ON SOME APPLICATIONS OF CONTACT POTENTIALS

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7th International Workshop on New Challenges in Quantum Mechanics: Integrability and Supersymmetry

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SUMMARY

1.- Introduction

2.- 1D non-relativistic $V(x) = -a\,\delta(x) + b\,\delta'(x)$

3.- $V(x) = \displaystyle\sum_{n=-\infty}^{\infty} \Big(V_0\,\delta(x-na) + aV_1\,\delta'(x-na)\Big)$

4.- Radial potential: $V(r) = -a\,\delta(r-r_0) + b\,\delta'(r-r_0)$

5.- 3D: application in Nuclear Physics

1.- INTRODUCTION

1D non-relativistic QM **contact potentials** are relevant in Theor. Phys.

Easy to deal with them to analyze basic quantum properties: bound states, resonances or scattering.

Used to model point defects in materials, thin structures, heterostructures (abrupt effective mass change), and topological insulators.

In nanophysics: to model sharply peaked impurities inside quantum dots.

In scalar QFT on a line: used to model impurities and external singular backgrounds.

Contact interactions $\delta(x)$ or $\delta'(x)$: analyze perturbations of a free kinetic Schrödinger Hamiltonian, the harmonic oscillator, a constant electric field, the infinite square well, the conical oscillator, etc.

The relativistic counterparts have not yet attracted much attention. We are working in this direction.

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GENERAL PURPOSE:

The analysis of the 1D quantum Hamiltonian

$$H = \frac{p^2}{2m} - a\,\delta(x) + b\,\delta'(x), \quad a > 0,\ b \in \mathbb{R}.$$

If $b = 0 \to$ known. If $a = 0 \to$ incompatible with $\delta(x)$.

SOME ASPECTS THAT CAN BE STUDIED:

- Existence of bound states

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2. Transmission and reflexion coefficients

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(1) H is self-adjoint in $\mathcal{D}$, continuous functions $\psi(x)$ such that

$$\psi'(0^+) - \psi'(0-) = -2m\,a\,\psi(0)/\hbar^2.$$

(2) There is a unique bound state $E = -\frac{ma^2}{2\hbar^2}$.

The case $b \neq 0$ was not considered before 2009 in the literature.

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$\psi(x) : \mathbb{R} \to \mathbb{C}$ in (Sobolev) space of continuous functions, except for a finite jump at the origin ($W_2^2(\mathbb{R}/\{0\})$) such that:

(i) $\psi(x) \in W_2^2(\mathbb{R}/\{0\}) \Rightarrow \psi'(x)$ continuous, except at the origin

(ii) $\psi''(x)$ exists almost everywhere

(iii) $\psi(x)$ and $\psi(x)''(x)$ are square integrable:

$$\int_{-\infty}^{\infty} \{|\psi(x)|^2 + |\psi''(x)|^2\}\, dx < \infty\,.$$

In addition, at $x = 0$, Kurasov's matching conditions must be satisfied:

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For different values of E, we have to solve

$$-\frac{\hbar^2}{2m}\psi'' - a\,\delta(x)\psi(x) + b\,\delta'(x)\psi(x) = E\psi(x)\ \ldots$$

- We must properly define $\delta(x)\psi(x)$, $\delta'(x)\psi(x)$ as distributions
- The $\delta'(x)$ term forces $\psi(x)$ to be discontinuous at $x = 0$ (!!)

ACTION OF $\delta(x)$ AND $\delta'(x)$ ON DISCONTINUOUS FUNCT.

Following Kurasov's proposal:

$$\psi(x)\delta(x) = \frac{\psi(0+) + \psi(0-)}{2}\,\delta(x),$$

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$$-\frac{\hbar^2}{2m}\psi'' - a\,\delta(x)\psi(x) + b\,\delta'(x)\psi(x) = E\psi(x)\ \ldots$$

- We must properly define $\delta(x)\psi(x)$, $\delta'(x)\psi(x)$ as distributions
- The $\delta'(x)$ term forces $\psi(x)$ to be discontinuous at $x = 0$ (!!)

ACTION OF $\delta(x)$ AND $\delta'(x)$ ON DISCONTINUOUS FUNCT.

Following Kurasov's proposal:

$$\psi(x)\delta(x) = \frac{\psi(0+) + \psi(0-)}{2}\,\delta(x),$$

$$\psi(x)\delta'(x) = \frac{\psi(0+) + \psi(0-)}{2}\,\delta'(x) - \frac{\psi'(0+) + \psi'(0-)}{2}\,\delta(x)\,.$$

The Schrödinger equation

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Bound sates

Schrödinger equation with $E < 0$

As $V(x) = 0$ for $x \neq 0$, the solution that vanishes as $x \to \pm\infty$ is

$$\psi(x) = \alpha\, e^{\kappa x}\,\Theta(-x) + \beta\, e^{-\kappa x}\,\Theta(x), \quad \kappa = \sqrt{-2mE/\hbar^2},$$

where $\psi(0^-) = \alpha \neq \beta = \psi(0^+)$ and $\Theta(x)$ is the Heaviside step function.

To assure self-adjointness, we impose Kurasov's matching conditions:

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$$E = -\frac{1}{2}\,\frac{m a^2 \hbar^6}{(\hbar^4 + b^2\, m^2)^2}$$

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SCHRÖDINGER EQUATION WITH $E < 0$ $(\hbar = 1)$

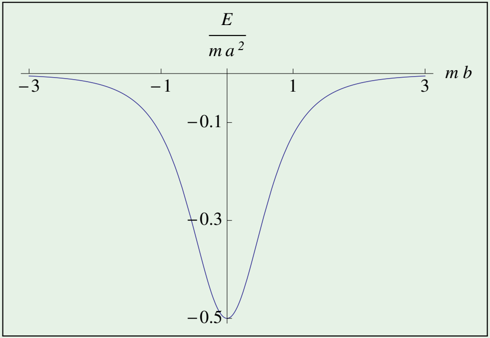

FIGURE: Energy of the only bound state as a function of $m\,b$

Bound sates

Schrödinger equation with $E < 0$ $(\hbar = 1)$

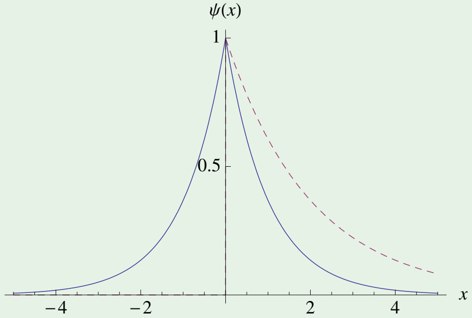

FIGURE: **Dashed line** $b = 1/m$: the function vanishes on the negative real axis. **Solid line** $b = 0$.

T and R coefficients

Incoming wave e^{ikx}, $\quad k = \sqrt{2mE/\hbar^2}$, $\quad E \geq 0$:

For $x < 0$: $\quad \psi(x) = e^{ikx} + Re^{-ikx}$ $\qquad$ For $x > 0$: $\quad \psi(x) = Te^{ikx}$

where R and T are the reflection and transmission coefficients

Using the matching conditions at the origin:

$$\begin{pmatrix} T \\ ikT \end{pmatrix} = \begin{pmatrix} \dfrac{\hbar^2 + mb}{\hbar^2 - mb} & 0 \\ \dfrac{-2am\hbar^2}{\hbar^4 - m^2 b^2} & \dfrac{\hbar^2 - mb}{\hbar^2 + mb} \end{pmatrix} \begin{pmatrix} 1 + R \\ ik(1 - R) \end{pmatrix}$$

$$R(k) = \frac{-(am + 2mbk\,i)}{am + (1 + m^2 b^2)k\,i} \qquad T(k) = \frac{(1 - m^2 b^2)k\,i}{am + (1 + m^2 b^2)k\,i}$$

satisfying $|R(k)|^2 + |T(k)|^2 = 1$ $\qquad (\hbar = 1)$

If $b = \pm 1/m$ there is no transmission

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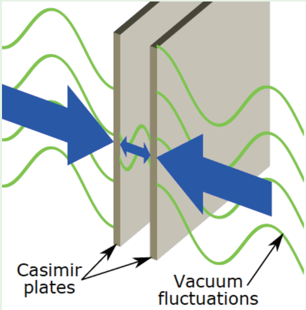
THE DOUBLE $\delta - \delta'$ WELL

JM MUÑOZ AND J MATEOS, PRD **91** (2015) 025028

Quantum vacuum interaction in a Casimir setup



by mimicking the plates as two contact interactions of the form:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + a_1\delta(x+q) + b_1\delta'(x+q) + a_2\delta(x-q) + b_2\delta'(x-q)$$

Then, it is natural to consider ...

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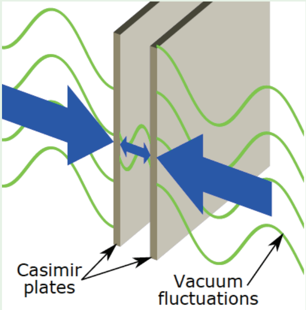

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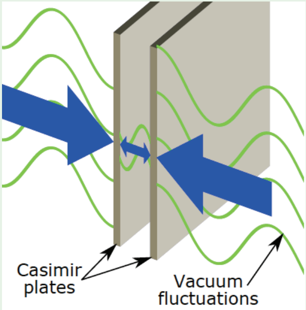

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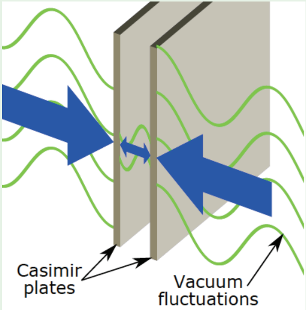

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Kronig-Penney model: exactly solvable periodic potential, used in Solid State Physics, which describes electron motion in a period array of **rectangular** barriers.

The **Dirac comb** is obtained by taking the appropriate limit in KP model, such that the rectangular barriers become Dirac delta distributions:

$$V_{DKP}(x) = V_0 \sum_{n=-\infty}^{\infty} \delta(x - na), \; V_0 > 0.$$

We consider a modified Dirac comb, adding a δ' in every singular point:

$$V_1(x) = \sum_{n=-\infty}^{\infty} \Big(V_0\, \delta(x - na) + aV_1\, \delta'(x - na) \Big), \; a, V_0 > 0, V_1 \in \mathbb{R}.$$

Periodic array of charges and dipoles.

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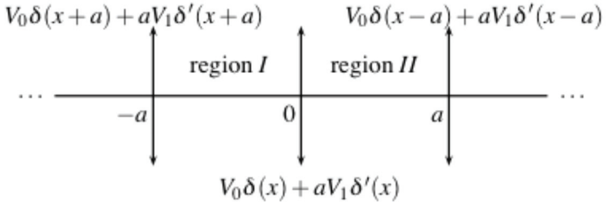

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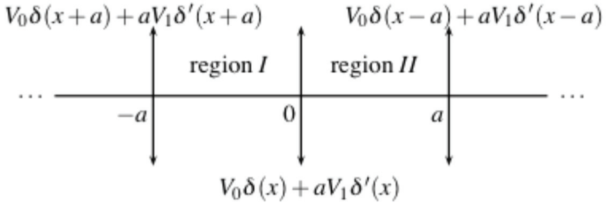

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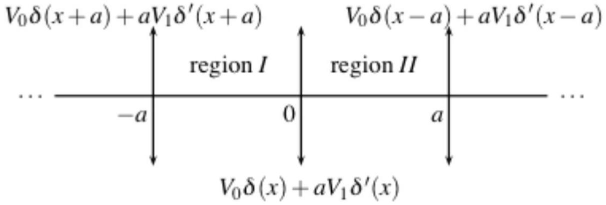

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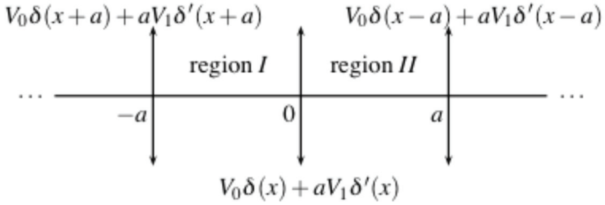

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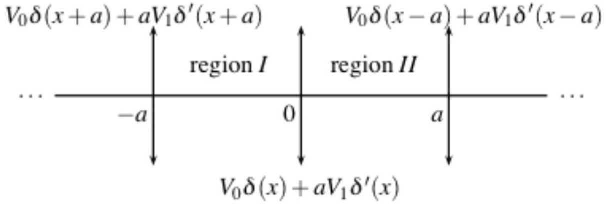

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SOLVING THE SCHRÖDINGER EQN. IN REGIONS I AND II:

$$\psi_I(x) = A_I\, e^{ikx} + B_I\, e^{-ikx}, \qquad \psi_{II}(x) = A_{II}\, e^{ikx} + B_{II}\, e^{-ikx},$$

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being $k = \frac{\sqrt{2mE}}{\hbar} > 0$.

In matrix compact form:

$$\vec{\psi}_J(x) = \begin{pmatrix} \psi_J(x) \\ \psi_J'(x) \end{pmatrix} = \mathbb{K}\mathbb{M}_x \begin{pmatrix} A_J \\ B_J \end{pmatrix}, \; J = I, II,$$

where

$$\mathbb{K} = \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix}, \qquad \mathbb{M}_x = \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix}.$$

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IMPOSING THE MATCHING CONDITIONS AT $x = 0$

Using the notation just introduced, Kurasov's matching conditions are

$$\vec{\psi}_{II}(0^+) = \begin{pmatrix} \frac{\hbar^2+maV_1}{\hbar^2-maV_1} & 0 \\ \frac{2m\hbar^2 V_0}{\hbar^4-m^2a^2V_1^2} & \frac{\hbar^2-maV_1}{\hbar^2+maV_1} \end{pmatrix} \vec{\psi}_I(0^-),$$

if $V_1 \neq \pm\hbar^2/(ma)$.

They can be written as $\vec{\psi}_{II}(0^+) = \mathbb{T}_U\, \vec{\psi}_I(0^-)$, with

$$\mathbb{T}_U = \begin{pmatrix} \frac{1+U_1}{1-U_1} & 0 \\ \frac{2U_0/a}{1-U_1^2} & \frac{1-U_1}{1+U_1} \end{pmatrix}, \; U_0 = \frac{maV_0}{\hbar^2}, U_1 = \frac{maV_1}{\hbar^2}.$$

And finally

$$\begin{pmatrix} A_{II} \\ B_{II} \end{pmatrix} = \mathbb{K}^{-1}\mathbb{T}_U\mathbb{K} \begin{pmatrix} A_I \\ B_I \end{pmatrix}.$$

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$$\vec{\psi}_{II}(0^+) = \begin{pmatrix} \frac{\hbar^2+maV_1}{\hbar^2-maV_1} & 0 \\ \frac{2m\hbar^2 V_0}{\hbar^4-m^2a^2V_1^2} & \frac{\hbar^2-maV_1}{\hbar^2+maV_1} \end{pmatrix} \vec{\psi}_I(0^-),$$

if $V_1 \neq \pm\hbar^2/(ma)$.

They can be written as $\vec{\psi}_{II}(0^+) = \mathbb{T}_U\,\vec{\psi}_I(0^-)$, with

$$\mathbb{T}_U = \begin{pmatrix} \frac{1+U_1}{1-U_1} & 0 \\ \frac{2U_0/a}{1-U_1^2} & \frac{1-U_1}{1+U_1} \end{pmatrix}, \; U_0 = \frac{maV_0}{\hbar^2}, U_1 = \frac{maV_1}{\hbar^2}.$$

And finally

$$\begin{pmatrix} A_{II} \\ B_{II} \end{pmatrix} = \mathbb{K}^{-1}\mathbb{T}_U\mathbb{K}\begin{pmatrix} A_I \\ B_I \end{pmatrix}.$$

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THE FLOQUET-BLOCH THEOREM

Now we must take into account the periodicity of the potential: the Floquet-Bloch theorem forces

$$\psi(x+a) = e^{iqa}\psi(x) \;\Rightarrow\; \psi'(x+a) = e^{iqa}\psi'(x),$$

where q is the momentum of the crystal.

In matrix form, for $x \in (-a, 0)$,

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As the matrices $\mathbb{M}_x$ and $\mathbb{K}$ are invertible:

$$[\mathbb{M}_a\mathbb{K}^{-1}\mathbb{T}_U\mathbb{K} - e^{iqa}\,\mathbb{I}]\begin{pmatrix} A_I \\ B_I \end{pmatrix} = \vec{0},$$

and hence

$$\boxed{\det\left[\mathbb{T}_U - e^{iqa}\mathbb{K}\mathbb{M}_a^{-1}\mathbb{K}^{-1}\right] = 0.}$$

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THE DISPERSION RELATION:

Defining $\tilde{q} = aq$, $\tilde{k} = ka$, we get the dispersion relation (DR)

$$\boxed{\cos\tilde{q} = f(U_1)\left[\cos\tilde{k} + U_0\, g(U_1)\,\frac{\sin\tilde{k}}{\tilde{k}}\right]}$$

$$f(U_1) = \frac{1+{U_1}^2}{1-{U_1}^2}, \qquad g(U_1) = \frac{1}{1+{U_1}^2}.$$

The DR is an even function of $U_1 \propto V_1$, the coefficient of δ'.

• If there is no δ'-terms, then $f(0) = g(0) = 1$ and it reduces to the well know band structure of the standard Dirac comb

$$\cos\tilde{q} = \cos\tilde{k} + U_0\,\frac{\sin\tilde{k}}{\tilde{k}}.$$

• If $U_1 \neq 0$ we also have an interesting band structure. Plots are shown in the next Figures.

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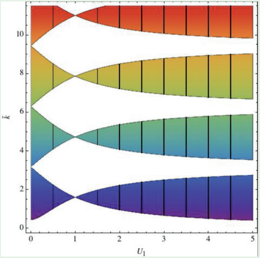
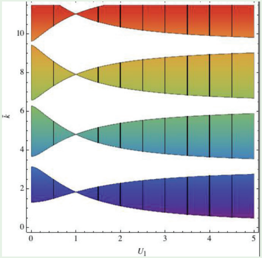
ALLOWED (COLOR) AND FORBIDDEN (WHITE) BANDS

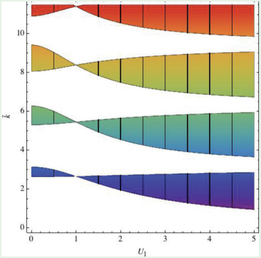
Case $U_0 = 0.1$:



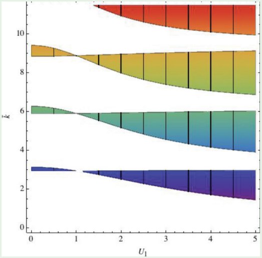
Case $U_0 = 1$:



Case $U_0 = 10$:



Case $U_0 = 30$:



4.- HYPERSPHERICAL δ-δ'

THE PROBLEM TO BE SOLVED:

A non-relativistic quantum particle in $\mathbb{R}^d$ with hyper-spherical potential

$$\widehat{V}_{\delta\text{-}\delta'}(r) = a\,\delta(r - r_0) + b\,\delta'(r - r_0), \quad a, b \in \mathbb{R},\ r_0 > 0.$$

The quantum Hamiltonian operator is

$$\mathbf{H} = \frac{-\hbar^2}{2\,m}\widehat{\Delta}_d + \widehat{V}_{\delta\text{-}\delta'}(r),$$

If we introduce dimensionless quantities:

$$\mathbf{h} \equiv \frac{2}{mc^2}\mathbf{H}, \quad w_0 \equiv \frac{2a}{\hbar c}, \quad w_1 \equiv \frac{bm}{\hbar^2}, \quad x \equiv \frac{mc}{\hbar}r,$$

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HYPERSPHERICAL COORDINATES

The d-dimensional Laplace operator Δ_d is

$$\Delta_d = \frac{1}{x^{d-1}}\frac{\partial}{\partial x}\left(x^{d-1}\frac{\partial}{\partial x}\right) + \frac{\Delta_{S^{d-1}}}{x^2}$$

$\Delta_{S^{d-1}} = -\mathbf{L}_d^2$ is the Laplace-Beltrami operator in S^{d-1}.

$\mathbf{L}_d^2$ is the square of the generalised angular momentum operator.

The eigenvalue equation for $\mathbf{h}$ is separable in hyperspherical coordinates

$$\psi_{\lambda\ell}(x, \Omega_d) = R_{\lambda\ell}(x) Y_\ell(\Omega_d),$$

$R_{\lambda\ell}(x)$ is the radial wave function.

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RADIAL PROBLEM:

The radial wave function is

$$\left[-\frac{d^2}{dx^2}-\frac{d-1}{x}\frac{d}{dx}+\frac{\ell(\ell+d-2)}{x^2}+V_{\delta\text{-}\delta'}(x)\right]R_{\lambda\ell}(x)=\lambda R_{\lambda\ell},$$

being

$$V_{\delta\text{-}\delta'}(x)=w_0\delta(x-x_0)+2w_1\delta'(x-x_0).$$

We introduce the reduced radial function

$$u_{\lambda\ell}(x)\equiv x^{\frac{d-1}{2}}R_{\lambda\ell}(x),$$

to remove the first derivative, and we get

$$\left(\mathcal{H}_0+V_{\delta\text{-}\delta'}(x)\right)u_{\lambda\ell}(x)=\lambda_\ell u_{\lambda\ell}(x),$$

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MATCHING CONDITIONS

The domain of the selfadjoint extension $\mathcal{H}_0 + V_{\delta\text{-}\delta'}$ of the operator $\mathcal{H}_0$ defined on $\mathbb{R}_{x_0}$ is given by the square integrable functions such that

$$\begin{pmatrix} f(x_0^+) \\ f'(x_0^+) \end{pmatrix} = \begin{pmatrix} \frac{1+w_1}{1-w_1} & 0 \\ \frac{w_0}{1-w_1^2} & \frac{1-w_1}{1+w_1} \end{pmatrix} \begin{pmatrix} f(x_0^-) \\ f'(x_0^-) \end{pmatrix}.$$

The matching conditions for the radial wave function $R_{\lambda\ell}$:

$$\begin{pmatrix} R_{\lambda\ell}(x_0^+) \\ R'_{\lambda\ell}(x_0^+) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \begin{pmatrix} R_{\lambda\ell}(x_0^-) \\ R'_{\lambda\ell}(x_0^-) \end{pmatrix},$$

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BOUND STATES

The solutions of the radial wave function is

$$R_{\kappa\ell}(x) = \begin{cases} A_1\, \mathcal{I}_\ell(\kappa x) + B_1\, \mathcal{K}_\ell(\kappa x) & \text{if} \quad x \in (0, x_0), \\ A_2\, \mathcal{I}_\ell(\kappa x) + B_2\, \mathcal{K}_\ell(\kappa x) & \text{if} \quad x \in (x_0, \infty), \end{cases}$$

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being I_ℓ and K_ℓ modified Bessel functions of the first and second kind.

The matching conditions give the secular equation

$$F(\kappa x_0) = (d-2)\left(\alpha - \alpha^{-1}\right) + \beta x_0,$$

where

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BOUND STATES

The solutions of the radial wave function is

$$R_{\kappa\ell}(x) = \begin{cases} A_1\, \mathcal{I}_\ell(\kappa x) + B_1\, \mathcal{K}_\ell(\kappa x) & \text{if} \quad x \in (0, x_0), \\ A_2\, \mathcal{I}_\ell(\kappa x) + B_2\, \mathcal{K}_\ell(\kappa x) & \text{if} \quad x \in (x_0, \infty), \end{cases}$$

where

$$\mathcal{I}_\ell(z) \equiv \frac{1}{z^\nu}\, I_{\ell+\nu}(z), \quad \mathcal{K}_\ell(z) \equiv \frac{1}{z^\nu}\, K_{\ell+\nu}(z) \quad \text{with} \quad \nu \equiv \frac{d-2}{2},$$

being I_ℓ and K_ℓ modified Bessel functions of the first and second kind.

The matching conditions give the secular equation

$$F(\kappa x_0) = (d-2)\left(\alpha - \alpha^{-1}\right) + \beta x_0,$$

where

$$\boxed{F(\kappa x_0) \equiv -\kappa x_0 \left(\frac{I_{\nu+\ell-1}(\kappa x_0)}{\alpha\, I_{\nu+\ell}(\kappa x_0)} + \frac{\alpha\, K_{\nu+\ell-1}(\kappa x_0)}{K_{\nu+\ell}(\kappa x_0)} \right) - \left(\alpha - \alpha^{-1}\right)\ell}$$

PLOTS:

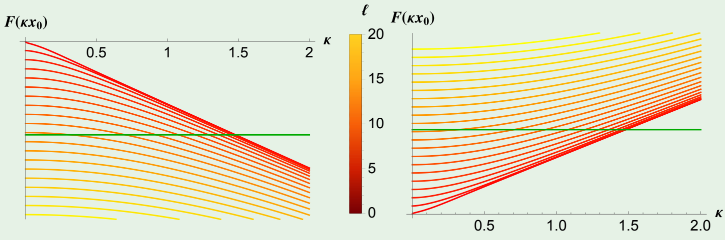

FIGURE: Each curve represents $F(\kappa\, x_0)$ for different values of the angular momentum, and $d = 2$. The green horizontal line is the constant on the rhs. LEFT: $\alpha = 0.8$, $\beta = -3$ and $x_0 = 7$. RIGHT: $\alpha = -0.8$, $\beta = 3$ and $x_0 = 7$.

MORE:

- Scattering states
- Existence of zero modes (states of $E = 0$)

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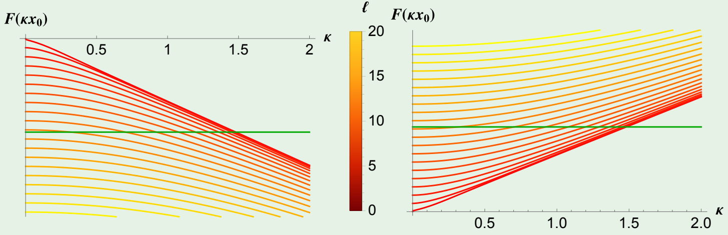

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5.- Nuclear Physics

δ-δ' interaction may be obtained as a limit of a regular mean-field nuclear potential with volume, surface, and spin-orbit parts:

$$H(\mathbf{r}) = -\frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{r}} + U_0(r) + U_{so}(r)(\mathbf{L}\cdot\mathbf{S}) + U_q(r)$$

μ is the reduced. $U_0(r)$, $U_{so}(r)$, and $U_q(r)$ comes from Woods-Saxon:

$$\begin{aligned}
U_0(r) &= -V_0\, f(r) = -V_0\,\frac{1}{1+e^{(r-R)/a}},\\
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a is the thickness of the nuclear surface.
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RADIAL & ANGULAR PARTS:

The eigenfunction $\psi(\mathbf{r}) = \frac{u_{n\ell j}(r)}{r}\,\mathcal{Y}_{\ell jm}(\theta,\phi)$:

$$\mathbf{L}^2\mathcal{Y}_{\ell jm}(\theta,\phi) = \hbar^2\ell(\ell+1)\mathcal{Y}_{\ell jm}(\theta,\phi),\ (\mathbf{L}\cdot\mathbf{S})\mathcal{Y}_{\ell jm}(\theta,\phi) = \hbar^2\xi_{\ell,j}\mathcal{Y}_{\ell jm}(\theta,\phi)$$

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$$\xi_{\ell,j} \equiv \begin{cases} \frac{\ell}{2} & \text{for} \quad j = \ell + \frac{1}{2}, \\ -\frac{(\ell+1)}{2} & \text{for} \quad j = \ell - \frac{1}{2}, \end{cases} \qquad \ell \in \mathbb{N}\cup\{0\}.$$

$\mathcal{Y}_{\ell jm}(\theta,\phi)$ is a simultaneous eigenfunction of the operators

$$\mathbf{L}^2, \quad \mathbf{S}^2, \quad \mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2.$$

The 3D Schrödinger equation reduces to $H(\mathbf{r})\psi(\mathbf{r}) = E\,\psi(\mathbf{r})$ is

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$$\lim_{a\to 0} U_0(r) = -V_0 \lim_{a\to 0} f(r) = V_0\,[\Theta(r-R)-1], \qquad r \geq 0.$$

Consequently, we have that

$$\lim_{a\to 0} V_{so}\,\xi_{\ell,j}\,f'(r) = -V_{so}\,\xi_{\ell,j}\,\delta(r-R)\,.$$

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We get the following the singular Hamiltonian

$$H_{sing}(r) = -\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right] + V_0[\theta(r-R)-1] - V_{so}\xi_{\ell,j}\,\delta(r-R) - V_q\delta'(r-R).$$

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$$\frac{d^2u(r)}{dr^2}+\left\{\frac{2\mu E}{\hbar^2}-\frac{2\mu V_0}{\hbar^2}[\theta(r-R)-1]+\alpha\,\delta(r-R)+\beta\,\delta'(r-R)-\frac{\ell(\ell+1)}{r^2}\right\}u(r)=0\,,$$

where

$$\alpha=\frac{2\mu}{\hbar^2}V_{so}\,\xi_{\ell,j},\qquad \beta=\frac{2\mu}{\hbar^2}V_q\,.$$

Wave equation inside the nucleus ($0 \le r < R$): in this region the square integrable solution is

$$u_{1,\ell}(r)=A_\ell\sqrt{\gamma r}\,J_{\ell+\frac{1}{2}}(\gamma r)\,,\quad \gamma=\frac{\sqrt{2\mu(V_0+E)}}{\hbar}\,,\quad r\in[0,R)\,.$$

Wave equation outside the nucleus ($r > R$): in this region the square integrable solution is

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MATCHING CONDITIONS AT $r = R$

To fix a self-adjoint extension of the relevant operator

$$\begin{pmatrix} u_{2,\ell}(R^+) \\ u'_{2,\ell}(R^+) \end{pmatrix} = \begin{pmatrix} \dfrac{2+\beta}{2-\beta} & 0 \\ \dfrac{4\alpha}{4-\beta^2} & \dfrac{2-\beta}{2+\beta} \end{pmatrix} \begin{pmatrix} u_{1,\ell}(R^-) \\ u'_{1,\ell}(R^-) \end{pmatrix},$$

From here, the **secular equation**

$$\frac{\chi\, J_{\ell+\frac{3}{2}}(\chi)}{J_{\ell+\frac{1}{2}}(\chi)} = \frac{(2+\beta)^2}{(2-\beta)^2}\, \frac{\sigma\, K_{\ell+\frac{3}{2}}(\sigma)}{K_{\ell+\frac{1}{2}}(\sigma)} - \frac{8\beta(\ell+1)}{(2-\beta)^2} + \frac{w_0}{(2-\beta)^2} = \phi(\sigma),$$

where

$$\chi := v_0\sqrt{1-\varepsilon}\,, \qquad \sigma := v_0\sqrt{\varepsilon}\,,$$

$$v_0 \equiv \sqrt{\frac{2\mu R^2 V_0}{\hbar^2}} > 0\,, \quad w_0 \equiv 4\alpha R = \frac{8\mu V_{so}\xi_{\ell,j} R}{\hbar^2}\,,$$

$$\varepsilon \equiv \frac{|E|}{V_0} \in (0,1)\,.$$

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$$v_0 \equiv \sqrt{\frac{2\mu R^2 V_0}{\hbar^2}} > 0\,, \quad w_0 \equiv 4\alpha R = \frac{8\mu V_{so}\xi_{\ell,j}R}{\hbar^2}\,,$$

$$\varepsilon \equiv \frac{|E|}{V_0} \in (0,1)\,.$$

MATCHING CONDITIONS AT $r = R$

To fix a self-adjoint extension of the relevant operator

$$\begin{pmatrix} u_{2,\ell}(R^+) \\ u'_{2,\ell}(R^+) \end{pmatrix} = \begin{pmatrix} \dfrac{2+\beta}{2-\beta} & 0 \\ \dfrac{4\alpha}{4-\beta^2} & \dfrac{2-\beta}{2+\beta} \end{pmatrix} \begin{pmatrix} u_{1,\ell}(R^-) \\ u'_{1,\ell}(R^-) \end{pmatrix},$$

From here, the **secular equation**

$$\frac{\chi\, J_{\ell+\frac{3}{2}}(\chi)}{J_{\ell+\frac{1}{2}}(\chi)} = \frac{(2+\beta)^2}{(2-\beta)^2}\, \frac{\sigma\, K_{\ell+\frac{3}{2}}(\sigma)}{K_{\ell+\frac{1}{2}}(\sigma)} - \frac{8\beta(\ell+1)}{(2-\beta)^2} + \frac{w_0}{(2-\beta)^2} = \phi(\sigma),$$

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Physical example

For the isotope ^{209}Pb, the relevant parameters describing the lowest experimental energy states are

$V_0 = 44.4$ MeV, $V_{so} = 16.5$ MeV fm, $R = 7.525$ fm,

$a = 0.7$ fm, and $\frac{2\mu}{\hbar^2} = 0.0480253$ MeV^{-1} fm^{-2}.

Then: $v_0 = 10.98, \quad w_0 = 23.83\,\xi_{\ell,j}$.

	$\beta = 0$		$\beta = 1$	
State	Numerical	Model	Numerical	Model
$0s_{1/2}$	-41.35	-41.36	-40.97	-40.85
$1s_{1/2}$	-32.27	-32.31	-31.11	-30.23
$2s_{1/2}$	-17.53	-17.61	-18.11	-12.92
$0p_{3/2}$	-38.21	-37.96	-37.48	-37.12
$1p_{3/2}$	-26.29	-25.53	-25.30	-22.97
$2p_{3/2}$	-9.17	-7.71	-13.30	-2.78
$0p_{1/2}$	-38.08	-39.16	-37.34	-37.19
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RESONANCES:

Resonance state functions (Gamow functions) are not square integrable.

We may write this solution for $r > R$ in terms of the Hänkel functions of first (1) and second kind (2) as

$$u_l(r) := \sqrt{\kappa r}\left(C_\ell H^{(1)}_{\ell+\frac{1}{2}}(\kappa r) + D_\ell H^{(2)}_{\ell+\frac{1}{2}}(\kappa r)\right),\ \kappa = \frac{\sqrt{2\mu E}}{\hbar},\ E > 0.$$

Resonances are given by the **purely outgoing boundary condition**: only the outgoing wave function survives.

Asymptotic behavior: $H^{(1)}_{\ell+\frac{1}{2}}(\kappa r)$ is outgoing; $H^{(2)}_{\ell+\frac{1}{2}}(\kappa r)$ is incoming.

This is condition is satisfied if and only if $D_\ell = 0$. Imposing the matching condition between the outgoing function and the wave function inside the potential well (nucleus):

$$H^{(1)}_{\ell+\frac{1}{2}}(R\kappa)\left[8(\alpha R-\beta)J_{\ell+\frac{1}{2}}(R\gamma)-(\beta-2)^2R\gamma J_{\ell+\frac{3}{2}}(R\gamma)+(\beta-2)^2R\gamma J_{\ell-\frac{1}{2}}(R\gamma)\right]$$
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Types of solutions:

Are classified in three categories:

- (i) Simple solutions on the positive imaginary axis that correspond to the **bound states**.
- (ii) Simple solutions on the negative part of the imaginary axis, that show the presence of **antibound or virtual states**.
- (iii) Pairs of solutions on the lower half plane, symmetrically located with respect to the imaginary axis: **resonances**.

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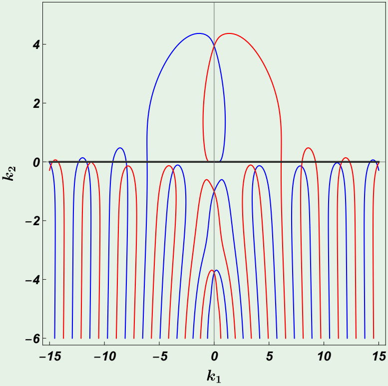
PLOTS:

In blue Re $F(k_1, k_2) = 0$ and in red Im $F(k_1, k_2) = 0$, from the key equation. Bound states and resonances correspond to intersection of red and blue curves. The parameters are $v_0 = 5$, $w_0 = 10$ and $\beta = 1$.

$\ell = 0$:



More plots: $\ell = 1, 2, 3, 4$