Supergrassmanians, superflags and the conformal superspace

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Let G be a ss Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h}$ a CSA and $\Delta = \Delta^+ + \Delta^-$ the set of (positive and negative) roots. Let

$$\mathfrak{n}_+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- = \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha ,$$

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

$\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ are *Borel subalgebras*. All Borel subalgebras of $\mathfrak{g}$ are related by conjugation.

A *parabolic subalgebra* $\mathfrak{p} \subset \mathfrak{g}$ is a subalgebra that contains the Borel subalgebra but it is not $\mathfrak{g}$ itself.

Example. For $\mathrm{SL}_4(\mathbb{C})$, Borel and parabolic subalgebras are, respectively,

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \qquad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Projective geometry

Theorem. Let $\mathfrak{g}$ be a complex ss Lie algebra, $\mathfrak{p}$ a parabolic subalgebra, G a connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$ and P a parabolic subgroup of G with $\mathrm{Lie}(P) = \mathfrak{p}$. Then, the generalized flag manifold G/P is a compact Kähler manifold and a projective algebraic variety. Moreover, P is parabolic if and only if G/P is a projective algebraic variety. ■

Embeddings of a variety into projective spaces are in one to one correspondence with *very ample line bundles*: bundles that have enough global sections so to span, at each point, the fiber. These global sections are used as projective coordinates.

Example. *Projective space.* $\mathbf{P}^n = \{\text{Lines in } \mathbb{C}^{n+1}\}$.
The order one polynomials $(x^0, \ldots, x^n)$ span globally a line bundle, since they are not simultaneously 0. It is a very ample line bundle.

■

Projective geometry

For every parabolic subalgebra $\mathfrak{p}$ one can associate a k-grading of $\mathfrak{g}$

$$\mathfrak{g} = \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \cdots$$

such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2} \oplus \cdots$.

One denotes as $\mathfrak{g}_\pm = \mathfrak{g}_{\pm 1} \oplus \mathfrak{g}_{\pm 2} \oplus \cdots$.

We consider now G/P and $\pi : G \to G/P$. Its sheaf of regular functions is given, for $U \subset_{\text{open}} G/P$, by

$$\mathcal{R}_{G/P}(U) = \{f \in \mathcal{O}_G(\pi^{-1}(U)) \mid f(gp) = f(g)$$

$$\forall g \in \pi^{-1}(U),\ p \in P\}.$$

Projective geometry

The Lie subgroup $G_0 \subset G$ whose Lie algebra is $\mathfrak{g}_0$ is the *Levi subgroup of G* with respect to this grading. We consider the one dimensional representation (determinant of the Adjoint on $\mathfrak{p}_-$)

$$\begin{array}{ccc} G_0 & \xrightarrow{\chi} & \mathbb{C} \\ g & \longrightarrow & |\det(\mathrm{Ad}_-(g))|^{-\frac{1}{d}} \, . \end{array}$$

This representation of G_0 one can be extended to P just by letting it act trivially on $\mathfrak{g}_+$.

Then, by the method of induced representations, we extend it to the full G. As the representation space we shall consider the set of sections of the line bundle $G \times_P \mathbb{C}$ over G/P, defined by

$$\Gamma(G \times_P \mathbb{C}) = \{f \in \mathcal{O}(G) \otimes \mathbb{C} \mid f(gp) = \chi(p)^{-1} f(g)$$

$$\forall g \in G, \; p \in P\} \, .$$

Projective geometry.

The *complexified, conformal space* is the Grassmannian

$$G(2,4) = \mathrm{SL}_4(\mathbb{C})/P = \{\text{2-planes in } \mathbb{C}^4\} \text{ where}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} l & q \\ 0 & r \end{pmatrix} \right\}, \text{ with grading}$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} l & 0 \\ 0 & r \end{pmatrix} \right\}, \ \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \right\}, \ \mathfrak{g}_{+1} = \left\{ \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \right\}.$$

Remark. The big cell of $G(2,4)$ is the complexified Minkowsi space. We identify it by studying the action of the subgroup of $\mathrm{SL}(4,\mathbb{C})$ corresponding to the Poincaré group. When taking the physical real form, however, the conformal group is $\mathrm{SU}(2,2)$ and the real space on which it acts is NOT a real Grassmannian.

■

The Levi subgroup is the double cover of the Lorentz group times dilations $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times$, acting on the *twistor space*. The induced line bundle is the bundle of *conformal densities*.

Projective geometry

Proposition. *Let*

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad p = \begin{pmatrix} L & Q \\ 0 & R \end{pmatrix}.$$

Then, the bundle of conformal densities of weight 1 over $\mathrm{SL}_4(\mathbb{C})/P$*, defined by the character* $\det L^{-1}$ *of* P *is very ample.* ■

In fact, the determinants formed by minors corresponding to the first two columns of g are equivariant functions on $\mathrm{SL}_4(\mathbb{C})$:

$$gp = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} L & Q \\ 0 & R \end{pmatrix} = \begin{pmatrix} AL & AQ + BR \\ CL & CQ + DR \end{pmatrix}$$

and they cannot all be simultaneously 0.

This embedding is exactly the Plücker embedding $G(2,4)$ into $\mathbf{P}(\wedge^2\mathbb{C}^4) = \mathbf{P}^5$.

The first two columns of g can be taken as two vectors spanning a plane in $G(2,4)$. The minors d_{ij}, $i < j = 1, \ldots 4$ (columns 1, 2 and rows i, j) are homogeneous coordinates for $\mathbf{P}^5$.

As it is well known, the image of the embedding is given by the *Klein quadric*

$$d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23} = 0\,.$$

Super projective geometry

There is no generalization of the former theorem. There are superflag manifolds that are not projective. This happens when certain sheaf-cohomology group is $\neq 0$. So there can be topological obstructions to super projectiveness.

Example. The Grassmann super variety $G(1|1, 2|2)$ cannot be embedded in a projective superspace. ■

Still, for the ones that are projective, there is a one to one correspondence between super projective embeddings and very ample superline bundles (rank $1|0$ bundles).

Projective superspace. We consider homogeneous polyomials in

$$\mathbb{C}[x_0, \dots, x_n; \xi_0, \dots, \xi_n].$$

The projective coordinates $(x^0, \dots, x^n; \xi_0, \dots, \xi_n)$ form a basis of the set of global sections of a super line bundle.

Antichiral superspace $G(2|0,4|1)$

We consider the ss super Lie algebra

$$\mathfrak{g} = \mathfrak{sl}(4|1) = \left\{ \begin{pmatrix} l & q & \nu \\ p & r & \alpha \\ \mu & \beta & s \end{pmatrix} \;\middle|\; \mathrm{tr}\, l + \mathrm{tr}\, r = s \right\}.$$

Three examples of projective superflag manifolds with projective embedding that have physical meaning.

Antichiral superspace $\mathrm{Gr}_1 = G(2|0,4|1)$

$$\mathfrak{p}_1 = \left\{ \begin{pmatrix} l & q & \nu \\ 0 & r & \alpha \\ 0 & \beta & s \end{pmatrix} \right\},$$ with grading

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{g}_0 = \left\{ \begin{pmatrix} l & 0 & 0 \\ 0 & r & \alpha \\ 0 & \beta & s \end{pmatrix} \right\},$$

$$\mathfrak{g}_{+1} = \left\{ \begin{pmatrix} 0 & q & \nu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

Chiral superspace $\mathrm{Gr}_2 = G(2|1,4|1)$

$$\mathfrak{p}_2 = \left\{ \begin{pmatrix} l & q & \nu \\ 0 & r & 0 \\ \mu & \beta & s \end{pmatrix} \right\}, \text{ with grading}$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} \right\}, \ \mathfrak{g}_0 = \left\{ \begin{pmatrix} l & 0 & \nu \\ 0 & r & 0 \\ \mu & 0 & s \end{pmatrix} \right\},$$

$$\mathfrak{g}_{+1} = \left\{ \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \right\}.$$

Conformal superspace with real form $\mathrm{F} = F(2|0, 2|1, 4|1)$

$$\mathfrak{p}_u = \mathfrak{p}_1 \cap \mathfrak{p}_2 = \left\{ \begin{pmatrix} l & q & \nu \\ 0 & r & 0 \\ 0 & \beta & s \end{pmatrix} \right\}, \text{ with grading} .$$

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \; \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ \mu & 0 & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} l & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{pmatrix} \right\},$$

$$\mathfrak{g}_{+2} = \left\{ \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \; \mathfrak{g}_{+1} = \left\{ \begin{pmatrix} 0 & 0 & \nu \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \right\}$$

Super Plücker embedding

Let $E = \wedge^2(\mathbb{C}^{4|1}) \approx \mathbb{C}^{7|4}$ and A an arbitrary superalgebra (functor of points approach). We consider the map:

$$\begin{array}{ccc} \mathrm{Gr}_1(A) & \xrightarrow{p_A} & \mathbf{P}(E)(A) \\ W_1(A) = \mathrm{span}\,\{a_1, a_2\} & \longrightarrow & [a_1 \wedge a_2]\,. \end{array}$$

We ask then when a generic even vector

$$w = d + \delta \wedge \mathcal{E}_5 + d_{55}\mathcal{E}_5 \wedge \mathcal{E}_5 \ \in \ E(A)$$

$$\begin{aligned} d := & d_{12} e_1 \wedge e_2 + d_{13} e_1 \wedge e_3 + d_{14} e_1 \wedge e_4 + d_{23} e_2 \wedge e_3 + \\ & d_{24} e_2 \wedge e_4 + d_{34} e_3 \wedge e_4 , \\ \delta := & \delta_{15} e_1 + \delta_{25} e_2 + \delta_{35} e_3 + \delta_{45} e_4 . \end{aligned}$$

is in the image of this map. The result is that it has to be *decomposable*. This gives algebraic relations:

$$\begin{aligned} & d_{12} d_{34} - d_{13} d_{24} + d_{14} d_{23} = 0, && \text{(classical Plücker relation)} \\ & d_{ij} \delta_{k5} - d_{ik} \delta_{j5} + d_{jk} \delta_{i5} = 0, && 1 \leq i < j < k \leq 4, \\ & \delta_{i5} \delta_{j5} = d_{55} d_{ij} , && 1 \leq i < j \leq 4 . \end{aligned}$$

They are the *super Plücker relations.*

This projective embedding is realized in terms of a very ample line bundle by taking the one dimensional representation d_{12} of G_0.

The coordinate rings of Gr_1, Gr_2 and F can be seen as subrings of the coordinate ring of $\mathrm{SL}(4|1)$ (crucial point for the quantization). Let

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix} \in \mathrm{SL}(4|1)(A).$$

Let now

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad ge_1 = \begin{pmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{41} \\ \gamma_{51} \end{pmatrix}, \quad ge_2 = \begin{pmatrix} g_{12} \\ g_{22} \\ g_{32} \\ g_{42} \\ \gamma_{52} \end{pmatrix},$$

be two vectors of the standard basis e_1 and e_2 and the result of applying g to them.This selects the first two columns of g, which are the two independent vectors generating the subspace $W_1(A)$. The coordinate ring of $\mathrm{SL}(4|1)$ is

$$\mathbb{C}[\mathrm{SL}(4|1)] = \mathbb{C}[g_{ij}, \gamma_{i5}, \gamma_{5i}]/(\mathrm{Ber}\, g - 1),$$

We are now interpreting the entries as (even and odd) indeterminates.

One can show that $\mathbb{C}[\mathrm{Gr}_1]$ is the subring of $\mathbb{C}[\mathrm{SL}(4|1)]$ generated by the 2×2 determinants

$$d_{ij}^{12} := g_{i1}g_{j2} - g_{i2}g_{j1}, \quad \delta_{i5}^{12} := g_{i1}\gamma_{52} - g_{i2}\gamma_{51}, \quad d_{55}^{12} = \gamma_{51}\gamma_{52}\,,$$

with $i, j = 1, \ldots, 4$. We will suppress the superindex indicating columns 1 and 2.

Super Plücker embedding

For $G(2|1,4|1)$ we use a duality relation. The Grassmannian of $2|1$-planes on $\mathbb{C}^{4|1}$ (*super twistor space*) is isomorphic to the Grassmannian of $2|0$-planes on $(\mathbb{C}^{4|1})^*$. One just has to substitute g by $(g^{-1})^T$. The super Plücker relations are

$$
\begin{aligned}
&d^*_{12}d^*_{34} - d^*_{13}d^*_{24} + d^*_{14}d^*_{23} = 0, && \text{(classical Plücker relations)}\\
&d^*_{ij}\delta^*_{k5} - d^*_{ik}\delta^*_{j5} + d^*_{jk}\delta^*_{i5} = 0, && 1 \le i < j < k \le 4,\\
&\delta^*_{i5}\delta^*_{j5} = d^*_{55}d^*_{ij}, && 1 \le i < j \le 4
\end{aligned}
$$

The line bundle is constructed with d^*_{12}.

We have $G(2|0,4|1) \times G(2|1,4|1) \subset \mathbf{P}^{6|4} \times (\mathbf{P}^{6|4})^*$.

The flag supervariety $F(2|0,2|1,4|1) \subset G(2|0,4|1) \times G(2|1,4|1)$.

One has to impose the condition that the $2|0$ subspace is inside the $2|1$ subspace. This gives the *incidence relations*

$$\sum_{j=1}^{5} d^*_{ij}d_{jk} = 0, \qquad \forall\, i,k = 1,\ldots,5\,.$$

Summarizing:

Theorem. *There is an embedding of the superflag*

$$\mathrm{F} \subset \mathrm{Gr}_1 \times \mathrm{Gr}_2 \subset \mathbf{P}(E) \times \mathbf{P}(E^*)\,,$$

with $E = \wedge^2(T) \cong \mathbb{C}^{7|4}$, $E^* = \wedge^2(T^*) \cong \mathbb{C}^{7|4}$ *and* $T \cong C^{4|1}$. *With respect to such embedding, the coordinate ring of* F *is given by*

$$\mathbb{C}[F] := \mathbb{C}[d_{ij}, \delta_{i5}, d_{55}, d^*_{ij}, \delta^*_{i5}, d^*_{55}] \,/\, (I_{\mathrm{Gr}_1} + I_{\mathrm{Gr}_2} + I_{\mathrm{inc}})$$

where I_{inc}, *is the ideal generated by the 25 incidence relations*

■

Super Segre embedding

It maps the product of two projective spaces into one super projective space.

$$\begin{array}{ccc} \mathbf{P}^n(\mathbb{C}) \times \mathbf{P}^d(\mathbb{C}) & \xrightarrow{\psi} & \mathbf{P}^N(\mathbb{C}) \\ ([x_0, \ldots, x_n], [y_0, \ldots, y_d]) & \longrightarrow & [x_0 y_0, x_0 y_1, \ldots, x_i y_j, \ldots, x_n y_d] \end{array}$$

with $i = 0, .., n$, $j = 0, .., d$ and $N = (n+1)(d+1) - 1$.
Let z_{ij} be the homogeneous coordinates of $\mathbf{P}^N(\mathbb{C})$. As for the Plücker embedding, one can show that the image is an algebraic projective variety given as the zero locus of the 2×2 minors of the matrix

$$\begin{pmatrix} z_{00} & z_{02} & \cdots & z_{0d} \\ z_{20} & & & \\ \vdots & & \ddots & . \\ z_{n0} & \cdots & & z_{nd} \end{pmatrix}$$

The line bundle is given by the character $d_{12} d_{12}^*$.

Super Segre embedding

The Segre embedding can be generalized to the super case. Composing with it, we will embed F into $\mathbf{P}^{M|N}$, where $M|N = 64|56$. Explicitly, we get:

$$\begin{array}{ccc} \mathbf{P}(E)(A) \times \mathbf{P}(E^*)(A) & \xrightarrow{\Psi} & \mathbf{P}^{M|N}(A) \\ ([z_{ij}, z_{55} \,|\, \zeta_{i5}], [z^*_{ij}, z^*_{55} \,|\, \zeta^*_{i5}]) & \longrightarrow & [z_{ij} z^*_{kl}, z_{55} z^*_{55}, z_{ij} z^*_{55}, z_{55} z^*_{kl}, \zeta_{i5} \zeta^*_{k5} \,| \\ & & z_{ij} \zeta^*_{k5}, z_{55} \zeta^*_{k5}, \zeta_{i5} z^*_{kl}, \zeta_{i5} z^*_{55}] \,. \end{array}$$

Let us denote $I, K = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)$: Then, we can organize the coordinates on $\mathbb{C}^{M+1|N}$ in matrix form:

$$\left(\begin{array}{cc|c} z_I z^*_K & z_{55} z^*_K & z_I \zeta^*_k \\ z_I z^*_{55} & z_{55} z^*_{55} & z_{55} \zeta^*_k \\ \hline \zeta_{i5} z^*_K & \zeta_{i5} z^*_{55} & \zeta_{i5} \zeta^*_{k5} \end{array} \right) .$$

Super Segre embedding

The image of this map is a projective algebraic variety in the generators

$$\left(\begin{array}{cc|c} Z_{IK} & Z_{5K} & \Lambda_{Ik} \\ Z_{I5} & Z_{55} & \Lambda_{5k} \\ \hline \Gamma_{iK} & \Gamma_{i5} & T_{ik} \end{array}\right),$$

satisfying homogeneous polynomial relations

The classical section

As in the non super case, the global sections of the superline bundle on G/P can be thought as elements of $\mathcal{O}(G)$ with an the equivariance condition. Since $\mathbb{V} = \mathbb{C}$ we identify $\mathcal{O}(G) \otimes \mathbb{V} \cong \mathcal{O}(G)$. If $I(P)$ is an ideal in $\mathcal{O}(G)$ defining P we have $\pi : \mathcal{O}(G) \to \mathcal{O}(P) = \mathcal{O}(G)/I(P)$.
Let $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ be the coproduct in $\mathcal{O}(G)$. Then the equivariant condition becomes

$$\mathcal{O}(G/P)_1 = \left\{ f \in \mathcal{O}(G) \;\middle|\; (1\!\!1 \otimes \pi)\Delta(f) = f \otimes S(\chi) \right\}.$$

(S is the antipode).

The classical section

Let $t \in \mathcal{O}(G)$ such that $\pi(t) = S(\chi)$. If $\mathcal{L}$ is very ample –it corresponds to a projective embedding– we have the following important result (Fioresi'15).

Proposition. Let the supervariety G/P be embedded into some projective superspace via the line bundle $\mathcal{L}$ defined by χ. Let Δ be the coproduct in $\mathcal{O}(G)$ and $\pi : \mathcal{O}(G) \to \mathcal{O}(P) = \mathcal{O}(G)/I(P)$. Then, there exists an element $t \in \mathcal{O}(G)$, with $\pi(t) = S(\chi)$, such that

$$((\mathbb{1} \otimes \pi) \circ \Delta)(t) = t \otimes \pi(t), \quad \pi\big(t^m\big) \neq \pi\big(t^n\big) \quad \forall m \neq n \in \mathbb{N},$$
$$\mathcal{O}(G/P)_n = \Big\{ f \in \mathcal{O}(G) \;\Big|\; (\mathbb{1} \otimes \pi)\Delta(f) = f \otimes \pi\big(t^n\big) \Big\},$$
$$\mathcal{O}(G/P) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}(G)_n,$$

and $\mathcal{O}(G/P)$ is generated in degree 1, namely by $\mathcal{O}(G/P)_1$.

∎

The classical section

We call t the *classical section* associated to the super line bundle $\mathcal{L}$.

The following are the relevant examples:

1. For Gr_1, $t = d_{12} \in \mathcal{O}(\mathrm{SL}(4|1))$.
2. For Gr_2, $t = d_{12}^* \in \mathcal{O}(\mathrm{SL}(4|1))$.
3. For F (super Segre embedding), $t = d_{12} d_{12}^* \in \mathcal{O}(\mathrm{SL}(4|1))$.

We have then achieved a description of the coordinate ring of the projective embedding of F in $\mathbf{P}^{64|56}$ as a (graded) subring of $\mathcal{O}(\mathrm{SL}(4|1)$.

Quantum version

We have managed to give a presentation of the ring of the flag manifold in terms of generators (d_{ij}, d^*_{ij}) and relations (super Plücker and incidence relations).
Now we want to replace $\mathrm{SL}(4|1)$ by the quantum group $\mathrm{SL}_q(4|1)$. (Manin relations)

$$M_q(m|n) =_{\mathrm{def}} \mathbb{C}_q\langle a_{ij}\rangle / I_M , \qquad i, j = 1, \dots , n \, ,$$

where $\mathbb{C}_q\langle a_{ij}\rangle$ denotes the free algebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the homogeneous variables a_{ij} and the ideal I_M is generated by the Manin relations:

Quantum version

$$
\begin{aligned}
a_{ij}a_{il} &= (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, && j < l \\
a_{ij}a_{kj} &= (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, && i < k \\
a_{ij}a_{kl} &= (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, && i < k, j > l \quad \text{or} \quad i > k, j < l \\
a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij} &= \eta(q^{-1} - q)a_{kj}a_{il} && i < k, j < l
\end{aligned}
$$

where

$i, j, k, l = 1, \ldots m + n, \qquad \eta = (-1)^{p(k)p(l)+p(j)p(l)+p(k)p(j)},$
$p(i) = 0$ if $1 \leq i \leq m$, $p(i) = 1$ if $m + 1 \leq i \leq n + m$ and
$\pi(a_{ij}) = p(i) + p(j)$.

(Manin relations)

Quantum version

The determinants d_{ij} become quantum determinants $D_{q\,ij}$ and one can manage (difficult! It involves long calculations) to show that they generate, inside the quantum supergroup, a subalgebra that is a non commutative version (deformation) of the algebra of Gr_1 and Gr_2. So we can safely define a quantum Grassmannians ${\mathrm{Gr}_1}_q$, ${\mathrm{Gr}_2}_q$.

It is very difficult ($\sim$ impossible) to compute CR among $D_{q\,ij}$ and $D_{q\,ij}^{*}$. Also the incidence relations involve both types of coordinates. The strategy will be to give a *quantum super line bundle* realizing a quantum super Segre embedding with character $D_{q\,12}D_{q\,12}^{*}$.

The quantum projective supervariety so obtained is a deformation of $F(2|0,2|1,4|1)$: the quantum superflag $F_q(2|0,2|1,4|1)$. We can call it the *quantum super conformal space*.

Quantum version

Definition Let $\mathcal{L}$ be the super line bundle on G/P given by the classical section t. A *quantum section* or *quantization of t* is an element $d \in \mathcal{O}_q(G)$ such that

1. $(\mathbb{1} \otimes \pi)\Delta(d) = d \otimes \pi(d)$.
2. $t = d \mod (q-1)\,\mathcal{O}_q(G)$

■

Definition. Let d be a quantum section of $\mathcal{L}$. We define

$$\mathcal{O}_q(G/P) := \oplus_{n\in\mathbb{N}} \mathcal{O}_q(G/P)_n, \quad \text{where}$$

$$\mathcal{O}_q(G/P)_n := \left\{\, f \in \mathcal{O}_q(G) \,\middle|\, (\mathbb{1} \otimes \pi)\Delta(f) = f \otimes \pi(d^n) \,\right\}.$$

■

We say that $\mathcal{O}_q(G/P)$ is a *quantum projective supervariety*. It is an homogeneous space under the coaction of $\mathcal{O}_q(G)$

Quantum version

Proposition The element $d = D_{12}D^*_{12} \in \mathrm{SL}_q(4|1)$ is a quantum section, with respect to the super line bundle $\mathcal{L}$ on $\mathrm{SL}(4|1)/P_u$ given by $t = d_{12}d^*_{12}$.

■

Corollary. The $\mathbb{Z}$-graded subalgebra

$$C_q := \mathcal{O}_q(G/P) \subset \mathrm{SL}_q(4|1), \qquad G = \mathrm{SL}(4|1),\ P = P_u$$

defined by the quantum section $d = D_{12}D^*_{12}$ is a quantum deformation of the graded subalgebra of $\mathrm{SL}_q(4|1)$ obtained via the classical section $t = d_{12}d^*_{12}$.
Furthermore, C_q has a natural coaction of the supergroup $\mathrm{SL}_q(4|1)$. Therefore it is a quantum homogeneous superspace.

■