Perfectly invisible $\mathcal{PT}$-symmetric zero-gap systems and exotic nonlinear super-conformal symmetry

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The talk is based on joint work with Juan Mateos Guilarte:

- "Perfectly invisible $\mathcal{PT}$-symmetric zero-gap systems, conformal field theoretical kinks, and exotic nonlinear supersymmetry", JHEP **12** (2017) 061 [arXiv: 1602.02179]

- "Nonlinear symmetries of perfectly invisible $\mathcal{PT}$-regularized conformal and superconformal mechanics systems", JHEP **01** (2019) 194 [arXiv: 1710.00356]

and related works with
Francisco Correa, Vit Jakubsky, Luismi Nieto, Mariano del Olmo, Adrian Arancibia, Olaf Lechtenfeld, José Cariñena, Luis Inzunza

Plan:

- General aspects (Summary)
- Some technical details
  - Extreme (killing) waves
  - Two examples of exotic supersymmetry
  - Extension (generalization) for the case of conformal symmetry
- Concluding comments

• Peculiar properties of various classical and quantum systems can be related to/derived from those of a free particle

• Free particle $\to$ Darboux deformations $\to$ reflectionless quantum systems

• Darboux covariance of Lax pair representation of the KdV equation

•$\Longrightarrow$ Reflectionless systems can be promoted to multi-soliton solutions of the KdV equation

$\star$ $\star$ $\star$ any reflectionless system is a snapshot of a multi-soliton solution to the KdV equation

• By periodization of reflectionless systems, (at least some of) finite-gap quantum systems can be obtained; they are solutions of stationary equations of the KdV hierarchy

• Darboux covariance of Lax representation can be applied to finite-gap quantum systems to promote them into solutions of the KdV equation

(Inverse scattering theory: KdV evolution = isospectral deformation of the corresponding multi-soliton Schrödinger potential)

• Darboux transformations also can be applied to finite-gap systems to produce new solutions to the KdV and mKdV equations. Such systems represent soliton defects propagating in the periodic finite-gap background

⋆ • ⋆ With all these quantum systems exotic nonlinear supersymmetry can be associated via Darboux transformations

⋆ • ⋆ In the simplest case, instead of $\mathcal{N}=2$ conventional supersymmetry described by Lie superalgebraic structure, there emerges exotic $\mathcal{N}=4$ nonlinear Poncaré supersymmetry, which involves Lax-Novikov integrals of stationary equations of the KdV hierarchy

⋆ ⋆ ⋆ Any two reflectionless (any two isospectral, or almost isospectral finite-gap) systems can be related by two different Darboux-Crum transformations

•⟹ Extended system is described by four supercharges instead of conventional two supercharges, and their anitcommutators generate not only (polynomial in) Hamiltonian of the extended system, but also Lax-Novikov integrals of their subsystems

• Lax-Novikov integrals = additional bosonic generators are higher odd order differential operators.

• They separate left-right moving deformed plane waves of reflectionless systems in the continuous part of their spectra, or left- and right- moving Bloch states of finite-gap systems, and detect all the bound, or edge states in them

• Lax-Novikov integral of any reflectionless system is a Darboux dressed form of the momentum operator of the free particle

• On the other hand, Calogero systems govern the dynamics of moving poles of rational solutions of the KdV equation. These systems can be obtained via appropriate limit procedure from multi-soliton solutions, and by employing Galilean symmetry of the KdV equation

• They also can be obtained from the free particle via singular Darboux-Crum transformations

• Simplest case of two-particle Calogero system corresponds to conformal mechanics of de Alfaro, Fubini, Furlan

• Schrödinger symmetry of a free particle reduces to conformal $sl(2,\mathbb{R})$ symmetry

⋆ ⋆ ⋆ Lax-Novikov integral is lost, exotic SUSY is lost ⋆ ⋆ ⋆

• Schrödinger symmetry of two-particle Calogero system can be recuperated via $\mathcal{PT}$-regularization $x \to x + i\alpha$, where $\alpha \in \mathbb{R}$

•⟹ $\mathcal{PT}$-regularization of Darboux transformations allows to obtain perfectly invisible systems in which transmission coefficient, but not only its absolute value, equals one

•⟹ Another peculiarity: each of these systems contains a unique bound state at the very edge of the continuous doubly degenerate continuous part of the spectrum

• Lax-Novikov integrals in the perfectly invisible PT-regularized zero-gap quantum conformal and superconformal quantum mechanics systems affect on their (super)-conformal symmetries:

• Lax-Novikov integral modifies and extends conformal symmetry into a nonlinearly extended generalized Shrödinger algebra

• Exotic supersymmetric structure can be recuperated via $\mathcal{PT}$-regularization of Darboux transformations

• The $\mathcal{PT}$-regularized superconformal mechanics systems in the phase of the unbroken exotic nonlinear $\mathcal{N}=4$ super-Poincaré symmetry are described by nonlinearly super-extended Schrödinger algebra that involves the $osp(2|2)$ as a sub-superalgebra

• In the partially broken phase, the *scaling dimension* of all odd integrals is *indefinite*, and the $osp(2|2)$ *is not contained* as a sub-superalgebra.

• Additional bonus of the construction: extreme (killing) wave solutions can be generated proceeding from complexified KdV equation and higher equations of its hierarchy

• Some of the $\mathcal{PT}$-regularized conformal systems control stability properties of the kink-type solutions in the field-theoretical Liouville and $SU(3)$ conformal Toda systems

• Another peculiarity: Jordan states corresponding to zero energy play essential role in the construction

Example

Seed state for the Darboux transformation :

$$\psi^{(1)}_{\alpha,\gamma} = \gamma\xi^{-1} + \xi^2\,, \qquad \xi = x + i\alpha\,, \qquad \alpha \in \mathbb{R}\,,$$

$\gamma = 12t + i\nu\alpha^3$, $\nu \in (1,\infty)$; $\psi^{(1)}_{\alpha,\gamma}$ is a linear combination of the bound state ξ^{-1} of $H^{\alpha}_1 = -\frac{d^2}{dx^2} + \frac{2}{\xi^2}$ of zero eigenvalue and of its non-physical partner ξ^2.

$$\mathcal{W}^{(1)}_{\alpha,\gamma} = -\frac{d}{dx}\left(\ln\psi^{(1)}_{\alpha,\gamma}\right) = \frac{1}{\xi} - \frac{3\xi^2}{\xi^3+\gamma}\,,$$

$V_\pm = (\mathcal{W}^{(1)}_{\alpha,\gamma})^2 \pm (\mathcal{W}^{(1)}_{\alpha,\gamma})'$, $V_- = 2\xi^{-2}$ and

$$V_+ = -2\left(\ln W(\xi, -\gamma+\xi^3)\right)'' = \frac{6}{\xi^2} - 6\gamma\frac{4\xi^3+\gamma}{\xi^2(\xi^3+\gamma)^2}\,.$$

$V_+(x;\alpha,\gamma)$ satisfies complexified KdV equation
$u_t - 6uu_x + u_{xxx} = 0$.
Its real and imaginary parts, $u(x,t) = v(x,t) + iw(x,t)$, satisfy the system of coupled equations

$$v_t - 3(v^2 - w^2)_x + v_{xxx} = 0\,, \qquad w_t - 6(vw)_x + w_{xxx} = 0$$

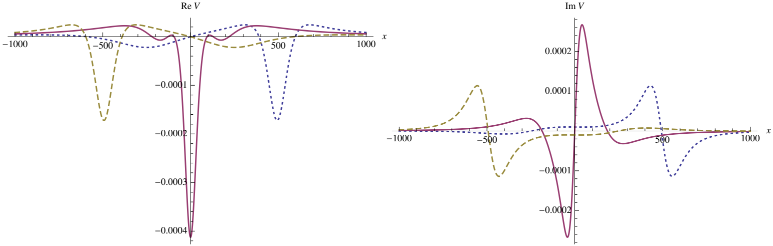

Figure: Evolution of real (on the left) and imaginary (on the right) parts of potential $u(x,t) = V_+(x;\alpha,\gamma(t))$ as a complex $\mathcal{PT}$-symmetric solution of the KdV equation at $\alpha = 100$, $\nu = 5$; dashed lines: $t = -10^7$, continuous lines: $t = 0$, dotted lines: $t = 10^7$.

$\psi^{(2)}_{\alpha,\gamma} = \gamma\xi^{-2} + \xi^3$ = zero energy eigenstate of $H_2^{\alpha} = -\frac{d^2}{dx^2} + \frac{6}{\xi^2}$ as a seed state for Darboux transformation

$$V^{(2)}_{+}(x;\alpha,\gamma) = -2\left(\ln W(\xi, \xi^3, \tfrac{8}{3}\gamma + \xi^5)\right)'' = \frac{12}{\xi^2} - 10\gamma\frac{6\xi^5+\gamma}{\xi^2(\xi^5+\gamma)^2}$$

With $\gamma = -720t + i\nu\alpha^5$, $\nu \in (24, \infty)$, it satisfies higher order equation of the KdV hierarchy

$$u_t + 30u^2u_x - 20u_xu_{xx} - 10uu_{xxx} + u_{xxxxx} = 0$$

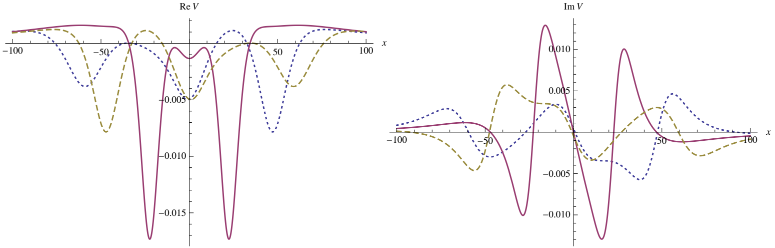

Figure: Evolution of real (on the left) and imaginary (on the right) parts of potential $V^{(2)}_{+}(x;\alpha,\gamma(t))$ at $\alpha = 20$, $\nu = 25$; dashed lines: $t = -10^6$, continuous lines: $t = 0$, dotted lines: $t = 10^6$

Simplest extended system :

$$\mathcal{H} = \begin{pmatrix} H_1^\alpha & 0 \\ 0 & H_0 \end{pmatrix}$$

Supercharges $([\mathcal{H}, Q_a] = 0\,,\ [\mathcal{H}, S_a] = 0)$:

$$Q_1 = \begin{pmatrix} 0 & D_1 \\ D_1^\# & 0 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 0 & -iD_1\mathcal{P}_0 \\ i\mathcal{P}_0 D_1^\# & 0 \end{pmatrix},$$

$Q_2 = i\sigma_3 Q_1$, $S_2 = i\sigma_3 S_1$, where $D_1 = \xi\frac{d}{dx}\xi^{-1} = \frac{d}{dx} - \xi^{-1}$, $D_1^\# = -\xi^{-1}\frac{d}{dx}\xi = -\frac{d}{dx} - \xi^{-1}$, $\mathcal{P}_0 = -i\frac{d}{dx}$ is the momentum operator of the free particle $H_0 = -\frac{d^2}{dx^2}$.

$$\{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}\,, \quad \{S_a, S_b\} = 2\delta_{ab}\mathcal{H}^2\,, \quad \{Q_a, S_b\} = 2\epsilon_{ab}\mathcal{L}_1\,,$$

$$\mathcal{L}_1 = \begin{pmatrix} \mathcal{P}_1^\alpha = D_1\mathcal{P}_0 D_1^\# & 0 \\ 0 & H_0\mathcal{P}_0 \end{pmatrix}$$

is the bosonic integral of motion = central charge of supersalgebra, $\ker \mathcal{P}_1^\alpha = \text{span}\,\{\xi^{-1}, \xi, \xi^3\}$.

One also can consider a bosonic integral $\mathcal{L}_2 = \sigma_3 \mathcal{L}_1$. It transforms mutually the first and second order supercharges:

$$[\mathcal{L}_2, Q_a] = 2i\mathcal{H}S_a\,, \qquad [\mathcal{L}_2, S_a] = -2i\mathcal{H}^2 Q_a\,.$$

Another example of superextended system

$$\mathcal{H} = \begin{pmatrix} H_1^{\alpha_2} & 0 \\ 0 & H_1^{\alpha_1} \end{pmatrix}, \qquad \alpha_1 > \alpha_2$$

Subsystems $H_1^{\alpha_1}$ and $H_1^{\alpha_2}$ can be intertwined by the second order differential operators $D_{\alpha_1} D_{\alpha_2}^{\#}$ and $D_{\alpha_2} D_{\alpha_1}^{\#}$ via the 'virtual' free particle system, $(D_{\alpha_1} D_{\alpha_2}^{\#}) H_1^{\alpha_2} = H_1^{\alpha_1} (D_{\alpha_1} D_{\alpha_2}^{\#})\,,$ $(D_{\alpha_2} D_{\alpha_1}^{\#}) H_1^{\alpha_1} = H_1^{\alpha_2} (D_{\alpha_2} D_{\alpha_1}^{\#})$. However, there also exists the first order intertwiners, $D = \frac{d}{dx} + \mathcal{W}\,,$ $D^{\#} = -\frac{d}{dx} + \mathcal{W}\,,$ where $\mathcal{W} = \frac{1}{\xi_1} - \frac{1}{\xi_2} - \frac{1}{\xi_1 - \xi_2}$, $\xi_j = x + i\alpha_j$: $D H_1^{\alpha_1} = H_1^{\alpha_2} D\,,$ $D^{\#} H_1^{\alpha_2} = H_1^{\alpha_1} D^{\#}$. They also satisfy the relations $D^{\#} D = H_1^{\alpha_1} - \Delta^2\,,$ $D D^{\#} = H_1^{\alpha_2} - \Delta^2\,,$ where $\Delta = (\alpha_1 - \alpha_2)^{-1}\,.$

Integrals:

$$Q_1 = \begin{pmatrix} 0 & D \\ D^{\#} & 0 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 0 & D_{\alpha_2} D^{\#}_{\alpha_1} \\ D_{\alpha_1} D^{\#}_{\alpha_2} & 0 \end{pmatrix},$$

$$\mathcal{L}_1 = \begin{pmatrix} \mathcal{P}^{\alpha_2} & 0 \\ 0 & \mathcal{P}^{\alpha_1} \end{pmatrix},$$

$Q_2 = \sigma_3 Q_1$, $S_2 = \sigma_3 s_1$, $\mathcal{L}_2 = \sigma_3 \mathcal{L}_1$. Nontrivial superalgebraic relations:

$$\{Q_a, Q_b\} = 2\delta_{ab}(\mathcal{H} - \Delta^2), \qquad \{S_a, S_b\} = 2\delta_{ab}\mathcal{H}^2,$$

$$\{Q_a, S_b\} = 2\left(\epsilon_{ab}\mathcal{L}_1 + i\delta_{ab}\Delta\mathcal{H}\right),$$

$$[\mathcal{L}_2, Q_a] = 2(i(\mathcal{H}-\Delta^2)S_a + \Delta\cdot\mathcal{H}Q_a), \; [\mathcal{L}_2, S_a] = -2(i\mathcal{H}^2 Q_a + \Delta\cdot\mathcal{H}S_a).$$

Exotic supersymmetry is in partially spontaneously broken phase: the doublet of bound states $\Psi_0^{\pm} = (D_{\alpha_2}1, \pm D_{\alpha_1}1)^t = (-\xi_2^{-1}, \mp\xi_1^{-1})^t$ of zero energy are not annihilated by the first order supercharges: $Q_1\Psi_0^{\pm} = \pm i\Delta\Psi_0^{\pm}$.

Superconformal symmetry of the system $\mathcal{H} = \mathrm{diag}(H_1^\alpha, H_0)$.

Integrals: $\mathcal{Q}_a$, $\mathcal{S}_a$, $\mathcal{L} = \mathcal{L}_1$, $\mathcal{D} = \mathrm{diag}\,(D_1^\alpha, D_0^\alpha)$,
$\mathcal{K} = \mathrm{diag}\,(K_1^\alpha, K_0^\alpha)$, $D_0^\alpha = D_0(x + i\alpha)$, $K_0^\alpha = K_0(x + i\alpha)$,
$D_0 = \frac{1}{4}\{G_0, \mathcal{P}_0\}$, $K_0 = G_0^2$, $G_0 = x - 2t\mathcal{P}_0$,

$$\mathcal{G} = \mathrm{diag}\,\left(G_1^\alpha, \tfrac{1}{2}\{G_0^\alpha, H_0\}\right)\,,\ \mathcal{V} = i\xi^2 - (\tfrac{d}{dx} + \xi^{-1})\mathcal{I} - 4t\mathcal{G} - 4t^2\mathcal{L}\,,$$

$$\mathcal{R} = \xi^3\mathcal{I} - 6t\mathcal{V} - 12t^2\mathcal{G} - 8t^3\mathcal{L}\,,$$

$$\mathcal{P}_- = \tfrac{1}{2}(1 - \sigma_3)\mathcal{P}_0\,, \qquad \mathcal{G}_- = \tfrac{1}{2}(1 - \sigma_3)G_0^\alpha\,,$$

$$\lambda_1 = \begin{pmatrix} 0 & i\xi \\ -i\xi & 0 \end{pmatrix} - 2t\mathcal{Q}_1\,, \qquad \lambda_2 = i\sigma_3\lambda_1\,,$$

$$\mu_1 = \begin{pmatrix} 0 & \xi\mathcal{P}_0 \\ \mathcal{P}_0\xi & 0 \end{pmatrix} - 2t\mathcal{S}_1\,, \qquad \mu_2 = i\sigma_3\mu_1\,,$$

$$\kappa_1 = \begin{pmatrix} 0 & \xi^2 \\ \xi^2 & 0 \end{pmatrix} - 4t\mu_1 - 4t^2\mathcal{S}_1\,, \qquad \kappa_2 = i\sigma_3\kappa_1\,.$$

Nonlinear extension of $osp(2|2)$ superalgebra with coefficients to be of order not higher than two in generators

Bosonic integrals
$\mathcal{L}$, $\mathcal{H}$, $\mathcal{G}$, $\mathcal{P}_-$, $\mathcal{I} = \text{diag}(1,1)$, $\Sigma = \sigma_3$, $\mathcal{D}$, $\mathcal{V}$, $\mathcal{G}_-$, $\mathcal{K}$, $\mathcal{R}$
are eigenstates of $-i\mathcal{D}$, $[-i\mathcal{D}, \mathcal{O}] = s_{\mathcal{O}}\mathcal{O}$, of the eigenvalues
$s_{\mathcal{O}} = (3/2, 1, 1/2, 1/2, 0, 0, 0, -1/2, -1, -1, -3/2)$

In the case of the system $\mathcal{H} = \text{diag}(H_1^{\alpha_2}, H_1^{\alpha_1})$, superconformal algebra is more complicated.

The number of generattors is the same, but no odd fermionic generator has a definite scaling dimension (is not eigenstate of the operartor $-i\mathcal{D}$.

$osp(2|2)$ is not contained as a sub-superalgebra.

Concluding comments:

• Solutions to the KdV equation: Lax-Novikov integral is a higher order differential operator of odd order factorizable into two differential operators of the even and odd orders

• Extended system constructed by Darboux transformations based on those factorizing operators is described by the exotic $\mathcal{N} = 4$ nonlinear supersymmetry that involves Lax-Novikov integrals

• Quantum harmonic oscillator and AFF model with confining potential term: rational extensions of the "finite-gap" structure can be constructed by two dual Darboux transformations with intertwining operators to be differential operators of the even and odd orders, but dual Darboux schemes produce the same system up to a nonzero additive constant shift

$\bullet\!\!\Longrightarrow$ Instead of Lax-Novikov type integrals, nontrivial ladder operators are generated, which allow to connect "valence bands" with equidistant part of the spectrum

$\bullet\!\!\Longrightarrow$ Three pairs of the ladder operators encode the spectral peculiarities of a system and form a complete spectrum-generating set of the ladder operators

Thank you