Integrable Hierarchies, Solitons and Infinite Dimensional Algebras

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Aim

- Discuss the general structure of *Integrable non Linear evolution equations* associated to graded Lie algebraic structure.

- Integrable Hierarchies
  1. Systematic construction Soliton Equations
  2. Systematic construction of Soliton solutions.
  3. Systematic construction of Backlund transformation and Defects.

Exemples:

$$
\begin{aligned}
\partial_{t_1} v &= v_x, \qquad v_x = \partial_x v, \\
4\partial_{t_3} v &= v_{3x} - 6v^2 v_x, \qquad mKdV \\
16\partial_{t_5} v &= v_{5x} - 10v^2 v_{3x} - 40 v v_x v_{2x} - 10 v_x^3 + 30 v^4 v_x, \\
64\partial_{t_7} v &= v_{7x} - 182 v_x v_{2x}^2 - 126 v_x^2 v_{3x} - 140 v v_{2x} v_{3x} \\
&- 84 v v_x v_{4x} - 14 v^2 v_{5x} + 420 v^2 v_{3x} + 560 v^3 v_x v_{2x} \\
&+ 70 v^4 v_{3x} - 140 v^6 v_x
\end{aligned}
$$

$\cdots$ *etc*

Series of *Integrable Non linear equations* presenting a **Universal Structure** based upon **Affine Lie Algebras** leading to the following properties:

- Vacuum Solution: $v = const = 0$ is solution
- One soliton Solution

$$v(x, t_N) = \partial_x ln(\frac{1+\rho}{1-\rho}), \qquad \rho = e^{kx+k^N t_N}, \quad N = 3, 5, \cdots$$

- Eqs. of Motion are written in Zero Curvature Repres., i.e.,

$$\partial_x A_{t_N} - \partial_{t_N} A_x + [A_x, A_{t_N}] = 0 \to eqs. \quad motion$$

where $A_\mu \in \hat{\mathcal{G}}$.

Lie Algebraic Structure

- A_x and $A_{t_N} \in \hat{\mathcal{G}}$, *graded Affine Lie Algebra*.
- *Grading operator* Q define graded subspaces,

$$[Q, \mathcal{G}_a] = a\, \mathcal{G}_a, \quad \text{and} \quad [\mathcal{G}_a, \mathcal{G}_b] \subset \mathcal{G}_{a+b}$$

- Decomposition of Affine Lie Algebra into graded subspaces, e.g.,

$$\hat{\mathcal{G}} = \oplus \mathcal{G}_a,$$

Example:

- $\mathcal{G} = sl(2)$ case (Angular Momentum Algebra)

$$[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \qquad [E_\alpha, E_{-\alpha}] = h,$$

- **Affine** $\hat{\mathcal{G}} = \hat{s}l(2) = \{h^{(m)} = \lambda^m h, \quad E_{\pm}^{(m)} = \lambda^m E_{\pm\alpha}, \}, \;\; m = 0, \pm 1, \pm 2, \dots$

- **Principal Gradation**, $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2} h$,

$$\mathcal{G}_{2m} = \{\lambda^m h\}, \qquad \mathcal{G}_{2m+1} = \{\lambda^m E_\alpha, \; \lambda^{m+1} E_{-\alpha}\}$$

- Choose *semi-simple* element $E = E^{(1)} \equiv E_{\alpha} + \lambda E_{-\alpha}$, such that

$$\mathcal{K}_E = Kernel = \{x, /[x, E] = 0\}, \quad \text{and}$$

$$\mathcal{G} = \mathcal{K} + \mathcal{M}, \quad \text{where} \quad \mathcal{K} = \mathcal{K}_{2n+1} = \{\lambda^n (E_{\alpha} + \lambda E_{-\alpha})\}$$

has grade $2n + 1$ and $\mathcal{M}$ is the complement.

- Assume that E is *semi-simple* in the sense that this second decomposition is such that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}.$$

- Define **Lax operator**

$$L = \partial_x + E^{(1)} + A_0, \qquad A_0 = v(x,t)h \in \mathcal{M} \subset \mathcal{G}_0 \quad \text{Image}$$

- Zero Curvature Equation for **Positive Hierarchy**

$$[\partial_x + E^{(1)} + A_0, \partial_{t_N} + D^{(N)} + D^{(N-1)} + \cdots + D^{(0)}] = 0,$$

$$D^{(a)} \in \mathcal{G}_a,$$

which can be decomposed and solved grade by grade, i.e.,

- $$\begin{aligned} [E^{(1)}, D^{(N)}] &= 0, \\ [E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0, \\ \vdots \quad & \quad \vdots \\ [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0, \end{aligned}$$
- In particular, highest grade component, i.e.,

$$[E^{(1)}, D^{(N)}] = 0,$$

implies $D^{(N)} \in \mathcal{K}_{2n+1}$ and therefore $N = 2n + 1$.

Solving recursively for $D^{(a)}$ we get the eqn. of motion

$$\partial_{t_N} A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] = 0,$$

Exemples:

$$N = 1 \qquad \partial_t v = v_x$$

$$N = 3 \qquad 4\partial_{t_3} v = v_{3x} - 6v^2 v_x, \qquad mKdV$$

$$N = 5 \qquad 16\partial_{t_5} v = v_{5x} - 10v^2 v_{3x} - 40 v v_x v_{2x} - 10 v_x^3 + 30 v^4 v_x,$$

$$\begin{aligned} N = 7 \qquad 64\partial_{t_7} v &= v_{7x} - 182 v_x v_{2x}^2 - 126 v_x^2 v_{3x} - 140 v v_{2x} v_{3x} \\ &- 84 v v_x v_{4x} - 14 v^2 v_{5x} + 420 v^2 v_{3x} + 560 v^3 v_x v_{2x} \\ &+ 70 v^4 v_{3x} - 140 v^6 v_x \end{aligned}$$

$\cdots etc$

Remark

- Vacuum Solution is $v = const = 0$
- zero curvature for vacuum solution becomes

$$[\partial_x + E^{(1)}, \partial_{t_N} + E^{(N)}] = 0, \qquad [E^{(1)}, E^{(N)}] = 0$$

- and imply pure gauge potentials, i.e.,

$$A_{x,vac} = E^{(1)} = T_0^{-1}\partial_x T_0, \qquad A_{t_N,vac} = E^{(N)} = T_0^{-1}\partial_{t_N} T_0$$

- where

$$T_0 = e^{xE^{(1)}} e^{t_N E^{(N)}}$$

- Extend construction to **Negative Hierarchy**

  $$[\partial_x + E^{(1)} + A_0, \partial_{t_{-n}} + D^{(-n)} + D^{(-n+1)} + \cdots + D^{(-1)}] = 0.$$

- Lowest grade projection,

  $$\partial_x D^{(-n)} + [A_0, D^{(-n)}] = 0$$

  yields a nonlocal equation for $D^{(-n)}$. **No condition upon n.**

- The second lowest projection of grade $-n+1$ leads to

  $$\partial_x D^{(-n+1)} + [A_0, D^{(-n+1)}] + [E^{(1)}, D^{(-n)}] = 0$$

  and determines $D^{(-n+1)}$.

- The same mechanism works recursively until we reach the zero grade equation

$$\partial_{t_{-n}} A_0 + [E^{(1)}, D^{(-1)}] = 0$$

which gives the *time evolution* for the field in A_0 according to time t_{-n}.

- Simplest Example $t_{-n} = t_{-1}$.

$$\partial_x D^{(-1)} + [A_0, D^{(-1)}] = 0,$$

$$\partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] = 0.$$

- Solution for $A_0 = v(x,t)\, h$ is

$$D^{(-1)} = e^{2\int^x v(x')dx'} \lambda^{-1} E_\alpha + e^{-2\int^x v(x')dx'} E_{-\alpha}, \qquad A_0 = vh,$$

denoting $\int^x v(x')dx' \equiv \phi(x, t_N)$.

- The time evolution is given by the sinh-Gordon equation, (*relativistic*)

$$\partial_{t_{-1}} \partial_x \phi = e^{2\phi} - e^{-2\phi},$$

where $\quad t_{-1} = z, \quad x = \bar{z}$ are light cone coordinates.

- General t_{-1} (relativistic) solution is

$$D^{(-1)} = B^{-1} E^{(-1)} B, \qquad A_0 = B^{-1} \partial_x B, \qquad B = \exp(\mathcal{G}_0)$$

- The time evolution is then given by the Leznov-Saveliev equation (Affine Toda eqns.) ,

$$\partial_{t_{-1}} \left(B^{-1} \partial_x B \right) = [E^{(1)}, B^{-1} E^{(-1)} B]$$

for $B = e^{\phi_a h_a}$.

- Next simplest example $t_{-n} = t_{-2}$
  Z.Qiao and W. Strampp *Physica* **A 313**, (2002), 365 ;
  JFG, G Starvaggi França, G R de Melo and A H Zimerman, *J. of Phys.* **A42**,(2009), 445204

$$\begin{aligned} \partial_x D^{(-2)} + [A_0, D^{(-2)}] &= 0, \\ \partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] &= 0, \\ \partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned}$$

- Propose solution of the form

$$\begin{aligned} D^{(-2)} &= c_{-2}\lambda^{-1} h, \\ D^{(-1)} &= a_{-1}\left(\lambda^{-1} E_{\alpha} + E_{-\alpha}\right) + b_{-1}\left(\lambda^{-1} E_{\alpha} - E_{-\alpha}\right). \end{aligned}$$

- Get $c_{-2} = const$ and

$$\begin{aligned} a_{-1} + b_{-1} &= 2c_{-2}\exp(-2d^{-1}v)d^{-1}\left(\exp(2d^{-1}v)\right), \\ a_{-1} - b_{-1} &= -2c_{-2}\exp(2d^{-1}v)d^{-1}\left(\exp(-2d^{-1}v)\right), \end{aligned}$$

where $A_0 = vh$ and $d^{-1}v = \int^x v(x')dx'$.

- Equation of motion is (integral eqn.)

$$\partial_{t_{-2}} v = -2c_{-2} e^{-2\phi} d^{-1} \left(e^{2\phi} \right) - 2c_{-2} e^{2\phi} d^{-1} \left(e^{-2\phi} \right) \tag{1}$$

where $\phi \equiv d^{-1} v = \int^x v(x') dx'$.

- In the same way we find

$$\begin{aligned} \partial_{t_{-3}} v &= 4e^{-2\phi} d^{-1} \left(e^{2\phi} d^{-1} (\sinh 2\phi) \right) \\ &+ 4e^{2\phi} d^{-1} \left(e^{-2\phi} d^{-1} (\sinh 2\phi) \right) \end{aligned} \tag{2}$$

and so on ...

- Propose the simplest vacuum configuration, $v = 0$,

$$A_{x,vac} = T_0^{-1}\partial_x T_0 = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)},$$
$$A_{t_{2n+1},vac} = T_0^{-1}\partial_{t_{2n+1}} T_0 = E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)}$$

where

$$T_0 = e^{x(E_{\alpha}^{(0)} + E_{-\alpha}^{(1)})} \quad e^{t_{2n+1}(E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)})}.$$

$n = 0, \pm 1, \pm 2, \cdots$

The Soliton solutions are constructed from the vacuum solution by gauge transformation (which preserves the zero curvature condition), i.e.,

$$A_\mu = \Theta^{-1} A_{\mu,vac} \Theta + \Theta^{-1} \partial_\mu \Theta,$$

where

$$A_\mu = T^{-1} \partial_\mu T, \qquad T = T_0 \Theta, \qquad A_{\mu,vac} = T_0^{-1} \partial_\mu T_0$$

we may choose $\Theta = \Theta_+ = e^{\theta_0} e^{\theta_1} \cdots$ or $\Theta = \Theta_- = e^{\theta_{-1}} e^{\theta_{-2}} \cdots$, $\theta_i \in \mathcal{G}_i$.

It then follows that $T = \quad T_0\Theta_- = gT_0\Theta_+$ for g constant

$$\Theta_-\Theta_+^{-1} = T_0^{-1}gT_0, \qquad e^{\theta_0} = Be^{\nu\hat{c}}$$

where we have extended the loop algebra to the full **central extended** Kac-Moody in order to introduce highest weight states $|\lambda_i>, i = 0, 1$, i.e.,

$$\Theta_+|\lambda_i >= e^{\theta_0}|\lambda_i >, \qquad <\lambda_i|\Theta_- = 1 < \lambda_i|$$

It then follows,

$$<\lambda_i|Be^{\nu\hat{c}}|\lambda_i >=< \lambda_i|T_0^{-1}gT_0|\lambda_i >$$

- The solution for *odd grade mKdV sub-hierarchy* is then given by

$$
\begin{aligned}
e^{-\nu} &= <\lambda_0|T_0^{-1} g T_0|\lambda_0> \equiv \tau_0, \\
e^{-\phi-\nu} &= <\lambda_1|T_0^{-1} g T_0|\lambda_1> \equiv \tau_1
\end{aligned}
$$

and hence need to evaluate matrix elements in $\tau_i, i = 0, 1$,

$$
v = \partial_x \ln\left(\frac{\tau_0}{\tau_1}\right), \qquad v = \partial_x \phi.
$$

where

$$
T_0 = e^{xA_{x,vac}} e^{t_M A_{t_M,vac}}, \qquad g = e^{F(\gamma)},
$$

and $F(\gamma)$ is an eigenvector (**vertex operator**) of $b_M = A_{t_M,vac}$ and $b_1 = A_{x,vac}$, i.e.,

$$
[b_M, F(\gamma)] = w_M F(\gamma).
$$

- In order to construct explicit soliton solutions consider the **vertex operator**,

$$F(\gamma) = \sum_{n=-\infty}^{\infty} \gamma^{-2n} \left(h^{(n)} - \frac{1}{2}\delta_{n,0}\hat{c} + \gamma^{-1}(E_{\alpha}^{(n)} - E_{-\alpha}^{(n+1)}) \right).$$

A direct calculation shows that

$$[b_{2m+1}, F(\gamma)] \quad = \quad -2\gamma^{2m+1} F(\gamma).$$

If we now take, for the one-soliton solution,

$$g = \exp\{F(\gamma)\}, \qquad T_0^{-1} g T_0 = e^{\rho(x,t_N) F(\gamma)}$$

- We find that the one-soliton solution of the form,

$$v = \partial_x \ln \left(\frac{\tau_0}{\tau_1} \right), \qquad v = \partial_x \phi.$$

where

$$\tau_0 = 1 + C_0 \rho(\gamma), \qquad \tau_1 = 1 + C_1 \rho(\gamma)$$

where explicit space-time dependence is given by

$$\rho(\gamma) = \exp \left\{ 2\gamma x + 2\gamma^{2m+1} t_{2m+1} \} \right\}.$$

Solves all eqns. within the mKdV hierarchy for $t = t_{2m+1}, m = 0, \pm 1, \cdots$.

- Assume two field configurations ϕ_1 and ϕ_2 satisfying the sinh-Gordon eqn.,

$$\partial_x \partial_{t_{-1}} \phi_i = e^{2\phi_i} - e^{-2\phi_i}, \quad i = 1, 2$$

- The **Backlund transformation** is known to be (Albert Victor Backlund, 1880),

$$\begin{aligned} \partial_x(\phi_1 - \phi_2) &= \frac{4}{\beta} \sinh(\phi_1 + \phi_2), \\ \partial_{t_{-1}}(\phi_1 + \phi_2) &= \beta \sinh(\phi_1 - \phi_2) \end{aligned}$$

for β an arbitrary parameter.

- In **Lie algebraic language** need to construct a **Backlund-gauge transformation** $K(\phi_1, \phi_2)$, such that

$$K(\phi_1,\phi_2)A_x(\phi_1) = A_x(\phi_2)K(\phi_1,\phi_2) + \partial_x K(\phi_1,\phi_2),$$

which is satisfied by

$$K(\phi_1,\phi_2) = \begin{bmatrix} 1 & -\frac{\beta}{2\lambda}e^{-\phi_1+\phi_2} \\ -\frac{\beta}{2}e^{\phi_1+\phi_2} & 1 \end{bmatrix} \quad (3)$$

provided

$$\partial_x(\phi_1 - \phi_2) = \frac{4}{\beta}\sinh(\phi_1+\phi_2).$$

- Since A_x is **Universal** within the hierarchy, extend the same Backlund-gauge transformation to A_{t_N} (JFG, Retore and Zimerman, 2015, 2016), i.e.,

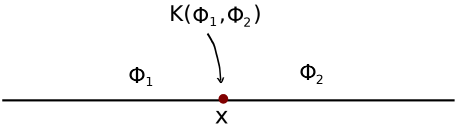
$$K(\phi_1, \phi_2) A_{t_N}(\phi_1) = A_{t_N}(\phi_2) K(\phi_1, \phi_2) + \partial_{t_N} K(\phi_1, \phi_2),$$

which yields for all values $N = 2n + 1$, in particular for $N = -1, 3$ corresponding to s-G and mKdV respec.,

$$\partial_{t_{-1}}(\phi_1 + \phi_2) = \beta \sinh(\phi_1 - \phi_2)$$

$$4\partial_{t_3}(\phi_2 - \phi_1) = 2\beta \partial_x^2 \phi_1 \cosh(\phi_1 + \phi_2) - 2\beta(\partial_x \phi_1)^2 \sinh(\phi_1 + \phi_2) + 2\beta^2(\partial_x \phi_1) + \beta^3 \sinh(\phi_1 + \phi_2)$$

- **Integrable Defects** interpolate two solutions of same eqn. of motion by Backlund Transformation (Bowcock, Corrigan and Zambon, 2003),i.e.,



- Eqns. of motion at the Defect position, x, is given by Backlund Transformation.
- Energy, Momentum, Topological Charges, etc are modified in order to include the Defect contribution (border effects).

$SL(3)$ Toda Model

- Eqns. Motion

$$\begin{aligned} \partial_x \partial_{t_{-1}} \phi_1 &= e^{2\phi_1 - \phi_2} - e^{-\phi_1 - \phi_2}, \\ \partial_x \partial_{t_{-1}} \phi_2 &= e^{2\phi_2 - \phi_1} - e^{-\phi_1 - \phi_2}. \end{aligned}$$

- Lax

$$L = \partial_x + E_{\alpha_1} + E_{\alpha_2} + \lambda E_{-\alpha_1 - \alpha_2} + \partial_x \phi_1 h_1 + \partial_x \phi_2 h_2$$

- Backlund-gauge Transformation

$$K(\phi_1, \phi_2, \psi_1, \psi_2) = \begin{bmatrix} 1 & 0 & \lambda^{-1} e^{-\phi_2 - \phi_1} \\ e^{\phi_1 + \psi_1 - \psi_2} & 1 & 0 \\ 0 & e^{-\phi_1 + \phi_2 + \psi_2} & 1 \end{bmatrix}$$

Leading to

$$\begin{aligned}\partial_x(\phi_1 - \psi_1) &= \lambda(e^{\phi_1+\psi_1-\psi_2} - e^{-\phi_2-\psi_1}) \\ \partial_x(\phi_2 - \psi_2) &= \lambda(e^{-\phi_1+\phi_2+\psi_2} - e^{-\phi_2-\psi_1})\end{aligned}$$

and

$$\begin{aligned}\partial_t(\phi_1 + \psi_1 - \psi_2) &= \lambda^{-1}(e^{\phi_1-\phi_2-\psi_1+\psi_2} - e^{-\phi_1+\psi_1}) \\ \partial_t(-\phi_1 + \phi_2 + \psi_2) &= \lambda^{-1}(e^{\phi_2-\psi_2} - e^{\phi_1-\phi_2-\psi_1+\psi_2})\end{aligned}$$

Easely generalized to $sl(n+1)$.

More General Backlund Transformations (Araujo, JFG,Zimerman 2011)

- e.g. Tzitzeica Model ($\phi_1 = \phi_2 = \phi$)

$$\partial_x \partial_{t_{-1}} \phi = e^{\phi} - e^{-2\phi}.$$

- Type II Backlund-gauge Transformation (Corrigan, Zambon,09)
- Involves non local auxiliary field, Λ
- diagonal $K_{i,i} \neq 1$,

$$\begin{aligned}
\partial_x q &= -\frac{i}{2\zeta} e^{\Lambda} - \zeta e^{-2\Lambda}(e^{q} + e^{-q})^2, \\
\partial_x(\Lambda - p) &= -\zeta e^{-2\Lambda}(e^{2q} - e^{-2q}), \\
\partial_t q &= -i\zeta e^{p-\Lambda}(e^{q} + e^{-q}) - \frac{1}{4\zeta} e^{2\Lambda - 2p}, \\
\partial_t \Lambda &= i\zeta e^{-\Lambda + p}(e^{-q} - e^{q})
\end{aligned} \tag{4}$$

for $q = \phi - \psi, \quad p = \phi + \psi$ and Λ is the auxiliary field.

Conclusions

- Introduced *Integrable Hierarchy* associated to Affine Lie Algebra.
- Proposed systematic construction of Soliton solutions in terms of vertex operators and representations of Affine Lie Algebras.
- Generalized to other integrable hierarchies allowing constant vacuum solutions, e.g., negative even sub-hierarchy

- Construct **generalized (type II) Backlund** transformation for $sl(3)$ and higher rank algebras involving $K_{ii} \neq 1$ and auxiliary fields.

- Adapt Dressing method to construct **periodic solutions** (Jacobi Theta functions).
  where

$$\tau_a \sim \sum_{k=-\infty}^{+\infty} e^{2\pi i \eta k^2} \rho^k, \qquad \eta = \text{deform.} \quad \text{parameter}$$

  c.f. soliton where

$$\tau_0 = 1 + \rho, \qquad \tau_1 = 1 - \rho$$