vdots \\ \varphi_2^{N1}(x,t) & \cdots & \varphi_2^{NN}(x,t) \end{pmatrix}$$

$$\varphi_1^{ab} = (\varphi_1^*)^{ba} \;\;,\;\; \varphi_2^{ab} = (\varphi_2^*)^{ba} \;\;;\;\; a, b = 1, 2, \cdots, N$$

- The non Abelian Goldstone-Wilczek Hamiltonian

$$H = -i\gamma_* \partial_x + \varphi_1 \gamma^0 + i\varphi_2 \gamma^1 \;\; \Rightarrow$$

$$H_\rho^2 = -\partial_x^2 + G^2 + i\gamma^1 \frac{d\varphi_1}{dx} - \gamma^0 \frac{d\varphi_2}{dx} + \varepsilon\left(2\rho G - i\gamma_*(2\rho\partial_1 + \frac{d\rho}{dx})\right) \;\;,\;\; G = \varphi_1\gamma^0 + i\varphi_2\gamma^1$$

- Let $\delta\varphi_1$ and $\delta\varphi_2$ kocal variations of the scalar fields

$$\delta H = \gamma^0 \delta\varphi_1 + i\gamma^1 \delta\varphi_2 \;\; \Rightarrow$$

$$\delta\eta(0,H) = -\frac{2}{\sqrt{\pi}} a_0(H^2, \delta H) = -\frac{1}{\pi}\int dx\, \mathrm{tr}\left(\gamma^0\delta\varphi_1 + i\gamma^1\delta\varphi_2\right) = 0$$

$\eta(0,H)$ is a topological invariant. i.e., it depends on the asymptotic values of φ_1 and φ_2 only.

High mass computation of the heat trace in the $N = 1$ Goldstone-Wilczek model

- Let M^2 be a mass gap parameter

$$\tilde{H}_\rho^2 - M^2 \;\Rightarrow\; E = (M^2 - \varphi_1^2 - \varphi_2^2) - i\gamma^1 \frac{d\varphi_1}{dx} + \gamma^0 \frac{d\varphi_2}{dx} - 2\rho\varepsilon(\varphi_1\gamma^0 + i\varphi_2\gamma^1) \qquad \omega_1 = i\gamma_*\varepsilon\rho$$

$\eta(0, H)$ depends only on the asymptotic values of φ_1, φ_2 at $x = \pm\infty$

- The lowest, non-null, linear in ε, heat trace coefficient

$$a_4(\tilde{H}_\rho^2) = \frac{1}{\sqrt{4\pi}2!}\int dx\,\mathrm{tr}E^2 = \frac{2}{\sqrt{\pi}}\varepsilon\int dx\,\rho(x)(\varphi_2\frac{d\varphi_1}{dx} - \varphi_1\frac{d\varphi_2}{dx}) \Rightarrow$$

$$\eta(0, H, \rho) \sim -\frac{1}{2\sqrt{\pi}}\Gamma(1)|M|^{-2}\frac{d}{d\varepsilon}|_{\varepsilon=0}a_4(\tilde{H}_\rho^2) = -\frac{1}{\pi}\frac{1}{|M|^2}\int dx\,\rho(x)(\varphi_2\frac{d\varphi_1}{dx} - \varphi_1\frac{d\varphi_2}{dx})$$

- Summation of the series, small derivatives approximation

$$a_{2(2+l)}(\tilde{H}_\rho^2) = \frac{1}{\sqrt{4\pi}(2+l)!}E^{2+l} \Rightarrow$$

$$\eta(0, H, \rho \to 1) \sim -\sum_{l=0}^{\infty}\frac{1}{\pi l!}\Gamma(l+1)|M|^{-2l}\int dx\,\frac{(M^2 - \varphi_1^2 - \varphi_2^2)^l}{|M|^2}\cdot(\varphi_2\frac{d\varphi_1}{dx} - \varphi_1\frac{d\varphi_2}{dx})$$

$$\eta(0, H) = -\frac{1}{\pi}\int dx\,\frac{\varphi_2\frac{d\varphi_1}{dx} - \varphi_1\frac{d\varphi_2}{dx}}{\varphi_1^2 + \varphi_2^2} = -\frac{1}{\pi}\left(\arctan\frac{\varphi_1}{\varphi_2}|_{x=\infty} - \arctan\frac{\varphi_1}{\varphi_2}|_{x=-\infty}\right)$$

The non-Abelian Goldstone-Wilczek formula

- The non Abelian GW potential

$$\tilde{H}_\rho^2 - M^2 \;\Rightarrow\; E = (M^2 - G^2) - i\gamma^1 \frac{d\varphi_1}{dx} + \gamma^0 \frac{d\varphi_2}{dx} - 2\rho\varepsilon G \qquad , \qquad G = \varphi_1\gamma^0 + i\varphi_2\gamma^1$$

$$G^2 = \varphi_1^2 + \varphi_2^2 + i\gamma_*[\varphi_1, \varphi_2]$$

$\eta(0, H)$ depends only on the asymptotic values of φ_1, φ_2 at $x = \pm\infty$

- The low derivative approximation to the η invariant

$$\eta(0, H) \sim -\sum_{l=0}^{\infty} \frac{1}{\pi l!} \Gamma(l+1) \left|M\right|^{-2l} \int dx \,\mathrm{tr}\left[(M^2 - G^2)^l(-2G)(-i\gamma^1 \frac{d\varphi_1}{dx} + \gamma^0 \frac{d\varphi_2}{dx})\right]$$

Computation of the trace in spinor indices of the integrand gives

$$\mathrm{tr}_* \left[(z_+^l + z_-^l)(\varphi_2 \frac{d\varphi_1}{dx} - \varphi_1 \frac{d\varphi_2}{dx}) + i(z_+^l - z_-^l)(\varphi_1 \frac{\varphi_1}{dx} + \varphi_2 \frac{d\varphi_2}{dx})\right]$$

$$z_\pm = M^2 - y_\pm, \qquad y_\pm = \varphi_1^2 + \varphi_2^2 \pm i[\varphi_1, \varphi_2] = (\varphi_1 \mp i\varphi_2)(\varphi_1 \mp i\varphi_2)^\dagger$$

- To compute the sum in $\eta(0, H)$ one finds the matrix series

$$\sum_{l=0}^{\infty} \frac{z_\pm^l}{M^{2l}} = \left(1 - \frac{z_\pm}{M^2}\right)^{-1} \text{ convergent if } , \|z_\pm\| = \|M^2 - y_\pm\| < M^2$$

Shift the roots of φ_1 away from the roots of φ_2 thus making the series convergent everywhere.

Fermi charge of non Abelian GW solitons

- Fermi charge in the non Abelian GW model

$$\eta(0,H) = -\frac{1}{2\pi}\int dx\,\mathrm{tr}_*\left[(y_+^{-1}+y_-^{-1})(\varphi_2\frac{d\varphi_1}{dx}-\varphi_1\frac{\varphi_2}{dx})+i(y_+^{-1}-y_-^{-1})(\varphi_1\frac{\varphi_1}{dx}+\varphi_2\frac{d\varphi_2}{dx})\right]$$

$$\eta(0,H) = -\frac{i}{2\pi}\mathrm{tr}_*\ \left[\ln(\varphi_1+i\varphi_2)-\ln(\varphi_1-i\varphi_2)\right]\Big|_{-\infty}^{+\infty}$$

This result is new.

Sum of the heat trace expansion for solitonic domain walls

- Heat trace coefficients linear in ε

$$\varphi_1(x^3)\ ,\ \varphi_2(x^3)\ ,\ A_1(x^1,x^2)\ ,\ A_2(x^1,x^2)\ ,\ A_3=0$$

$$a_{2(l+3)} \sim \frac{1}{(4\pi)^{\frac{3}{2}}\, l!}\int d^3x\,\mathrm{tr}\Big\{(M^2-\varphi_1^2-\varphi_2^2)^l(ieF_{12}\gamma^1\gamma^2)$$

$$\times\Big[(-i\gamma^3\partial_3\varphi_1)(-2i\beta\varepsilon\rho\gamma^5\varphi_2)+(\gamma^3\gamma^5\partial_3\varphi_2)(-2\beta\varepsilon\rho\varphi_1)\Big]\Big\}$$

$$=\frac{8e\varepsilon}{(4\pi)^{\frac{3}{2}}\, l!}\int d^3x\,(M^2-\varphi_1^2-\varphi_2^2)^l F_{12}(\varphi_2\partial_3\varphi_1-\varphi_1\partial_3\varphi_2)\rho$$

- Fermi fractionization of magnetically charged domain walls

$$\eta(0,H)\sim 8e\sum_{l=0}^{\infty}\frac{\Gamma(l+1)}{(4\pi)^{3/2}\, l!}|M|^{-2l}\int d^3x\,\frac{(M^2-\varphi_1^2-\varphi_2^2)^l}{M^2}F_{12}(x^1,x^2)(\varphi_2\partial_3\varphi_1-\varphi_1\partial_3\varphi_2)$$

$$\Rightarrow\ N=-\frac{e}{4\pi^2}\arctan(\varphi_1/\varphi_2)\Big|_{x^3=-\infty}^{x^3=+\infty}\cdot\int d^2x\ F_{12}(x^1,x^2)$$

- Fermi number and the chiral angle

$$\varphi_1=\varphi\cos\theta,\quad \varphi_2=\varphi\sin\theta,\quad \varphi=\sqrt{\varphi_1^2+\varphi_2^2},\quad \theta=\mathrm{arctg}(\varphi_2/\varphi_1).$$

N prop to $\theta^+-\theta^-$ with $\theta^\pm\equiv\lim_{x^3\to\pm\infty}\theta(x^3)$. Invariance under global chiral rotations

$$\theta(x)\to\theta(x)+\delta\theta,\qquad \psi\to\exp\left(-\tfrac{i}{2}\delta\theta\gamma^5\right)\psi.$$

Induced Chern-Simons term on an interface

- One-loop effective action for spinors to second order in A_μ: odd contribution

$$S_{\rm odd} = \int d^4x\, d^4y\, F(x,y) A_\mu(x) \partial^y_\nu A_\rho(y) \epsilon^{\mu\nu\rho 3},$$

$$F(x,y) \text{ : nonlocal form factor } x^3 \;,\; y^3 \text{ and } z^\alpha = x^\alpha - y^\alpha \;,\; \alpha = 0,1,2$$

$$S_{\rm odd} = \int d^3z^\alpha d^3y^\alpha dx^3 dy^3\, F(z^\alpha, x^3, y^3) A_\alpha(z^\alpha + y^\alpha, x^3) \partial^y_\beta A_\gamma(y^\alpha, y^3) \epsilon^{\alpha\beta\gamma 3}$$

- Long wavelength limit: Chern-Simons action on the domain wall

$$S_{\rm odd} = \frac{ke^2}{4\pi} \int d^3y^\alpha A_\alpha(y^\alpha, 0) \partial_\beta A_\gamma(y^\alpha, 0) \epsilon^{\alpha\beta\gamma 3}$$

$$\frac{ke^2}{4\pi} = \int d^3z^\alpha dy^3 dx^3 F(z^\alpha, x^3, y^3)$$

- Hall conductivity and the wall Fermi number. If $i, j = 1, 2$

$$J^0(x) = \frac{1}{e} \frac{\delta}{\delta A_0(x)} S_{\rm odd} = \frac{2}{e} \int d^4y\, F(x,y) \partial^y_i A_j(y) \epsilon^{0ij3}$$

$$N = \int d^3x\, J^0(x) = \frac{ek}{2\pi} \int F_{12} d^2x \;\Rightarrow\; k = -\frac{\theta^+ - \theta^-}{2\pi}$$

References

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