Benasque 09/19

From Yang-Mills in de Sitter space to electromagnetic knots

Olaf Lechtenfeld

Leibniz Universität Hannover, Germany

- Description of de Sitter space
- Reduction of Yang–Mills equations
- Yang–Mills configurations on de Sitter space
- Conformal equivalence to Minkowski space
- Maxwell solutions in Minkowski space
- All electromagnetic solutions
- Examples
- Gravitational backreaction (for the YM solutions)

PRL 119 (2017) 061601 JHEP11 (2017) 017 PLA 382 (2018) 1528

Description of de Sitter space

Four-dimensional de Sitter space dS_4 is a one-sheeted hyperboloid in $\mathbb{R}^{1,4}$ via

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \;=\; \ell^2$$

metric: $$\mathrm{d}s^2 \;=\; -\mathrm{d}Z_0^2 + \mathrm{d}Z_1^2 + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2$$

Topologically, $\mathrm{dS}_4 \simeq \mathbb{R} \times S^3$. Embed unit S^3 with metric $\mathrm{d}\Omega_3^2$ into $\mathbb{R}^4$:

$$(\chi, \theta, \phi) \;\mapsto\; \omega_A(\chi, \theta, \phi) \quad \text{with} \quad A = 1, 2, 3, 4 \quad \text{and} \quad \omega_A \omega_A = 1$$

Closed-slicing global coordinates (T, χ, θ, ϕ):

$$Z_0 = \ell \sinh T \qquad \text{and} \qquad Z_A = \ell\, \omega_A \cosh T \qquad \text{with} \quad T \in \mathbb{R}$$

$$\text{metric:} \qquad \mathrm{d}s^2 \;=\; \ell^2 \left(-\mathrm{d}T^2 + \cosh^2 T\, \mathrm{d}\Omega_3^2\right)$$

Switch to conformal time coordinates via

$$\sinh T = -\cot\tau \qquad \Leftrightarrow \qquad \frac{\mathrm{d}T}{\mathrm{d}\tau} \;=\; \cosh T \;=\; \frac{1}{\sin\tau}$$

Range: $T \in \mathbb{R} \quad \Leftrightarrow \quad \tau \in \mathcal{I} = (0, \pi)$ open interval

$$\text{metric:} \qquad \mathrm{d}s^2 \;=\; \frac{\ell^2}{\sin^2\tau}\left(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2\right) \;=\; \frac{\ell^2}{\sin^2\tau}\, \mathrm{d}s^2_{\mathrm{cyl}}$$

finite Lorentzian cylinder $\mathcal{I} \times S^3$

Reduction of Yang–Mills to matrix equations

Gauge potential $\mathcal{A}$ and gauge field $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ with values in a Lie algebra $\mathfrak{g}$

General form of the gauge potential in temporal gauge $\mathcal{A}_\tau = 0$ is

$$\mathcal{A} \;=\; \sum_{a=1}^{3} X_a(\tau,\omega)\; e^a \qquad \text{on} \quad \mathcal{I} \times S^3$$

where $X_a \in \mathfrak{g}$ and $\{e^a\}$ is a basis of left-invariant one-forms on $S^3 \simeq \mathrm{SU}(2)$, with

$$\mathrm{d}e^a + \varepsilon^a_{\;bc}\, e^b \wedge e^c \;=\; 0 \qquad \text{and} \qquad e^a e^a \;=\; \mathrm{d}\Omega_3^2$$

In terms of S^3 coordinates $(a,i,j,k = 1,2,3)$:

$$e^a \;=\; -\eta^a_{BC}\,\omega_B\,\mathrm{d}\omega_C \qquad \text{where} \quad \eta^i_{jk} = \varepsilon^i_{\;jk} \quad \text{and} \quad \eta^i_{j4} = -\eta^i_{4j} = \delta^i_j$$

Left-invariant right multiplication: $R_a = -\eta^a_{BC}\,\omega_B \dfrac{\partial}{\partial\omega_C} \quad \Rightarrow \quad [R_a, R_b] = 2\,\varepsilon_{abc}\,R_c$

An arbitrary function Φ on S^3 obeys $\qquad \mathrm{d}\Phi \;=\; e^a\,R_a\Phi$

Resulting gauge field: $\dot{X}_a := \mathrm{d}X_a/\mathrm{d}\tau$

$$\mathcal{F} \;=\; \mathcal{F}_{\tau a}\, e^\tau\wedge e^a + \tfrac12 \mathcal{F}_{bc}\, e^b\wedge e^c$$

$$=\; \dot{X}_a\, e^\tau\wedge e^a + \tfrac12\Big(R_{[b}X_{c]} - 2\varepsilon^a_{bc}X_a + [X_b,X_c]\Big)\, e^b\wedge e^c$$

Yang–Mills Lagrangian: $D_aX_b := R_aX_b + [X_a,X_b]$

$$\mathcal{L} \;=\; \tfrac18\,\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \;=\; -\tfrac14\,\mathrm{tr}\mathcal{F}_{\tau a}\mathcal{F}_{\tau a} \;+\; \tfrac18\,\mathrm{tr}\mathcal{F}_{ab}\mathcal{F}_{ab}$$

$$=\; -\tfrac12\mathrm{tr}\Big\{\tfrac12\dot{X}_a\dot{X}_a - 2X_aX_a + \varepsilon_{abc}X_aD_bX_c - \tfrac12(D_aX_b)(D_aX_b)\Big\}$$

Yang–Mills equations:

$$\ddot{X}_a \;=\; -\,4\,X_a + 2\,\varepsilon_{abc}R_bX_c + R_bR_{[b}X_{a]} + 3\,\varepsilon_{abc}\,[X_b,X_c]$$

$$+\,2[X_b,R_bX_a] - [X_b,R_aX_b] - [X_a,R_bX_b] - \Big[X_b,[X_a,X_b]\Big]$$

and $$0 \;=\; R_a\dot{X}_a + [X_a,\dot{X}_a]$$

Too hard to solve analytically?

Impose O(4) symmetry $\quad \Rightarrow \quad X_a(\tau,\omega) = X_a(\tau)$

YM equations become ordinary matrix differential equations:

$$\ddot{X}_a \;=\; -4\,X_a + 3\,\varepsilon_{abc}\,[X_b, X_c] - \Big[X_b, [X_a, X_b]\Big] \qquad \text{and} \qquad [\dot{X}_a, X_a] \;=\; 0$$

three coupled ordinary differential equations for three matrix functions $X_a(\tau)$

Still too complicated...?

Choose gauge group SU(2), hence $\mathfrak{g} = su(2)$ and spin-j representation

The three SU(2) generators T_a are normalized to $C(\frac{1}{2}) = \frac{1}{2}, \quad C(1) = 2$

$[T_b, T_c] = 2\,\varepsilon^a_{bc} T_a$ and $\mathrm{tr}(T_a T_b) = -4C(j)\,\delta_{ab}$ for $C(j) = \frac{1}{3}\,j(j{+}1)(2j{+}1)$

Simplest ansatz for the matrices X_a:

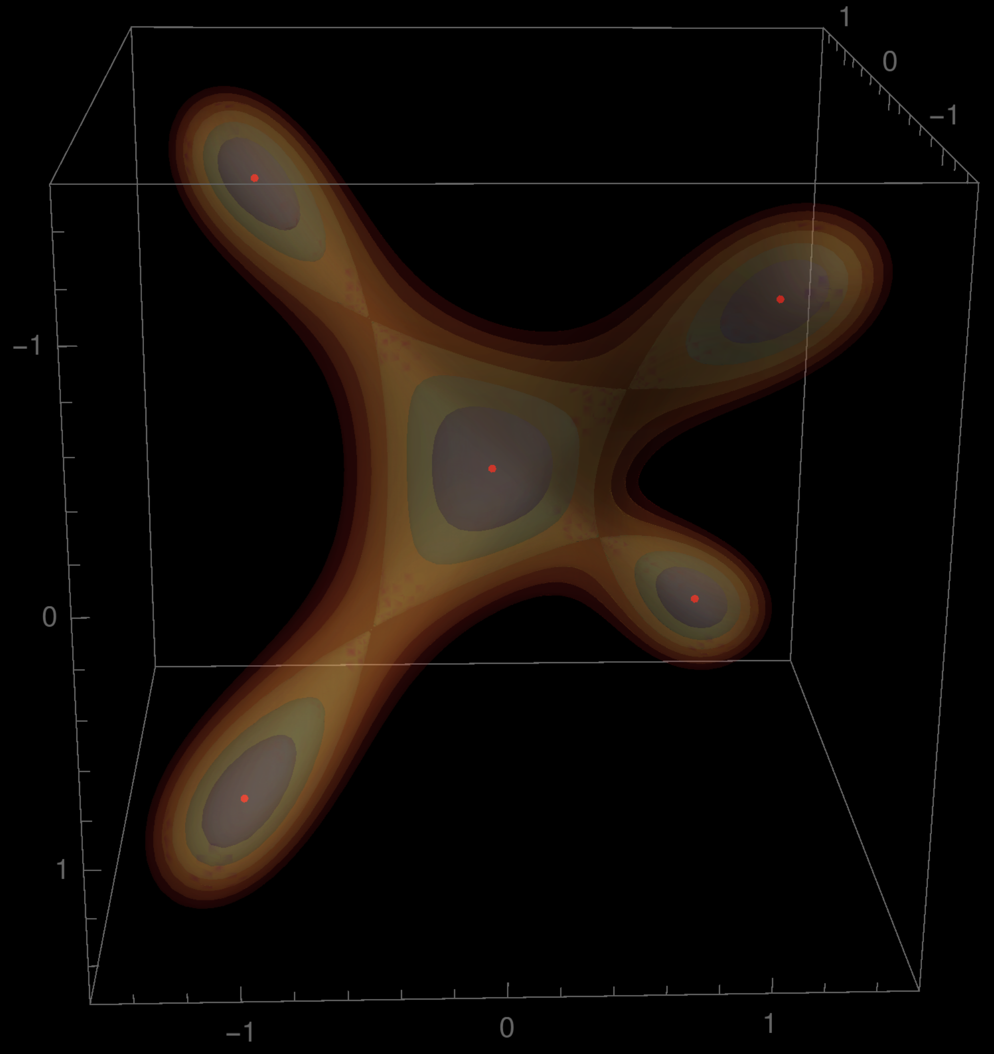
$$X_1 = \Psi_1 T_1\,, \quad X_2 = \Psi_2 T_2\,, \quad X_3 = \Psi_3 T_3 \qquad \text{with} \quad \Psi_a = \Psi_a(\tau) \in \mathbb{R}$$

Resulting simplification of Yang–Mills Lagrangian density:

$$\mathcal{L} \;=\; 4\,C(j)\left\{\tfrac{1}{4}\dot\Psi_a\dot\Psi_a - (\Psi_1 - \Psi_2\Psi_3)^2 - (\Psi_2 - \Psi_3\Psi_1)^2 - (\Psi_3 - \Psi_1\Psi_2)^2\right\}$$

Interpretation: $\{\Psi_a\}$ = particle coordinates in $\mathbb{R}^3$ $\Rightarrow$ Newtonian dynamics with

potential $\frac{1}{2}\mathcal{V}(\Psi) \;=\; (\Psi_1{-}\Psi_2\Psi_3)^2 + (\Psi_2{-}\Psi_3\Psi_1)^2 + (\Psi_3{-}\Psi_1\Psi_2)^2$



1
0
−1
−1
0
1
−1
0
1

Yang–Mills configurations on de Sitter space

Only analytic nonabelian solutions:

$$\Psi_1 = \Psi_2 = \Psi_3 =: \Psi \qquad \text{with} \qquad \ddot{\Psi} = 16\,\Psi\,(\Psi{-}1)(2\Psi{-}1)$$

$$\Rightarrow \qquad \mathcal{A} \;=\; \Psi(\tau)\,g^{-1}\mathrm{d}g \qquad \text{for} \quad g : S^3 \to \mathrm{SU(2)}$$

double well: $\Psi(\tau) = 0$ or 1 , $\Psi(\tau) = \frac{1}{2}$, $\Psi(\tau) = \text{bounce}$

Substitute solution $\Psi(\tau)$ into $\mathcal{F}$ and get SU(2) color electric and magnetic fields:

$$\mathcal{E}_a \;=\; \mathcal{F}_{\tau a} \;=\; \dot{\Psi}\,T_a \qquad \text{and} \qquad \mathcal{B}_a \;=\; \tfrac{1}{2}\varepsilon_{abc}\mathcal{F}_{bc} \;=\; 2\,\Psi\,(\Psi{-}1)\,T_a$$

Their total de Sitter energy and action is finite and proportional to double-well energy V_0

Related to solutions found by de Alfaro, Fubini, Furlan (1976) and Lüscher (1977)

Only other analytic solutions are abelian, generally: $X_a(\tau) = \widetilde{X}_a(\tau)\,T_3$

Conformal equivalence to Minkowski space

The $Z_0 + Z_4 < 0$ half of dS_4 is also conformally equivalent to future Minkowski space:

$$Z_0 = \frac{t^2 - r^2 - \ell^2}{2t}, \quad Z_1 = \ell \frac{x}{t}, \quad Z_2 = \ell \frac{y}{t}, \quad Z_3 = \ell \frac{z}{t}, \quad Z_4 = \frac{r^2 - t^2 - \ell^2}{2t}$$

with $x, y, z \in \mathbb{R}$ and $r^2 = x^2 + y^2 + z^2$ but $t \in \mathbb{R}_+$

since $t \in [0, \infty]$ corresponds to $Z_0 \in [-\infty, \infty]$ but $Z_0 + Z_4 < 0$

metric: $$\mathrm{d}s^2 = \frac{\ell^2}{t^2}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$$

Can cover whole $\mathbb{R}^{1,3}$ by gluing a second dS_4 copy and using patch $Z_0 + Z_4 > 0$

Direct relation between cylinder and Minkowski coordinates:

$$\cot\tau = \frac{r^2 - t^2 + \ell^2}{2\,\ell\, t}\,,\quad \omega_1 = \gamma\,\frac{x}{\ell}\,,\quad \omega_2 = \gamma\,\frac{y}{\ell}\,,\quad \omega_3 = \gamma\,\frac{z}{\ell}\,,\quad \omega_4 = \gamma\frac{r^2 - t^2 - \ell^2}{2\,\ell^2}$$

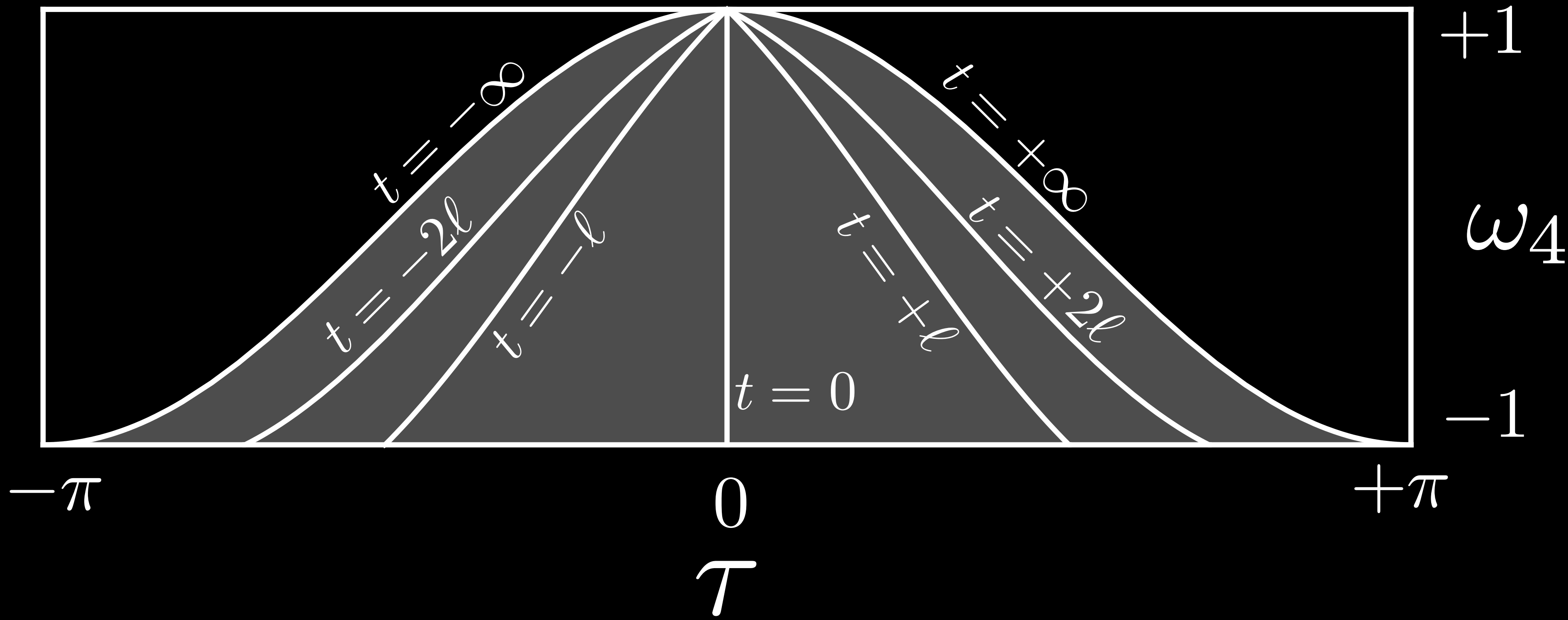
with the convenient abbreviation $\quad \gamma = \dfrac{2\,\ell^2}{\sqrt{4\,\ell^2 t^2 + (r^2 - t^2 + \ell^2)^2}}$

$t = -\infty, 0, \infty$ corresponds to $\tau = -\pi, 0, \pi$ so the cylinder is doubled to $2\mathcal{I} \times S^3$

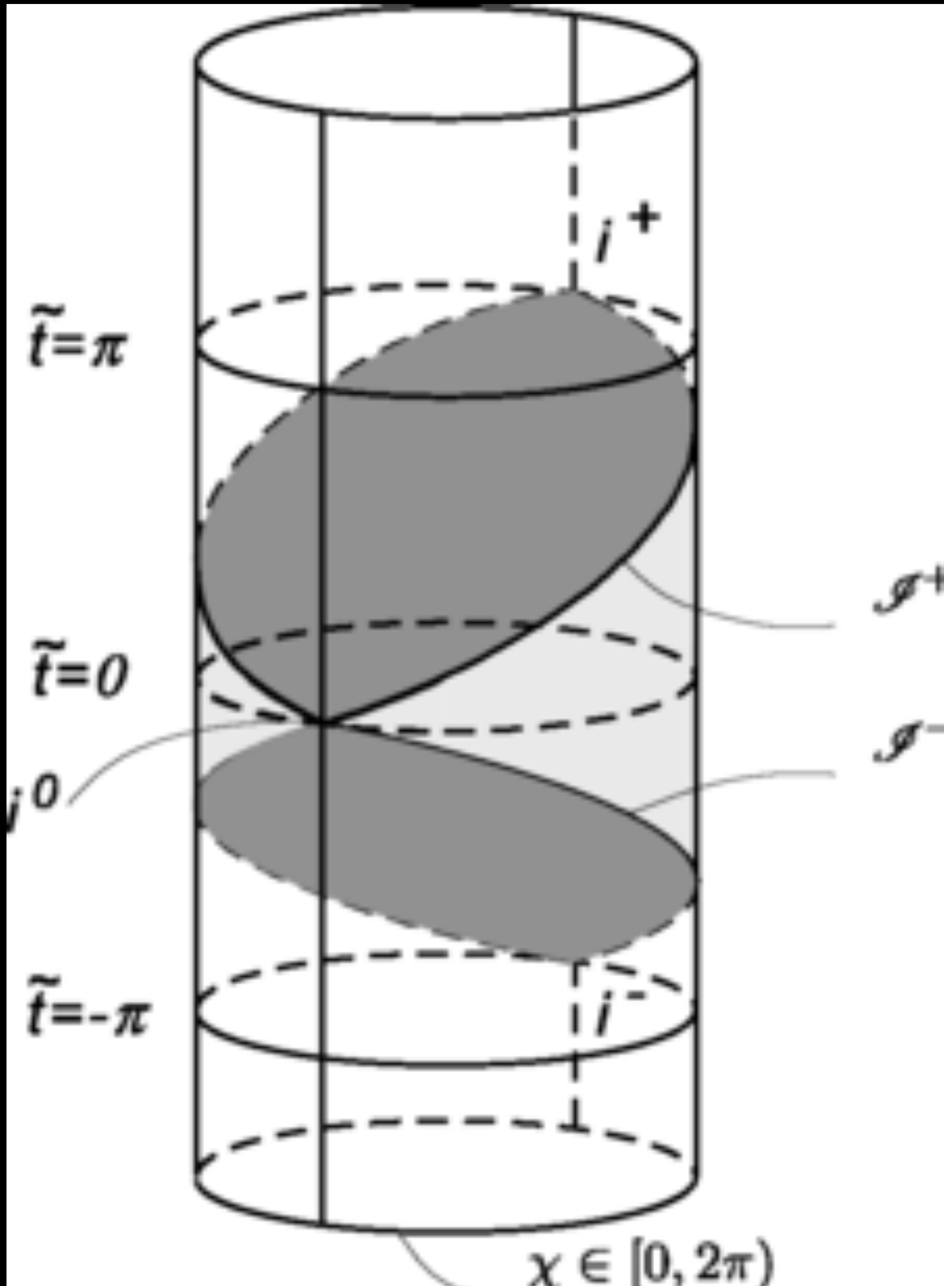
Full Minkowski space is covered by the cylinder patch $\quad \cos\chi \equiv \omega_4 \leq \cos\tau$

Cylinder time τ becomes a regular smooth function of (t, x, y, z), but more useful is

$$\exp(2\mathrm{i}\,\tau) = \frac{[(\ell + \mathrm{i}t)^2 + r^2]^2}{4\,\ell^2 t^2 + (r^2 - t^2 + \ell^2)^2}$$



$+1$
-1
ω_4
$-\pi$
0
$+\pi$
τ
$t = -\infty$
$t = -2\ell$
$t = -\ell$
$t = 0$
$t = +\ell$
$t = +2\ell$
$t = +\infty$



i^+
$\tilde{t}=\pi$
$\mathscr{I}^+$
$\tilde{t}=0$
$\mathscr{I}^-$
i^0
$\tilde{t}=-\pi$
i^-
$\chi \in [0, 2\pi)$

Maxwell solutions in Minkowski space

Yang–Mills and Maxwell are conformally invariant in four spacetime dimensions

$\Rightarrow$ may solve on cylinder $2\mathcal{I} \times S^3$ rather than directly on Minkowski space $\mathbb{R}^{1,3}$

Why? S^3 enables manifestly O(4)-covariant formalism!

Recall general form of the gauge potential in the $\mathcal{A}_\tau = 0$ gauge,

$$\mathcal{A} = \sum_{a=1}^{3} X_a(\tau,\omega)\, e^a \qquad \text{on} \quad 2\mathcal{I} \times S^3$$

where $X_a \in \mathfrak{g}$ and $\{e^a\}$ are left-invariant one-forms on S^3

Translate YM or Maxwell solutions from $2\mathcal{I} \times S^3$ to $\mathbb{R}^{1,3}$ simply by coordinate change

$$\tau = \tau(t,x,y,z) \qquad \text{and} \qquad \omega_A = \omega_A(t,x,y,z)$$

Helpful: $\exp(2\mathrm{i}\tau)$ is a rational function of t and r

Behavior at the boundary $\cos\tau = \omega_4$ yields fall-off properties at $t \to \pm\infty$

Need left-invariant one-forms in terms of Minkowski coordinates (calculation!):

$$e^\tau \;\equiv\; \mathrm{d}\tau \;=\; \frac{\gamma^2}{\ell^3}\Big(\tfrac12(t^2+r^2+\ell^2)\,\mathrm{d}t - t\,x^k\mathrm{d}x^k\Big)$$

$$e^a \;=\; -\eta^a_{BC}\omega_B\mathrm{d}\omega_C \;=\; \frac{\gamma^2}{\ell^3}\Big(t\,x^a\mathrm{d}t - \big(\tfrac12(t^2-r^2+\ell^2)\,\delta^a_k + x^a x^k + \ell\,\varepsilon^a{}_{jk}x^j\big)\,\mathrm{d}x^k\Big)$$

with notation $\quad (x^i) = (x,y,z) \quad$ and (for later) $\quad (x^\mu) = (x^0, x^i) = (t,x,y,z)$

Specialize to Maxwell theory, i.e. $\mathfrak{g} = \mathbb{R}$ and $X_a(\tau,\omega)$ are real functions $\Rightarrow$

$$\mathcal{A} \;=\; X_a\, e^a \quad , \qquad \mathcal{F} \;=\; \dot{X}_a\, e^\tau \wedge e^a + \tfrac{1}{2}\Big(R_{[b}X_{c]} - 2\varepsilon^{a}_{bc} X_a\Big)\, e^b \wedge e^c$$

$$\mathcal{L} \;=\; \tfrac{1}{4}\dot{X}_a\dot{X}_a - X_a X_a + \tfrac{1}{2}\varepsilon_{abc} X_a R_b X_c - \tfrac{1}{4}(R_a X_b)(R_a X_b)$$

$$\ddot{X}_a \;=\; -4\, X_a + 2\,\varepsilon_{abc} R_b X_c + R_b R_{[b} X_{a]} \qquad \text{and} \qquad R_a \dot{X}_a \;=\; 0$$

Start with O(4)-symmetric case $\Rightarrow$ $X_a(\tau,\omega) = X_a(\tau)$ $\Rightarrow$ $R_a X_b = 0$

$$\mathcal{L} \;=\; \tfrac{1}{4}\dot{X}_a\dot{X}_a - X_a X_a \qquad \Rightarrow \qquad \ddot{X}_a \;=\; -4\, X_a \qquad \Rightarrow \qquad \text{harmonic motion}$$

Oscillatory solutions $X_a(\tau) \;=\; c_a \,\cos\big(2(\tau-\tau_a)\big)$ for $\tau \in (-\pi,+\pi)$

Conversion to Minkowski solutions (with $x \equiv \{x^\mu\}$):

$$\mathcal{A} \;=\; X_a(\tau(x))\, e^a(x) \;=\; \mathcal{A}_\mu(x)\, \mathrm{d}x^\mu \qquad \text{yields } \mathcal{A}_\mu(x) \qquad \mathcal{A}_t \neq 0$$

$$\mathrm{d}\mathcal{A} \;=\; \dot{X}_a\, e^\tau \wedge e^a - \varepsilon^a_{\ bc} X_a\, e^b \wedge e^c \;=\; \tfrac{1}{2}\mathcal{F}_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{yields } \mathcal{F}_{\mu\nu}(x)$$

and hence electric and magnetic fields $E_i \;=\; \mathcal{F}_{it}$ and $B_i \;=\; \tfrac{1}{2}\varepsilon_{ijk}\mathcal{F}_{jk}$

Finite energy and zero action on de Sitter and on Minkowski space

An example (putting $\ell = 1$):

$$X_1(\tau) \;=\; -\tfrac{1}{8}\sin 2\tau\;, \qquad X_2(\tau) \;=\; -\tfrac{1}{8}\cos 2\tau\;, \qquad X_3(\tau) \;=\; 0$$

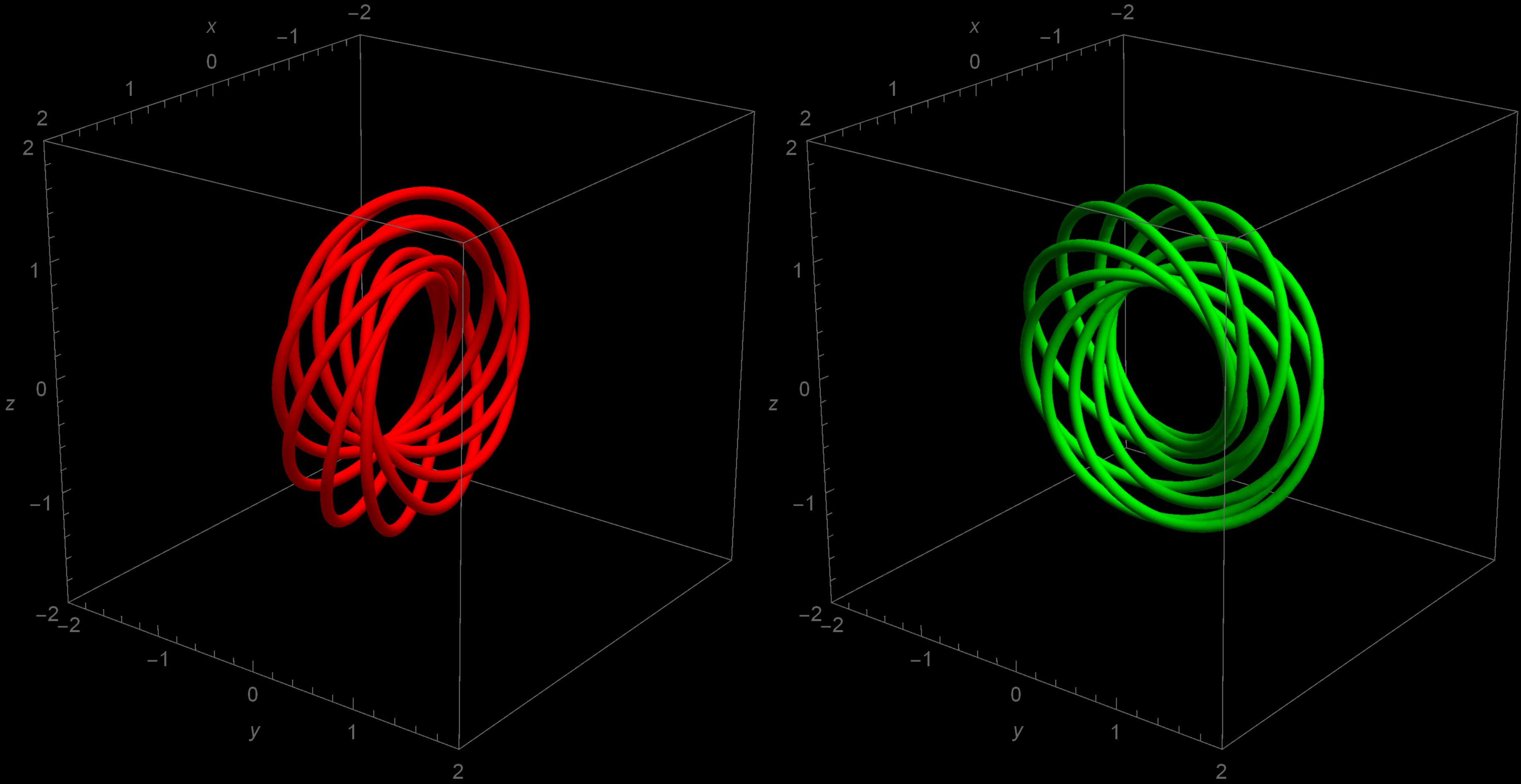
Result of short computation:

$$\vec{E} + \mathrm{i}\vec{B} \;=\; \frac{1}{\big((t-\mathrm{i})^2 - r^2\big)^3} \begin{pmatrix} (x-\mathrm{i}y)^2 - (t-\mathrm{i}-z)^2 \\ \mathrm{i}(x-\mathrm{i}y)^2 + \mathrm{i}(t-\mathrm{i}-z)^2 \\ -2\,(x-\mathrm{i}y)\,(t-\mathrm{i}-z) \end{pmatrix}$$

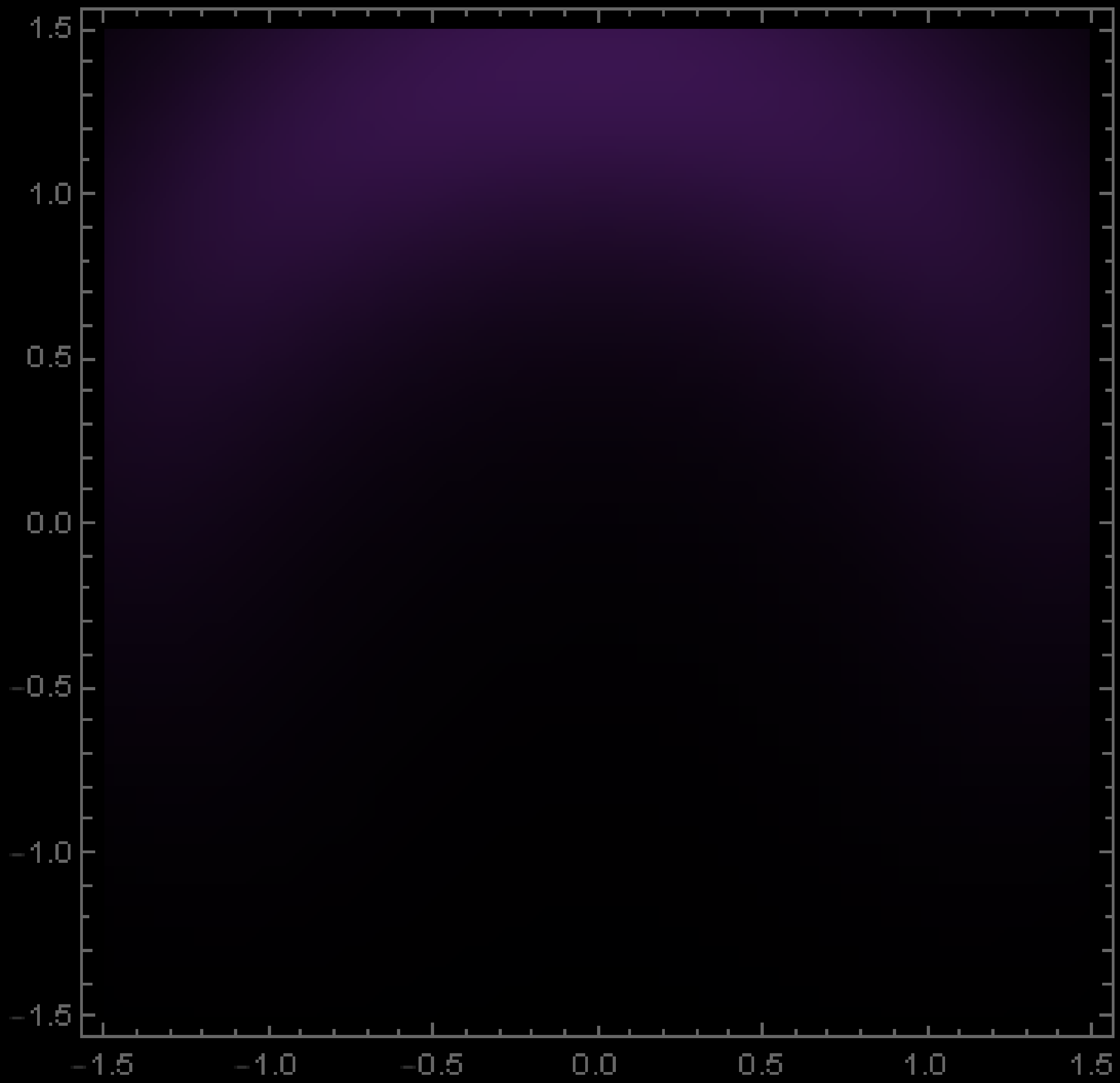
This is the celebrated Hopf–Rañada electromagnetic knot.

Our approach also yields its gauge potential.

Note that it is only axisymmetric in Minkowski space but O(4) covariant in de Sitter



Some magnetic (red) and electric (green) field lines at $t = 0$



Energy density in y=0 plane, changing with time

All electromagnetic solutions

Admit arbitrary O(4)-non-symmetric solutions $\Rightarrow$ $X_a = X_a(\tau,\omega)$

But capture the ω-dependence in an O(4)-covariant fashion!

Choose Coulomb gauge $R_a X_a = 0$ $\Rightarrow$ coupled wave equations:

$$\ddot{X}_a \;=\; (R^2-4)\,X_a + 2\,\varepsilon_{abc}R_b X_c$$

Expand $X_a(\tau,\omega) \;=\; \sum_{jmn} X_a^{j;m,n}(\tau)\,Y_{j;m,n}(\omega)$ in hyperspherical harmonics

$Y_{j;m,n}(\omega)$ with $m,n=-j,-j{+}1,\ldots,+j$ and $2j=0,1,2,\ldots$

subject to $-\frac{1}{4}R^2\,Y_{j;m,n} = j(j{+}1)\,Y_{j;m,n}$ and $\frac{\mathrm{i}}{2}\,R_3\,Y_{j;m,n} = n\,Y_{j;m,n}$

Two types of basis solutions $\quad (X_\pm = (X_1 \pm \mathrm{i} X_2)/\sqrt{2})$:

- type I : $\quad j \geq 0\,, \quad m = -j, \ldots, +j\,, \quad n = -j-1, \ldots, j+1\,, \quad \Omega^j = \pm 2(j+1)$

$$X_+ \;=\; \sqrt{(j-n)(j-n+1)/2}\; \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\; Y_{j;m,n+1}$$

$$X_3 \;=\; \sqrt{(j+1)^2 - n^2}\; \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\; Y_{j;m,n}$$

$$X_- \;=\; -\sqrt{(j+n)(j+n+1)/2}\; \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\; Y_{j;m,n-1}$$

- type II : $\quad j \geq 1\,, \quad m = -j, \ldots, +j\,, \quad n = -j+1, \ldots, j-1\,, \quad \Omega^j = \pm 2j$

$$X_+ \;=\; -\sqrt{(j+n)(j+n+1)/2}\; \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\; Y_{j;m,n+1}$$

$$X_3 \;=\; \sqrt{j^2 - n^2}\; \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\; Y_{j;m,n}$$

$$X_- \;=\; \sqrt{(j-n)(j-n+1)/2}\; \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\; Y_{j;m,n-1}$$

Each complex solution yields two real ones (real part and imaginary part)

Count for j fixed:

$2(2j{+}1)(2j{+}3)$ type-I solutions and $2(2j{+}1)(2j{-}1)$ type-II solutions ($j{>}0$)

together: $4(2j{+}1)^2$ solutions for $j{>}0$ and 6 solutions for $j{=}0$

Constant solutions ($\Omega = 0$) are not allowed; simplest are $j{=}0$ type I (Hopf–Rañada)

Spin j type I $\longleftarrow$ parity $(L \leftrightarrow R, m \leftrightarrow n)$ $\longrightarrow$ spin $j{+}1$ type II

Electromagnetic duality: shifting $|\Omega^j|\tau$ by $\pm\frac{\pi}{2}$ yields a dual solution A_D

Main technical task:

transform a chosen solution on $2\mathcal{I} \times S^3$ to Minkowski coordinates (t, x, y, z),

straightforward due to explicit formulæ for all ingredients $\Rightarrow$ only rational functions

Helicity $$h = \frac{1}{2} \int_{\mathbb{R}^3} \left(\mathcal{A} \wedge \mathcal{F} + \mathcal{A}_D \wedge \mathcal{F}_D \right)$$ is conserved

Energy $$E = \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 x \left(\vec{E}^2 + \vec{B}^2 \right)$$ is conserved and related to h

Best computed in “sphere frame” at $t = \tau = 0$: $$\mathcal{F} = \mathcal{E}_a\, e^a \wedge e^0 + \tfrac{1}{2} \mathcal{B}_a\, \varepsilon^a{}_{bc}\, e^b \wedge e^c$$

$$\int_{\mathbb{R}^3} \mathrm{d}^3 x\; \vec{E}^2 = \frac{1}{\ell} \int_{S^3} \mathrm{d}^3 \Omega_3 \; (1 - \omega_4)\, \mathcal{E}_a \mathcal{E}_a$$ and $$\int_{\mathbb{R}^3} \mathrm{d}^3 x\; \vec{B}^2 = \frac{1}{\ell} \int_{S^3} \mathrm{d}^3 \Omega_3 \; (1 - \omega_4)\, \mathcal{B}_a \mathcal{B}_a$$

exploiting orthogonality properties of the hyperspherical harmonics

Examples

Example 1: $(j;m,n)=(1,0,0)$, type I, combine $\mathrm{e}^{4\mathrm{i}\tau}+\mathrm{e}^{-4\mathrm{i}\tau}=2\cos 4\tau$

$$X_\pm = -\frac{\sqrt{3}}{\pi}(\omega_1\pm\mathrm{i}\omega_2)(\omega_3\pm\mathrm{i}\omega_4)\ \cos 4\tau\ , \qquad X_3 = -\frac{\sqrt{6}}{\pi}(\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2)\ \cos 4\tau$$

$\Rightarrow$ $h = 12$ and $E = 48/\ell$

Example 2: $(j;m,n)=(2;1,-1)$, type I $\qquad E = 6\,h/\ell$

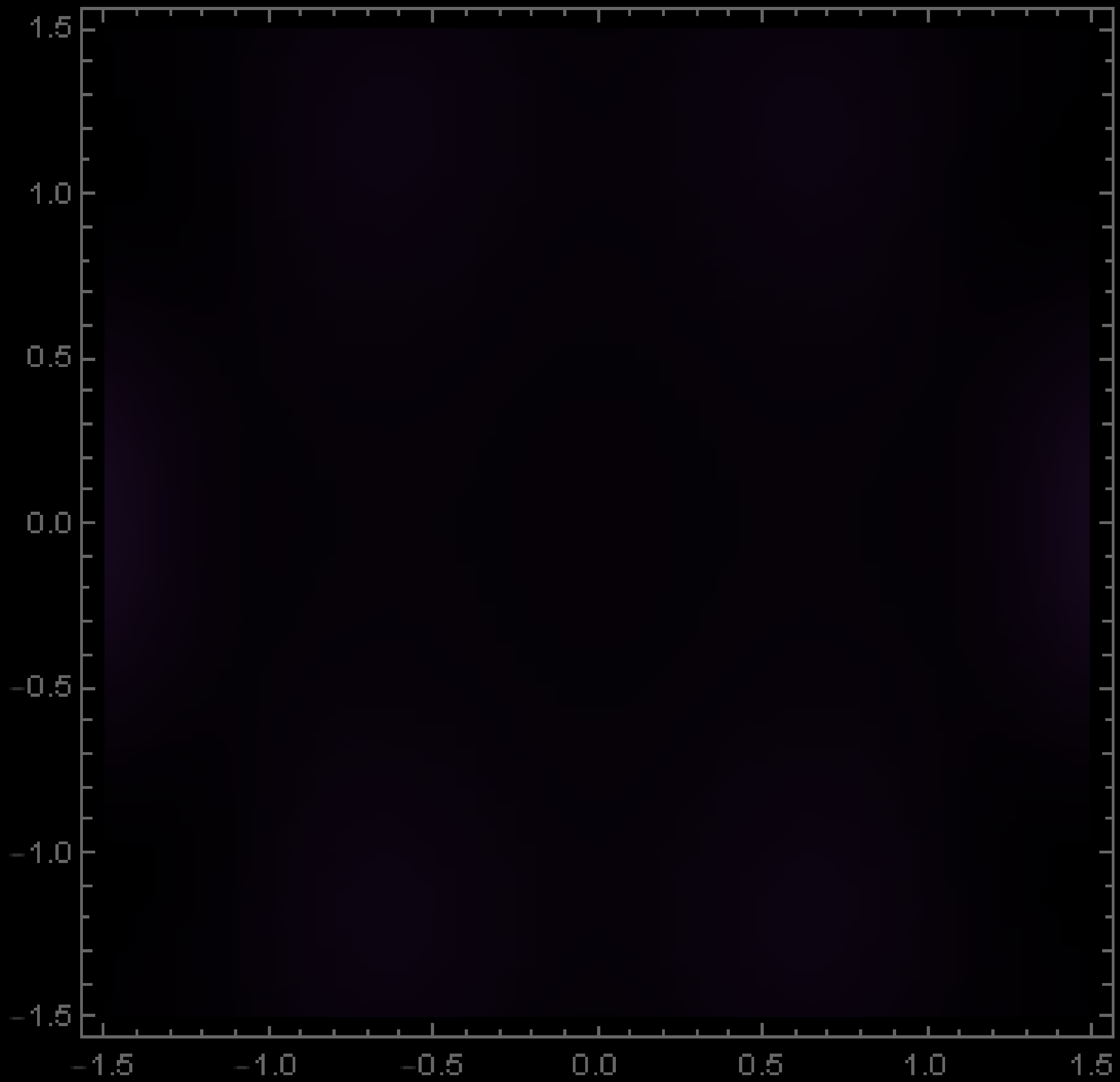
Example 3: $(j;m,n)=(\frac{5}{2};\frac{3}{2},\frac{1}{2})$, type I $\qquad E = 7\,h/\ell$

Example 1

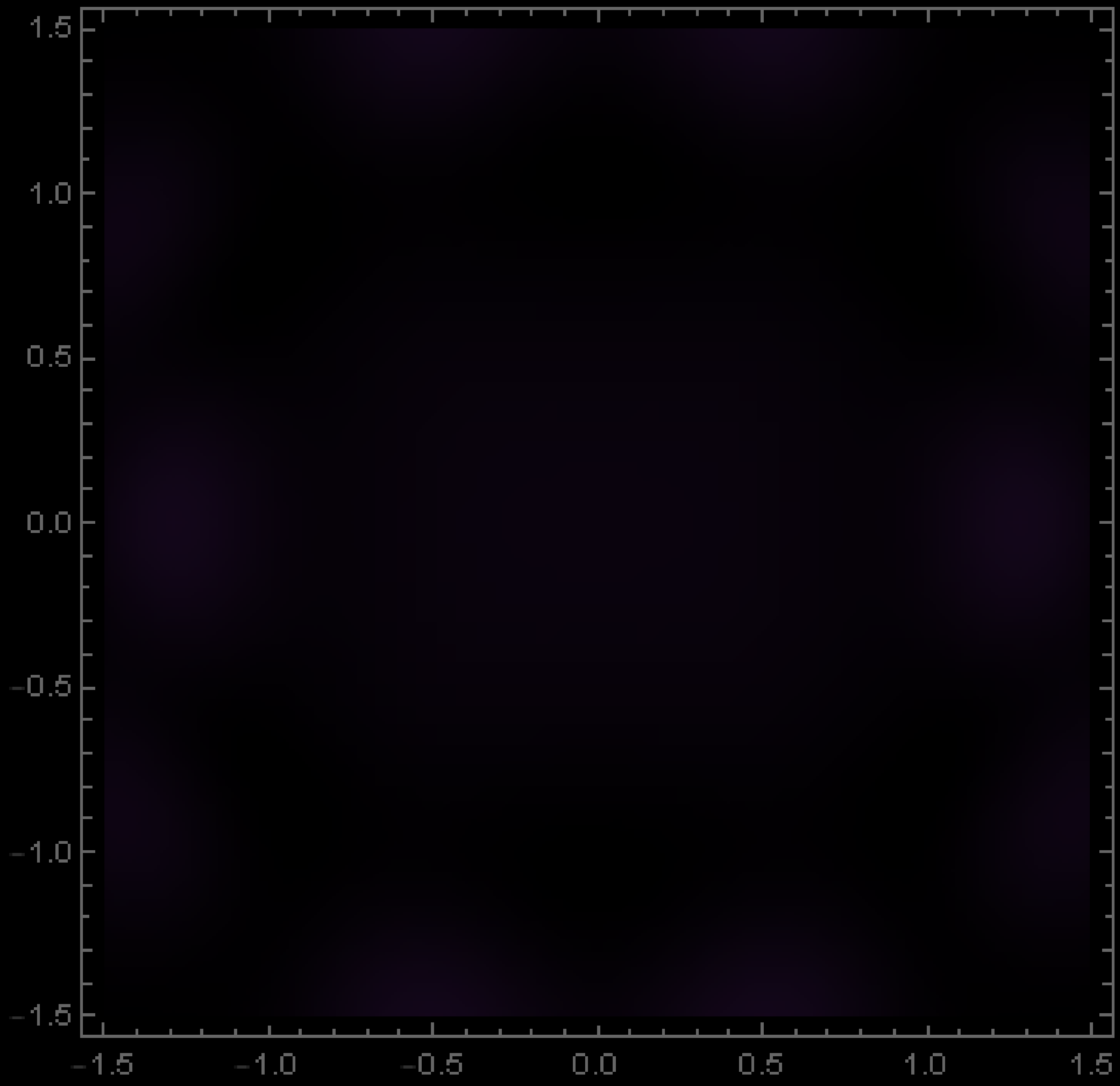
$$(E+\mathrm{i}B)_x = \frac{-2\mathrm{i}}{((t-\mathrm{i})^2 - x^2 - y^2 - z^2)^5} \times$$
$$\times \Big\{ 2y + 3\mathrm{i}ty - xz + 2t^2 y + 2\mathrm{i}txz - 8x^2 y - 8y^3 + 4yz^2$$
$$+\ 4\mathrm{i}t^3 y - 6t^2 xz - 8\mathrm{i}tx^2 y - 8\mathrm{i}ty^3 + 4\mathrm{i}tyz^2 + 10x^3 z + 10xy^2 z - 2xz^3$$
$$+\ 2(\mathrm{i}txz + x^2 y + y^3 + yz^2)(-t^2 + x^2 + y^2 + z^2) + (\mathrm{i}ty - xz)(-t^2 + x^2 + y^2 + z^2)^2 \Big\}$$

$$(E+\mathrm{i}B)_y = \frac{2\mathrm{i}}{((t-\mathrm{i})^2 - x^2 - y^2 - z^2)^5} \times$$
$$\times \Big\{ 2x + 3\mathrm{i}tx + yz + 2t^2 x - 2\mathrm{i}tyz - 8x^3 - 8xy^2 + 4xz^2$$
$$+\ 4\mathrm{i}t^3 x + 6t^2 yz - 8\mathrm{i}tx^3 - 8\mathrm{i}txy^2 + 4\mathrm{i}txz^2 - 10x^2 yz - 10y^3 z + 2yz^3$$
$$+\ 2(-\mathrm{i}tyz + x^3 + xy^2 + xz^2)(-t^2 + x^2 + y^2 + z^2) + (\mathrm{i}tx + yz)(-t^2 + x^2 + y^2 + z^2)^2 \Big\}$$

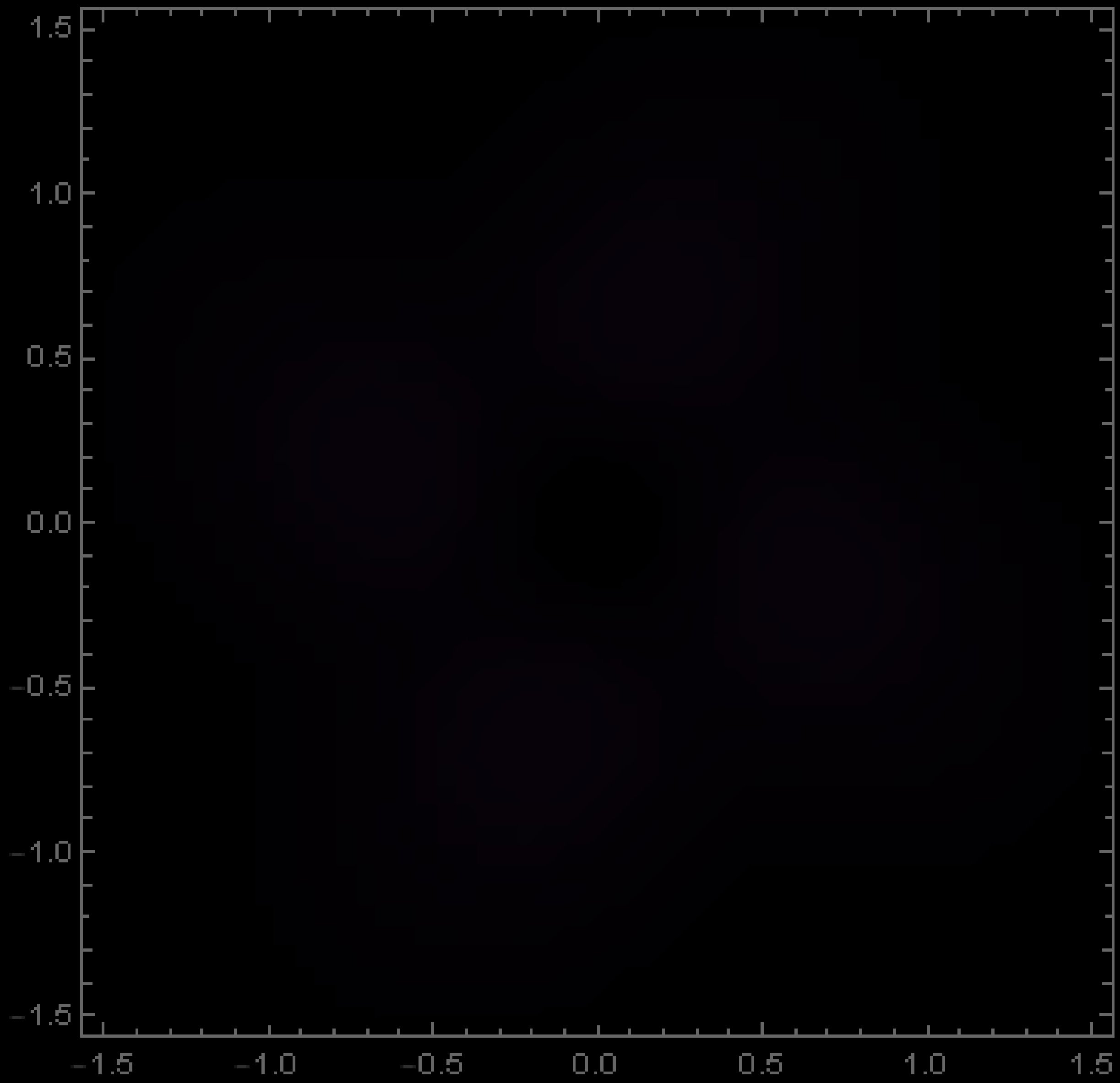
$$(E+\mathrm{i}B)_z = \frac{\mathrm{i}}{((t-\mathrm{i})^2 - x^2 - y^2 - z^2)^5} \times$$
$$\times \Big\{ 1 + 2\mathrm{i}t + t^2 - 11x^2 - 11y^2 + 3z^2 + 4\mathrm{i}t^3 - 16\mathrm{i}tx^2 - 16\mathrm{i}ty^2 + 4\mathrm{i}tz^2$$
$$-\ t^4 - 2t^2 x^2 - 2t^2 y^2 - 2t^2 z^2 + 11x^4 + 22x^2 y^2 + 10x^2 z^2 + 11y^4 - 10y^2 z^2 + 3z^4$$
$$+\ 2\mathrm{i}t(t^2 - 3x^2 - 3y^2 - z^2)(t^2 - x^2 - y^2 - z^2) - (t^2 + x^2 + y^2 - z^2)(-t^2 + x^2 + y^2 + z^2)^2 \Big\}$$



$(j;m,n) = (1;0,0)$: energy density in y=0 plane, changing with time



$(j; m, n) = (2; 1, -1)$: energy density in y=0 plane, changing with time



$(j;m,n) = (\frac{5}{2};\frac{3}{2},\frac{1}{2})$: energy density at $t{=}0$, scanning $z = \text{const}$ planes

Gravitational backreaction (for the YM solutions)

In FLRW spacetime $$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}T^2 + a(T)^2\,\mathrm{d}\Omega_3^2 \\ &= a(T(\tau))^2(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2) \end{aligned}$$ YM does not feel the geometry

$\Rightarrow$ our cylinder solutions remain valid for any cosmological scale factor $a(T)$

But Einstein's equations see the gauge-field energy-momentum

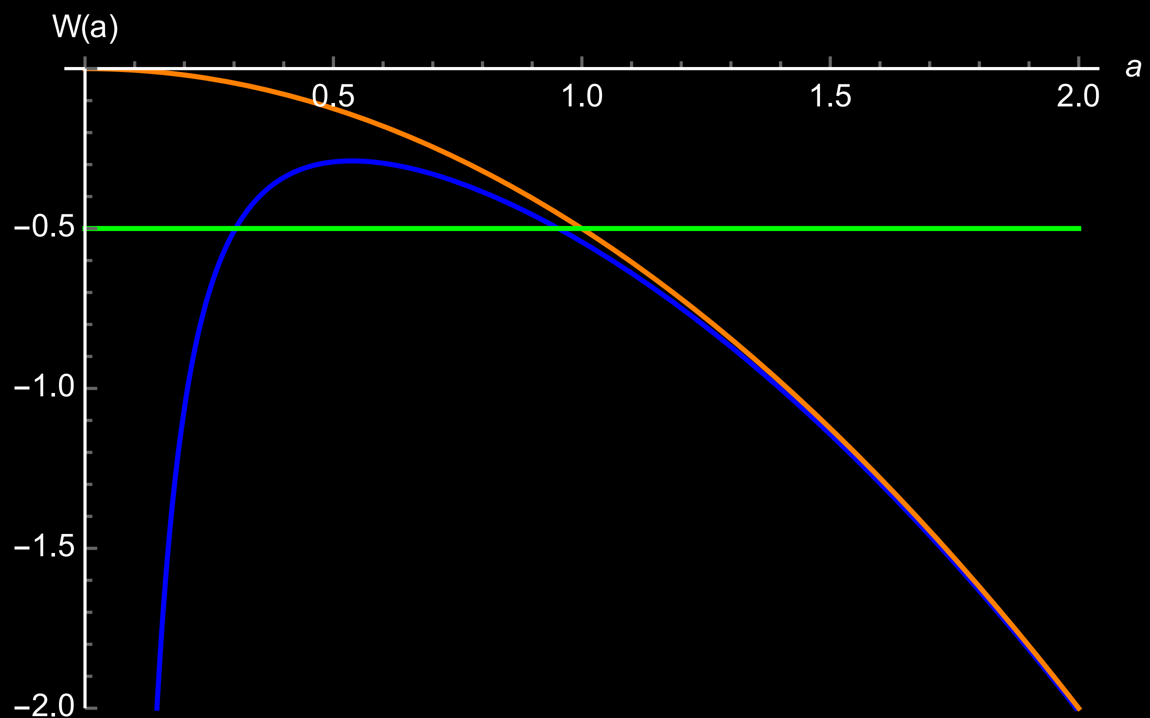
$$T_{TT} = \gamma\,a^{-4} = e = 3\,p \qquad \Rightarrow \qquad \mathrm{tr}\,T = 0$$

with the "double-well" energy density $\gamma = 3\,C(j)\,V_0/2g^2$ for YM coupling g

$$R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} + \Lambda\,g_{\mu\nu} = 8\pi G\,T_{\mu\nu} \qquad \Rightarrow \qquad \left\{ \begin{aligned} -R + 4\,\Lambda &= 0 \\ R_{TT} + \tfrac{1}{2}Ra^2 - \Lambda\,a^2 &= 8\pi G\,\gamma\,a^{-4} \end{aligned} \right\}$$

$$\Rightarrow \left\{ \begin{array}{c} a\,a'' + a'^2 + 1 - \frac{2}{3}\Lambda\,a^2 \;=\; 0 \\ a'^2 + 1 - \frac{1}{3}\Lambda\,a^2 \;=\; 8\pi G\,\frac{1}{3}\gamma\,a^{-2} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} a'' \;+\; \partial_a W(a) \;=\; 0 \\ \frac{1}{2}a'^2 \;+\; W(a) \;=\; -\frac{1}{2} \end{array} \right\}$$

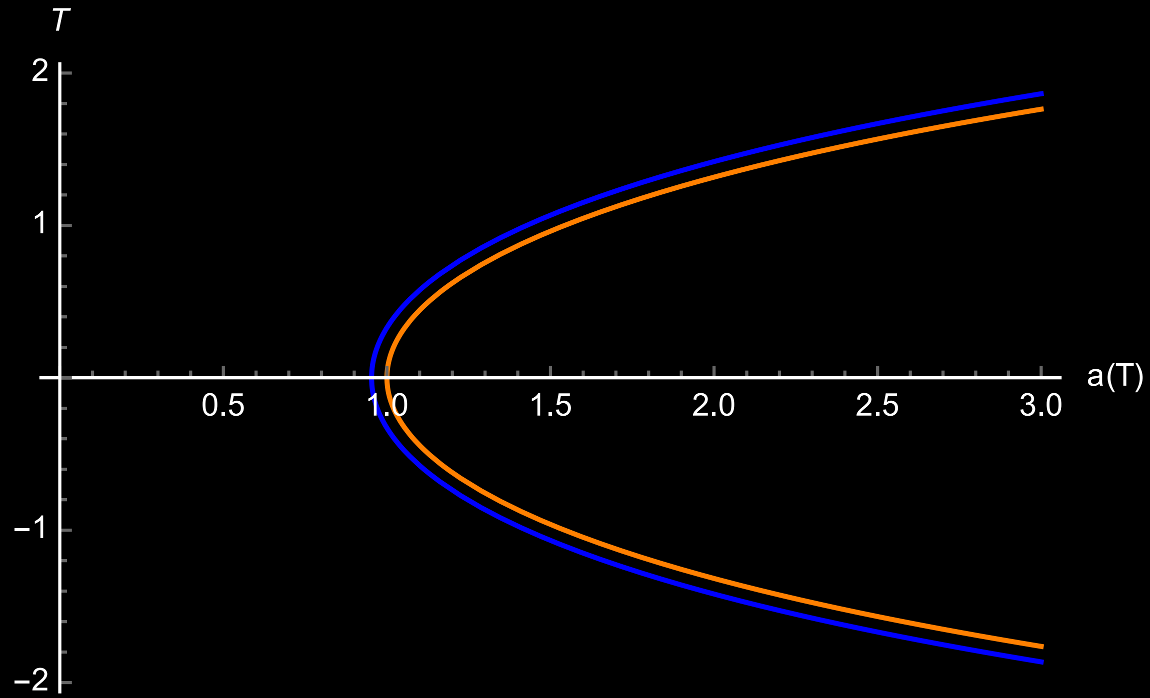
with “cosmological potential” $W(a) \;=\; -\frac{1}{6}\Lambda\,a^2 \;-\; 8\pi G\,\frac{1}{6}\gamma\,a^{-2}$



cosmological potential $W(a)$ and $W(a) = -\frac{1}{2}$ level for $8\pi G = 1$, $\Lambda = 3$ and $\gamma = 0, \frac{1}{4}$

Solution: $a(T)^2 = \frac{3}{2\Lambda} + \sqrt{\frac{9}{4\Lambda^2} - 8\pi G \frac{\gamma}{\Lambda}} \cosh\left(2\sqrt{\frac{\Lambda}{3}}T\right)$

Asymptotic de Sitter fixes $\Lambda = 3$ $\Rightarrow$ $a(T)^2 = \frac{1}{2} + \sqrt{\frac{1}{4} - 8\pi G \frac{\gamma}{3}} \cosh 2T$



cosmological scale factor $a(T)$ for same data as above

Regular solution requires $8\pi G \frac{4}{3}\gamma \leq 1$ $\Leftrightarrow$ $8\pi G \leq g^2/(2\, C(j)\, V_0)$

Action for reduced coupled system?

$a(T)$ analytic in de Sitter frame, $\Psi(\tau)$ analytic in conformal frame, but $T \leftrightarrow \tau$ unknown

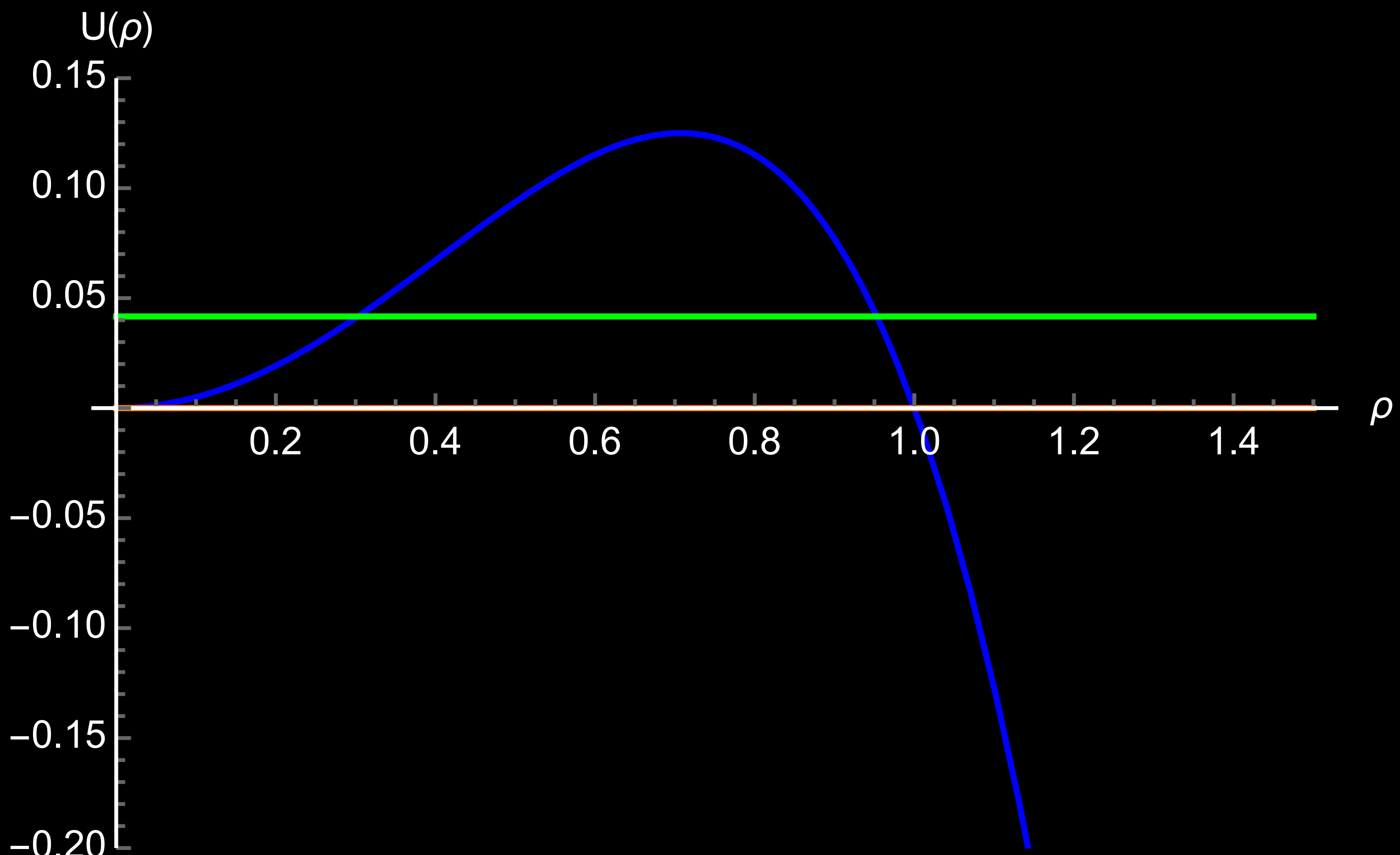
one-way decoupling only in conformal frame $\Rightarrow$ go to cylinder: $a(T(\tau)) =: \rho(\tau)$

$$\ddot{\rho} + \partial_\rho U(\rho) = 0 \quad \text{and} \quad \tfrac{1}{2}\dot{\rho}^2 + U(\rho) = 8\pi G\,\tfrac{\gamma}{6} \quad \text{for} \quad U(\rho) = \tfrac{1}{2}\rho^2 - \tfrac{\Lambda}{6}\rho^4$$

together with

$$\ddot{\Psi} + \partial_\Psi V(\Psi) = 0 \quad \text{and} \quad \tfrac{1}{2}\dot{\Psi}^2 + V(\Psi) = \frac{2g^2\gamma}{3\,C(j)} \quad \text{for} \quad V(\Psi) = 8\Psi^2(\Psi-1)^2$$

Coupling only via zero-sum of conserved energies $\frac{C(j)}{4g^2}\,\mathcal{E}_\Psi - \frac{1}{8\pi G}\,\mathcal{E}_\rho = \frac{\gamma}{6} - \frac{\gamma}{6} = 0$



cosmological potential $U(\rho)$ and $U(\rho) = \frac{\gamma}{6}$ levels for same data as above

Published for SISSA by Springer

Received: *January 3, 2014*
Accepted: *March 9, 2014*
Published: *April 24, 2014*

Nonlinear supersymmetry in the quantum Calogero model

Francisco Correea,[a] **Olaf Lechtenfeld**[b,c] **and Mikhail Plyushchay**[d]

[a] *Centro de Estudios Científicos (CECs),*
Casilla 1468, Valdivia, Chile
[b] *Institut für Theoretische Physik and Riemann Center for Geometry and Physics,*
Leibniz Universität Hannover,
Appelstraße 2, 30167 Hannover, Germany
[c] *Centre for Quantum Engineering and Space-Time Research,*
Leibniz Universität Hannover,
Welfengarten 1, 30167 Hannover, Germany
[d] *Departamento de Física, Universidad de Santiago de Chile,*
Casilla 307, Santiago 2, Chile

E-mail: correa@cecs.cl, lechtenf@itp.uni-hannover.de,
mikhail.plyushchay@usach.cl

JHEP04(2014)1




HAPPY BIRTHDAY MIKHAIL !