Lie and quasi-Lie systems in Quantum Mechanics

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New challenges in Quantum Mechanics: Integrability and Supersymmetry, Benasque, September 1-6, 2019

Abstract

After a quick presentation of the theory of Lie and quasi Lie systems from a geometric perspective, recent progresses on their applications when compatible geometric structures exist will be described with an special emphasis in the particular case of admissible Kähler structures, and therefore with applications in Quantum Mechanics. Finally we point out the relationship of time-independent Schrödinger equation with a special case of Lie system and its relation with the theory of Darboux transformations

This talk is based on collaborations with many people, among them J. Grabowski and G. Marmo, and more recently with:

J. de Lucas and C. Sardón

Outline

Introduction

Solution of systems of differential equations appearing in many physical problems is not an easy task. In geometric terms they are represented by vector fields, whose integral curves are the solutions of the system.

In order to find their solutions, i.e. the flow of the vector fields, as in a generic case there is not way of writing them in an explicit way, i.e. using fundamental functions, we are happy if, at least, we can express the solutions in terms of quadratures.

Characterisation of system admitting such type of solutions has received a lot of attention, and the answer is always based on the use of Lie algebras of vector fields containing the given one.

In general, in order to study such systems use is made of symmetry and reduction techniques. In such procedures the knowledge of some particular solutions of them or related systems may be useful.

For instance as far as Riccati equation is concerned one knows that

- ☐ If one particular solution is known, we can find the general solution by means of two quadratures.
- ☐ If two particular solutions are known, we can find the general solution by means of just one quadrature.
- ☐ If three particular solutions are known we can explicitly write the solution without any quadrature, by means of a superposition function.

We are now interested in this kind of systems admitting a superposition function, the so called Lie systems, which appear very often in many problems in science and engineering.

We will fix our attention in the particular case of quantum mechanics, where they are useful in:

- Studying the time evolution of a quantum system
- In particular cases of time-independent Schrödinger equation

Lie–Scheffers systems: a quick review

Lie–Scheffers systems = Non-autonomous systems of n first-order differential equations admitting a ...

Superposition rule: a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}, \quad x = \Phi(u_1, \ldots, u_m; k_1, \ldots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)(t)}; k_1, \ldots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ being a generic set of particular solutions of the system and where $k_1, \ldots, k_n$ are real numbers.

They are a generalisation of linear superposition rules for Squeezed States and Helmholtz spectra linear systems for which $m = n$ and $x = \Phi(x_{(1)}, \ldots, x_{(n)}; k_1, \ldots, k_n) = k_1\, x_{(1)} + \cdots + k_n\, x_{(n)}$ but

i) The number m may be different from the dimension n.

ii) The function Φ is nonlinear in this more general case.

They appear quite often in many different branches of science ranging from pure mathematics to classical and quantum physics, control theory, economy, etc. Forgotten for a long time they had a revival due to the work of Winternitz and coworkers.

One particular example is Riccati equation, of a fundamental importance in physics (for instance factorisation of second order differential operators, Darboux transformations and in general Supersymmetry in Quantum Mechanics) and in mathematics.

These systems are related with equations in Lie groups and in general connections in fibre bundles.

In the solution of such non-autonomous systems of first-order differential equations we can use techniques imported from group theory, for instance Wei–Norman method, and reduction techniques coming from the theory of connections.

Recent generalisations have also been shown to be useful for dealing with other systems of differential equations (e.g. Emden–Fowler equations, Abel equations).

The existence of additional compatible geometric structures, like symplectic or Poisson structures may be useful in the search for solutions.

Theorem: *Given a non-autonomous system of n first order differential eqns*

$$\frac{dx^i}{dt} = X^i(x^1, \dots, x^n, t), \quad i = 1 \dots, n,$$

a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n, \quad x = \Phi(u_1, \dots, u_m; k_1, \dots, k_n)$, $u_a \in \mathbb{R}^n$, *such that the general solution is*

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)(t)}; k_1, \dots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \dots, m\}$ *being a set of particular solutions of the system and where* $k_1, \dots, k_n$, *are* n *arbitrary constants, is that the system can be written as*

$$\frac{dx^i}{dt} = b_1(t)\xi_1^i(x) + \cdots + b_r(t)\xi_r^i(x), \qquad i = 1 \dots, n,$$

where $b_1, \dots, b_r$, *are* r *functions depending only on* t *and* ξ_α^i, $\alpha = 1, \dots, r$, *are functions of* $x = (x^1, \dots, x^n)$, *such that the* r *vector fields in* $\mathbb{R}^n$ *given by*

$$X_\alpha \equiv \sum_{i=1}^n \xi_\alpha^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i} \ , \qquad \alpha = 1, \dots, r,$$

close on a real finite-dimensional Lie algebra, i.e. the X_α are l.i. and there are r^3 real numbers, $c_{\alpha\beta}{}^\gamma$, such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma X_\gamma \ .$$

The number r satisfies $r \leq mn$.

The geometric concept of superposition rule is the following:

A superposition rule for a t-dependent vector field X in a n-dimensional manifold M is a map $\Phi : M^m \times M \to M$ such that if $\{x_{(1)}(t), \ldots, x_{(m)}(t)\}$ is a generic set of integral curves of X, then $x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t), k)$, with $k \in M$, is also integral curve of X, and each integral curve is obtained in this way.

The result of the Theorem in modern terms is that a t-dependent vector field X admits a superposition rule if there exist r fields $X_1, \ldots, X_r$ in M, closing on a Lie algebra, and functions $b_1(t), \ldots, b_r(t)$ such that $X(x,t)$ be a linear combination

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha(x).$$

The t-dependent vector field can be seen as a family of vector fields $\{X_t \mid t \in \mathbb{R}\}$.

Definition. *The minimal Lie algebra of such a t-dependent vector field X on a manifold M is the smallest real Lie algebra, V^X, containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$, namely $V^X = \mathrm{Lie}(\{X_t \mid t \in \mathbb{R}\})$.*

The vector field associated to a non-autonomous system X allows us to define a generalised distribution $\mathcal{D}^X : x \in M \mapsto \mathcal{D}_x^X \subset TM$, where $\mathcal{D}_x = \{Y_x \mid Y \in V^X\} \subset T_xM$, and X also gives rise to a generalised co-distribution $\mathcal{V} : x \in M \mapsto \mathcal{V}_x \subset T^*M$, where $\mathcal{V}_x = \{\omega_x \mid \omega_x(Y_x) = 0, \forall Y_x \in \mathcal{D}_x^X\}$.

Remark that the Lie–Scheffers theorem can be reformulated as follows:

Theorem: *A system X admits a superposition rule if and only if the minimal Lie algebra V^X is finite-dimensional.*

Some particular examples

A) Inhomogeneous linear systems:

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \ , \qquad i = 1, \ldots, n.$$

The time-dependent vector field is

$$X = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \right) \frac{\partial}{\partial x^i} \ ,$$

which is a linear combination with t-dependent coefficients,

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, Y_{ij} + \sum_{i=1}^{n} B^i(t)\, Y_i \ ,$$

of the $n^2 + n$ vector fields

$$Y_{ij} = x^j \frac{\partial}{\partial x^i}, \qquad Y_i = \frac{\partial}{\partial x^i} \ , \qquad i, j = 1, \ldots, n\,.$$

These last vector fields have the following commutation relations:

$$[Y_i, Y_k] = 0 \ , \qquad [Y_{ij}, Y_k] = -\delta_{kj}\, Y_i \ , \qquad \forall\, i, j, k = 1, \ldots, n\, .$$

- The set $\{Y_i \mid i = 1, \ldots, n\}$ generates an Abelian ideal.

- The set $\{Y_{ij} \mid i, j = 1, \ldots, n\}$ generates a Lie subalgebra.

- The Vessiot Lie algebra is isomorphic to the $(n^2 + n)$-dimensional Lie algebra of the affine group.

In this case $r = n^2 + n$ and using the general theory one can see that $m = n + 1$ and the equality $r = m\, n$ also follows.

The superposition function $\Phi : \mathbb{R}^{n(n+1)} \to \mathbb{R}^n$ is:

$$x = \Phi(u_1, \ldots, u_{n+1}; k_1, \ldots, k_n) = u_1 + k_1(u_2 - u_1) + \cdots + k_n(u_{n+1} - u_1),$$

i.e. the general solution can be written in terms of $n + 1$ generic solutions as:

$$\Phi(x_{(1)}, \ldots, x_{(n+1)}; k_1, \ldots, k_n) = x_{(1)} + k_1(x_{(2)} - x_{(1)}) + \cdots + k_n(x_{(n+1)} - x_{(1)}).$$

B) The Riccati equation $(n = 1)$

$$\frac{dx(t)}{dt} = a_2(t)\, x^2(t) + a_1(t)\, x(t) + a_0(t)\ .$$

Now $m = r = 3$ and the superposition principle comes from the relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k\ ,$$

or in other words,

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) + k\, x_2(t)(x_1(t) - x_3(t))}{(x_3(t) - x_2(t)) + k\,(x_1(t) - x_3(t))}\ ,$$

i.e. the superposition rule involves three different solutions, $m = 3$. The value $k = \infty$ must be accepted, otherwise we do not obtain the solution x_2.

The vector fields $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ are given by

$$Y^{(1)} = \frac{\partial}{\partial x}\,, \quad Y^{(2)} = x\,\frac{\partial}{\partial x}\,, \quad Y^{(3)} = x^2\,\frac{\partial}{\partial x}\,,$$

that close on a three-dimensional real Lie algebra, i.e. $r = 3$, with defining relations

$$[Y^{(1)}, Y^{(2)}] = Y^{(1)}\,, \quad [Y^{(1)}, Y^{(3)}] = 2Y^{(2)}\,, \quad [Y^{(2)}, Y^{(3)}] = Y^{(3)}\,.$$

Then, the associated Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$.

C) Lie–Scheffers systems on Lie groups

Consider the set of right–invariant vector fields in G spanning the opposite of the Lie algebra $\mathfrak{g}$. Similarly can be done with the set left–invariant vector fields in G, i.e. the Lie algebra $\mathfrak{g}$.

If $\{a_1, \ldots, a_r\}$ is a basis for the tangent space T_eG and X_α^R denotes the right-invariant vector field in G such that $X_\alpha^R(e) = a_\alpha$, a Lie–Scheffers system is

$$\dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R(g(t)) \ .$$

When applying $(R_{g(t)^{-1}})_{*g(t)}$ to both sides we obtain the equation on T_eG

$$(R_{g(t)^{-1}})_{*g(t)}(\dot{g}(t)) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ , \qquad (**)$$

This right-invariant equation is usually written with a slight abuse of notation:

$$(\dot{g}\, g^{-1})(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ .$$

If $\bar{g}(t)$ is a solution of (**) with initial condition $\bar{g}(0) = e$, the solution $g(t)$ with initial conditions $g(0) = g_0$ is given by $\bar{g}(t) g_0$.

Moreover, there is a superposition rule $\Phi : G \times G \to G$ involving one solution. If g_p is a particular solution, then its general solution is $g(t) = g_p(t)\, g_0$, i.e., there is a superposition rule

$$\Phi(g, g_0) = g\, g_0 .$$

D) Lie-Scheffers systems on homogeneous spaces for Lie groups

Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$. If $\tau : G \to G/H$ is the natural projection, the right–invariant vector fields X^R_α are τ-projectable and the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on M, $\tau_{*g} X^R_\alpha(g) = -X_\alpha(gH)$, and we will have an associated Lie–Scheffers system on M

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x) \ .$$

Therefore, a solution of this last system starting from x_0 will be: $x(t) = \Phi(g(t), x_0)$, with $g(t)$ being a solution of (**) starting from $e \in G$, $g(0) = e$.

The converse property is true: The complete vector fields of the Lie algebra $\mathfrak{g}$ of a Lie system can be seen as as fundamental vector fields relative to an action.

The reduction method

The important ingredient is an equation on a Lie group

$$\dot{g}(t)\, g(t)^{-1} = a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\, a_\alpha \in T_eG \text{ , with } g(0) = e \in G. \qquad (\bullet)$$

It may happen that the only different from zero coefficients b_α are those corresponding to those a_α of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the equation reduces to a simpler equation on a Lie subgroup, involving less coordinates.

The fundamental result is that if we know a particular solution of the problem associated in a homogeneous space, the original solution reduces to one on the subgroup.

Let us choose a curve $g'(t)$ in the group G, and define the curve $\overline{g}(t) = g'(t)g(t)$. If $g(t)$ is a solution of $(\bullet)$, then $\overline{g}(t)$, is solution of a new Lie system.

Indeed,

$$R_{\overline{g}(t)^{-1}*\overline{g}(t)}(\dot{\overline{g}}(t)) = R_{g'^{-1}(t)*g'(t)}(\dot{g}'(t)) - \sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{Ad}\,(g'(t))a_\alpha \ .$$

This defines an action of the group of curves in G on the set of Lie systems on the group. This can be used to reduce a given Lie system to a simpler one.

The aim is to choose the curve $g'(t)$ in such a way that the new equation be simpler. For instance, we can choose a subgroup H and look for a choice of $g'(t)$ such that the right hand side lies in T_eH, and hence $\overline{g}(t) \in H$ for all t.

If $\Psi : G \times M \to M$ is a transitive action of G on a homogeneous space M, identified with G/H by choosing a fixed point x_0, then the integral curves starting from the point x_0 associated to both Lie systems are related by

$$\overline{x}(t) = \Psi(\overline{g}(t), x_0) = \Psi(g'(t)g(t), x_0) = \Psi(g'(t), x(t)) \ .$$

Therefore, this gives an action of the group of curves in G on the set of associated Lie systems in homogeneous spaces.

The important result is that the knowledge of a particular solution of the associated Lie system in G/H allows us to reduce the problem to one in the subgroup H.

The Abel equation and Quasi-Lie systems

In order to deal with the Abel equation of the first kind:

$$\dot{x} = A_0(t) + A_1(t)x + A_2(t)x^2 + A_3(t)x^3$$

the theory of Lie systems must be generalised.

Note that then the linear space

$$V_{\mathrm{Abel}}(\mathbb{R}) = \left\langle \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}, x^3\frac{\partial}{\partial x} \right\rangle.$$

is not a finite dimensional real Lie algebra because

$$\left[x^2\,\frac{\partial}{\partial x}, x^3\,\frac{\partial}{\partial x}\right] = x^4\,\frac{\partial}{\partial x},$$

Let us remark that the vector fields $X_1, \ldots, X_r$, of the (Guldberg-Vessiot) Lie algebra of Lie Theorem play a double role: on one side as generators of a Lie algebra with an associated Lie group G, and on the other one, as defining the dynamics.

The important fact was that the group $\mathcal{G}$ of curves in G transforms elements of the Lie family among themselves, and that this fact may be used to simplify the problem.

This fact admits the following generalisation:

- The dynamics is, as before, $X = \sum_{0}^{r} b_\alpha(t) X_\alpha$, but the vector fields X_α do not close on a finite-dimensional Lie subalgebra anymore but only span a linear space.

- There is a Lie algebra W of (complete) vector fields $\{Y_a \mid a = 1, \ldots, l\}$ such that

$$[Y_a, X_\alpha] = \sum_{\beta=1}^{r} c_{a\alpha}{}^{\beta} X_\beta .$$

In this case the group $\mathcal{G}$ of curves in the group corresponding to the Lie algebra W also transforms each element of the family of systems into another one.

Appropriate curves can lead to simpler systems, e.g. when the image is included in a finite-dimensional Lie algebra, reducing the system to a Lie system.

As an instance, for the Abel equation we should determine a Lie algebra W leaving invariant $V_{\mathrm{Abel}}(\mathbb{R})$, i.e. such that $[W, V_{\mathrm{Abel}}] \subset V_{\mathrm{Abel}}$. A simple exercise shows that

$$W_{\mathrm{Abel}} = \left\langle \frac{\partial}{\partial x}, x\frac{\partial}{\partial x} \right\rangle .$$

We find in this way the Lie group G called *structure preserving group*, which turns out to be the affine group in one dimension.

Correspondingly, the set of first order Abel equations of the first kind is invariant under the group $\mathcal{G}$ of curves in G, i.e. that of all transformations of the following form:

$$\bar{x}(t) = \alpha(t)\, x(t) + \beta(t), \qquad \alpha(t) \neq 0,$$

The same procedure can be used to show that $W_{\mathrm{Abel}}(\mathbb{R})$ defines also the *structure preserving group* for generalised Abel equations of the first-kind, i.e.

$$\frac{dx}{dt} = A_0(t) + \cdots + A_n(t)x^n .$$

The action of the group of curves in the set of systems produces orbits characterised by different values of invariant functions, i.e. equivalence classes of Abel equations with the same properties of integrability or existence of superposition rules.

In some cases our systems belongs to an orbit containing a Lie system and therefore our problem can be related to a Lie system. This is the idea of the so called quasi-Lie systems

The theory has been fully developed for Abel equations, and it is mainly based on the determination of the Lie algebras contained in $V_{\rm Abel}$ reachable by the set of such transformations.

Observe that the coefficient of X_3 cannot be transformed to to be zero!. This result is a clear consequence of that $W_{\rm Abel}\subset V_{\rm Ricc}$.

In previous papers we have analysed other classical examples of quasi-Lie systems such as dissipative Milne–Pinney equations, nonlinear oscillators and Emden differential equations.

We only mention the classical nonlinear oscillator proposed by Perelomov whose evolution equation is given by

$$\ddot{x} = b(t)x + c(t)\,x^n, \quad n \neq 0, 1,$$

and we consider the corresponding system

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= b(t)x + c(t)x^n. \end{cases}$$

described by the t-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + (b(t)x + c(t)x^n)\frac{\partial}{\partial v},$$

which can be written as a linear combination

$$X_t = b(t)X_1 + c(t)X_2 + X_3$$

with

$$X_1 = x\frac{\partial}{\partial v}, \quad X_2 = x^n\frac{\partial}{\partial v}, \quad X_3 = v\frac{\partial}{\partial x},$$

where the linear space $V' = \langle X_1, X_2, X_3\rangle$ is not a Lie algebra, because $[X_2, X_3] \notin V'$. Moreover, there is no finite-dimensional Lie algebra including V', therefore it is not a Lie system.

Let us define the linear space V generated by V' together with

$$X_4 = v\frac{\partial}{\partial v}, \quad X_5 = x\frac{\partial}{\partial x},$$

and take the linear subspace $W \subset V$ generated by

$$Y_1 = X_4 = v\frac{\partial}{\partial v}, \quad Y_2 = X_1 = x\frac{\partial}{\partial v}, \quad Y_3 = X_5 = x\frac{\partial}{\partial x}.$$

Such W is a solvable Lie algebra of vector fields,

$$[Y_1, Y_2] = -Y_2\,, \quad [Y_1, Y_3] = 0\,, \quad [Y_2, Y_3] = -Y_2\,,$$

and taking into account that

$$\begin{array}{lll} [Y_1, X_2] = -X_2, & [Y_1, X_3] = X_3, & [Y_2, X_2] = 0, \\ [Y_2, X_3] = X_5 - X_4, & [Y_3, X_2] = nX_2, & [Y_3, X_3] = -X_3, \end{array}$$

we see that V is invariant under the action of W, i.e. $[W, V] \subset V$.

In this way we get the quasi-Lie scheme $S(W, V)$.

Therefore we can consider the set of transformations

$$g(\alpha(t), \beta(t), \gamma(t)) = \begin{cases} x = \gamma(t)x' \\ v = \alpha(t)v' + \beta(t)x' \end{cases}$$

with

$$\alpha(t), \gamma(t) > 0, \alpha(0) = \gamma(0) = 1, \beta(0) = 0$$

which lead to the new system

$$\begin{cases} \dfrac{dx'}{dt} &= \dfrac{1}{\gamma^2(t)} v', \\ \dfrac{dv'}{dt} &= (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n, \end{cases}$$

which are related to the second-order differential equations

$$\gamma^2(t)\ddot{x}' = -2\gamma(t)\dot{\gamma}(t)\dot{x}' + (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n \,.$$

We can try to transform a particular instance of such systems into a first-order differential equation associated with a nonlinear oscillator with a zero angular frequency

$$\begin{cases} \dfrac{dx'}{dt} &= f(t)v', \\ \dfrac{dv'}{dt} &= f(t)c_0 x'^n \,, \end{cases}$$

related to the nonlinear oscillator

$$\frac{d^2x'}{d\tau^2} = c_0 x'^n,$$

with $d\tau/dt = f(t)$.

The conditions ensuring such a transformation are

$$\gamma(t)b(t) - \ddot{\gamma}(t) = 0\,, \quad c(t) = c_0\gamma^{-(n+3)}(t),$$

with $f(t) = \gamma_1^{-2}(t)$, where γ_1 is a non-vanishing particular solution for $\gamma(t)b(t) - \ddot{\gamma}(t) = 0$.

The same scheme works for the case of the Hamiltonian

$$H(t) := \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + \omega^2(t)x_i^2\right) + c(t)U(x_1,\ldots,x_n)$$

where $c(t) \neq 0$ and $\omega^2(t)$ are two real functions and $U(x_1,\ldots,x_n)$ is a potential function given by an homogeneous polynomial of order k in the position coordinates.

More explicitely, the dynamical vector field is

$$X_t = \sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial x_i} + \left(-\omega^2(t)x_i - c(t)\frac{\partial U}{\partial x_i}\right)\frac{\partial}{\partial p_i}\right),$$

which can be written as a linear combination

$$X_t = b(t)X_1 + c(t)X_4 + X_3$$

where $b(t) = -\omega^2(t)$ and

$$X_1 = \sum_{i=1}^n x_i \frac{\partial}{\partial p_i}, \quad X_2 = \frac{1}{2}\sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i}\right), \quad X_3 = \sum_{i=1}^n p_i \frac{\partial}{\partial x_i},$$

which close on a Lie algebra structure because

$$[X_1, X_2] = -X_2, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = -X_3,$$

and

$$X_4 = -\sum_{i=1}^n \frac{\partial U}{\partial x_i} \frac{\partial}{\partial p_i},$$

with $[X_1, X_4] = 0$ and, as U is assumed homogeneous of order k, therefore $\partial U/\partial x_i$ is homogeneous of order $k-1$, the additional commutation relation

$$\begin{aligned} [X_2, X_4] &= \left[\frac{1}{2}\sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i}\right), -\sum_{j=1}^n \frac{\partial U}{\partial x_j} \frac{\partial}{\partial p_j}\right] \\ &= -\frac{1}{2}\sum_{i,j=1}^n \left(x_i \frac{\partial}{\partial x_i}\left(\frac{\partial U}{\partial x_j}\right) + \frac{\partial U}{\partial x_j}\right) \frac{\partial}{\partial p_j} = k\, X_4. \end{aligned}$$

This shows that a possible quasi- Lie scheme is defined by $V = \langle X_1, X_2, X_3, X_4\rangle$ and $W = \langle X_1, X_2\rangle$.

Structure preserving Lie systems

There are particularly interesting cases in which the manifold M is endowed with additional structures. For instance, let (M, Ω) be a symplectic manifold and the vector fields arising in the expression of the t-dependent vector field describing a Lie system are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra.

These vector fields correspond to a symplectic action of the Lie group G on (M, Ω).

The Hamiltonian functions of such vector fields, defined by $i(X_\alpha)\Omega = -dh_\alpha$, do not close on the same Lie algebra under Poisson bracket, but we can only say that

$$d\left(\{h_\alpha, h_\beta\} - h_{[X_\alpha, X_\beta]}\right) = 0\ ,$$

and then they span a Lie algebra extension of the original one.

The important fact is that we can define a t-dependent Hamiltonian

$$h_t = \sum_\alpha b_\alpha(t)\, h_\alpha,$$

with the functions h_α closing a Lie algebra, in such a wat hat $i(X_t)\Omega = -dh_t$.

As an example we can consider the differential equation of an n-dimensional Winternitz–Smorodinsky oscillator of the form

$$\left\{\begin{array}{rcl} \dot{x}_i & = & p_i, \\ \dot{p}_i & = & -\omega^2(t)x_i + \dfrac{k}{x_i^3}, \end{array}\right. \qquad i = 1, \ldots, n.$$

which describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}^n$

$$X_t = \sum_{i=1}^n \left[p_i \frac{\partial}{\partial x_i} + \left(-\omega^2(t)x_i + \frac{k}{x_i^3}\right)\frac{\partial}{\partial p_i}\right],$$

which can be written as $X_t = X_2 + \omega^2(t)X_1$ with X_1, X_2 and $X_3 = -[X_1, X_2]$ being given by

$$X_1 = -\sum_{i=1}^n x_i \frac{\partial}{\partial p_i}, \quad X_2 = \sum_{i=1}^n \left(p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3}\frac{\partial}{\partial p_i}\right), \quad X_3 = \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i}\right).$$

Note that X_t is a Lie system, because X_1, X_2 and X_3 close on a $\mathfrak{sl}(2, \mathbb{R})$ algebra:

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = X_1, \qquad [X_2, X_3] = -X_2.$$

Moreover, the preceding vector fields are Hamiltonian vector fields with respect to the usual symplectic form $\omega_0 = \sum_{i=1}^{n} dx^i \wedge dp_i$ with Hamiltonian functions

$$h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \qquad h_2 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + \frac{k}{x_i^2}\right), \qquad h_3 = \sum_{i=1}^{n} x_i p_i,$$

which obey that

$$\{h_1, h_2\} = h_3, \qquad \{h_1, h_3\} = -h_1, \qquad \{h_2, h_3\} = h_2.$$

Consequently, every curve h_t taking values in the Lie algebra $(W, \{\cdot,\cdot\})$ spanned by h_1, h_2 and h_3 gives rise to a Lie system which is Hamiltonian in $\mathrm{T}^*\mathbb{R}^n$ with respect to the symplectic structure ω_0 in such a way that

$$X_t = X_2 + \omega^2(t) X_1 = \widehat{\omega}_0^{-1}(dh_2 + \omega^2(t) dh_1),$$

i.e. the Hamiltonian is $h_t = h_2 + \omega^2(t) h_1$.

We can go a step further and consider Lie systems in (may be degenerate) Poisson manifolds, or even more generally in Dirac manifolds.

Geometric approach to Quantum Mechanics

The Schrödinger picture of Quantum mechanics admits a geometric interpretation similar to that of classical mechanics.

A separable complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ can be considered as a real linear space, to be then denoted $\mathcal{H}_\mathbb{R}$. The norm in $\mathcal{H}$ defines a norm in $\mathcal{H}_\mathbb{R}$, where $\|\psi\|_\mathbb{R} = \|\psi\|_\mathbb{C}$.

The linear real space $\mathcal{H}_\mathbb{R}$ is endowed with a natural symplectic structure as follows:

$$\omega(\psi_1, \psi_2) = 2\,\mathrm{Imag}\,\langle\psi_1, \psi_2\rangle.$$

The Hilbert $\mathcal{H}_\mathbb{R}$ can be considered as a real manifold modelled by a Banach space admitting a global chart.

The tangent space $T_\phi\mathcal{H}_\mathbb{R}$ at any point $\phi \in \mathcal{H}_\mathbb{R}$ can be identified with $\mathcal{H}_\mathbb{R}$ itself: the isomorphism associates $\psi \in \mathcal{H}_\mathbb{R}$ with the vector $\dot\psi \in T_\phi\mathcal{H}_\mathbb{R}$ given by:

$$\dot\psi f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi)\right)_{|t=0}, \qquad \forall f \in C^\infty(\mathcal{H}_\mathbb{R})\ .$$

The real manifold can be endowed with a symplectic 2-form ω:

$$\omega_\phi(\dot{\psi}, \dot{\psi}') = 2\,\mathrm{Imag}\,\langle \psi, \psi' \rangle\ .$$

One can see that the constant symplectic structure ω in $\mathcal{H}_\mathbb{R}$, considered as a Banach manifold, is exact, i.e., there exists a 1-form $\theta \in \bigwedge^1(\mathcal{H}_\mathbb{R})$ such that $\omega = -d\theta$. Such a 1-form $\theta \in \bigwedge^1(\mathcal{H})$ is, for instance, the one defined by

$$\theta(\psi_1)[\dot{\psi}_2] = -\mathrm{Imag}\,\langle \psi_1, \psi_2 \rangle.$$

This shows that the geometric framework for usual Schrödinger picture is that of symplectic mechanics, as in the classical case.

A continuous vector field in $\mathcal{H}_\mathbb{R}$ is a continuous map $X\colon \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$. For instance for each $\phi \in \mathcal{H}$, the constant vector field X_ϕ defined by

$$X_\phi(\psi) = \dot{\phi}.$$

It is the generator of the one-parameter subgroup of transformations of $\mathcal{H}_\mathbb{R}$ given by

$$\Phi(t, \psi) = \psi + t\,\phi\,.$$

As another particular example of vector field consider the vector field X_A defined by the $\mathbb{C}$-linear map $A : \mathcal{H} \to \mathcal{H}$, and in particular when A is skew-selfadjoint.

With the natural identification natural of $T\mathcal{H}_\mathbb{R} \approx \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}$, X_A is given by

$$X_A : \phi \mapsto (\phi, A\phi) \in \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}\,.$$

When $A = I$ the vector field X_I is the Liouville generator of dilations along the fibres, $\Delta = X_I$, usually denoted Δ given by $\Delta(\phi) = (\phi, \phi)$.

Given a selfadjoint operator A in $\mathcal{H}$ we can define a real function in $\mathcal{H}_\mathbb{R}$ by

$$a(\phi) = \langle \phi, A\phi \rangle\,,$$

i.e.,

$$a = \langle \Delta, X_A \rangle\,.$$

Then,

$$\begin{aligned} da_\phi(\psi) &= \frac{d}{dt} a(\phi + t\psi)_{t=0} = \frac{d}{dt}\left[\langle \phi + t\psi, A(\phi + t\psi)\rangle\right]_{|t=0} \\ &= 2\,\mathrm{Re}\,\langle \psi, A\phi\rangle = 2\,\mathrm{Imag}\,\langle -\mathrm{i}\,A\phi, \psi\rangle = \omega(-\mathrm{i}\,A\phi, \psi). \end{aligned}$$

If we recall that the Hamiltonian vector field defined by the function a is such that for each $\psi \in T_\phi\mathcal{H} = \mathcal{H}$,

$$da_\phi(\psi) = \omega(X_a(\phi), \psi)\,,$$

we see that

$$X_a(\phi) = -\mathrm{i}\, A\phi\,.$$

Therefore if A is the Hamiltonian H of a quantum system, the Schrödinger equation describing time-evolution plays the rôle of 'Hamilton equations' for the Hamiltonian dynamical system $(\mathcal{H}, \omega, h)$, where $h(\phi) = \langle \phi, H\phi\rangle$: the integral curves of X_h satisfy

$$\dot{\phi} = X_h(\phi) = -\mathrm{i}\, H\phi\,.$$

The real functions $a(\phi) = \langle \phi, A\phi\rangle$ and $b(\phi) = \langle \phi, B\phi\rangle$ corresponding to two selfadjoint operators A and B satisfy

$$\{a, b\}(\phi) = -\mathrm{i}\,\langle \phi, [A, B]\phi\rangle\,,$$

because

$$\{a, b\}(\phi) = [\omega(X_a, X_b)](\phi) = \omega_\phi(X_a(\phi), X_b(\phi)) = 2\,\mathrm{Imag}\,\langle A\phi, B\phi\rangle\,,$$

and taking into account that

$$2\,\mathrm{Imag}\,\langle A\phi, B\phi\rangle = -\mathrm{i}\,[\langle A\phi, B\phi\rangle - \langle B\phi, A\phi\rangle] = -\mathrm{i}\,[\langle \phi, AB\phi\rangle - \langle \phi, BA\phi\rangle]\,,$$

we find the above result.

In particular, on the integral curves of the vector field X_h defined by a Hamiltonian H,

$$\dot{a}(\phi) = \{a, h\}(\phi) = -\mathrm{i}\,\langle \phi, [A, H]\phi\rangle\,,$$

what is usually known as Ehrenfest theorem:

$$\frac{d}{dt}\langle \phi, A\phi\rangle = -\mathrm{i}\,\langle \phi, [A, H]\phi\rangle\,.$$

There is another relevant symmetric $(0, 2)$ tensor field which is given by the Real part of the inner product. It endows $\mathcal{H}_{\mathbb{R}}$ with a Riemann structure and we have also a complex structure J such that

$$g(v_1, v_2) = -\omega(Jv_1, v_2), \qquad \omega(v_1, v_2) = g(Jv_1, v_2),$$

together with

$$g(Jv_1, Jv_2) = g(v_1, v_2), \qquad \omega(Jv_1, Jv_2) = \omega(v_1, v_2)\,.$$

The triplet (g, J, ω) defines a Kähler structure in $\mathcal{H}_{\mathbb{R}}$ and the symmetry group of the theory must be the unitary group $U(\mathcal{H})$ whose elements preserve the inner product, or in an alternative but equivalent way (in the finite-dimensional case), by the intersection of the orthogonal group $O(2n, \mathbb{R})$ and the symplectic group $Sp(2n, \mathbb{R})$.

The time evolution from time t_0 to time t, even in the non-autonomous case, is described in terms of the evolution operator $U(t, t_0)$:

$$\psi(t) = U(t, t_0)\psi(t_0).$$

It must be a symmetry of the theory, i.e. for each fixed t_0, $U(t, t_0)$ is a curve in the unitary group $U(\mathcal{H})$.

Assume by simplicity that $\mathcal{H}$ is finite-dimensional, and then as

$$\frac{dU(t, t_0)}{dt} \in T_{U(t,t_0)}U(\mathcal{H}) \Longrightarrow \frac{dU(t, t_0)}{dt}(U(t, t_0))^{-1} \in T_I U(\mathcal{H}) \approx \mathfrak{u}(\mathcal{H}),$$

and therefore, there exists a curve $H(t)$ in $\mathrm{Herm}(n, \mathbb{C})$ such that

$$\frac{dU(t, t_0)}{dt} = -\mathrm{i}\, H(t)\, U(t, t_0).$$

In this equation $H(t)$ does not depend on t_0 because of the relation

$$U(t, t_0) = U(t, t_1)U(t_1, t_0),$$

which implies

$$\frac{dU(t, t_0)}{dt}(U(t, t_0))^{-1} = \frac{dU(t, t_1)}{dt}(U(t, t_1))^{-1}.$$

This is a Lie system in the unitary group $U(\mathcal{H})$ with associated Lie algebra $\mathfrak{u}(\mathcal{H})$ in the most general case. Sometimes however we can deal with some of its subalgebras.

Every curve $H(t)$ in $\mathfrak{u}(\mathcal{H})$ can be written as a linear combination of at most n^2 elements, those of a basis of $\mathfrak{u}(\mathcal{H})$, and therefore these (finite-dimensional) quantum systems are Lie systems.

As the elements of the Vessiot-Guldberg Lie algebra are skew-Hermitians, all of them define simultaneously Hamiltonian vector fields and Killing vector fields, and the system is a Lie-Kähler system.

As an example consider a Hamiltonian operator $H(t)$ that can be written as a linear combination, with some t-dependent real coefficients $b_1(t), \ldots, b_r(t)$, of some Hermitian operators,

$$H(t) = \sum_{k=1}^{r} b_k(t) H_k \,,$$

where the H_k form a basis of a real finite-dimensional Lie algebra V relative to the Lie bracket of observables, i.e. $[H_j, H_k] = \sum_{l=1}^{r} \mathrm{i}\, c_{jkl} H_l$, with $c_{jkl} \in \mathbb{R}$ and $j, k = 1, \ldots, r$.

It determines a t-dependent Schrödinger equation

$$\frac{d\psi}{dt} = -\mathrm{i}\, H(t)\psi = -\mathrm{i}\sum_{k=1}^{r} b_k(t) H_k \psi.$$

The vector fields X_k such that $X_k(\psi) = -\mathrm{i}\, H_k\, \psi$ are such that the t-dependent vector vector field X corresponding to the equation is $X = \sum_{k=1}^{r} b_k(t) X_k$ and

$$[X_j, X_k] = -\sum_{l=1}^{r} c_{jkl} X_l, \qquad j, k = 1, \ldots, r.$$

As an instance, if $\mathcal{H} = \mathbb{C}^2$, the time evolution is described by a curve $-\mathrm{i}H(t) := \dot{U}_t U_t^{-1}$ in the Lie algebra $\mathfrak{u}(2)$ of $U(2)$. Using the basis

$$I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and denoting $\mathbf{S} = (\sigma_1, \sigma_2, \sigma_3)/2$ and $\mathbf{B} := (B_1, B_2, B_3)$, the Hamiltonian can be written as

$$H(t) := B_0(t) I_0 + \mathbf{B}(t) \cdot \mathbf{S}.$$

Using the identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, the Schrödinger equation is

$$\begin{pmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \dot{p}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2B_0(t)+B_3(t) & -B_2(t) & B_1(t) \\ -2B_0(t)-B_3(t) & 0 & -B_1(t) & -B_2(t) \\ B_2(t) & B_1(t) & 0 & 2B_0(t)-B_3(t) \\ -B_1(t) & B_2(t) & B_3(t)-2B_0(t) & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}.$$

while the vector fields are now

$$\begin{aligned}
X_0 &= -\Gamma = p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2}, \\
X_1 &= \frac{1}{2}\left(p_2\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_2} - q_1\frac{\partial}{\partial p_2}\right), \\
X_2 &= \frac{1}{2}\left(-q_2\frac{\partial}{\partial q_1} - p_2\frac{\partial}{\partial p_1} + q_1\frac{\partial}{\partial q_2} + p_1\frac{\partial}{\partial p_2}\right), \\
X_3 &= \frac{1}{2}\left(p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial p_2}\right),
\end{aligned}$$

satisfying

$$[X_0,\cdot] = 0, \qquad [X_1, X_2] = -X_3, \qquad [X_2, X_3] = -X_1, \qquad [X_3, X_1] = -X_2.$$

The vector fields X_0, X_1, X_2, X_3 are Hamiltonian with Hamiltonian functions given by

$$\begin{aligned}
h_0(\psi) &= \tfrac{1}{2}\langle \psi, \psi\rangle = \tfrac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2), \\
h_1(\psi) &= \tfrac{1}{2}\langle \psi, S_1\psi\rangle = \tfrac{1}{2}(q_1 q_2 + p_1 p_2), \\
h_2(\psi) &= \tfrac{1}{2}\langle \psi, S_2\psi\rangle = \tfrac{1}{2}(q_1 p_2 - p_1 q_2), \\
h_3(\psi) &= \tfrac{1}{2}\langle \psi, S_3\psi\rangle = \tfrac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2).
\end{aligned}$$

h_1, h_2, h_3 are functionally independent, but $h_0^2 = 4(h_1^2 + h_2^2 + h_3^2)$.

When $\mathcal{H}$ is not finite-dimensional Lie system theory applies when the t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients of Hamiltonians H_i closing on, under the commutator bracket, a real finite-dimensional Lie algebra.

Note however that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension.

On the other hand, as the fundamental concept for measurements is the expectation value of observables, two vectors ψ_1 and ψ_2 of $\mathcal{H}$ such that

$$\frac{\langle \psi_2, A\psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle} = \frac{\langle \psi_1, A\psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle}, \quad \forall A \in \mathrm{Her}(\mathcal{H}),$$

should be considered as indistinguishable.

This is only possible when ψ_2 is proportional to ψ_1, and therefore we must consider rays rather than vectors the elements describing the quantum states.

The space of states is not $\mathbb{C}^n$ but the projective space $\mathbb{CP}^{n-1}$.

It is possible to define a Kähler structure on $\mathbb{CP}^{n-1}$ and therefore to study Lie-Kähler systems leading to superposition rules and to study time evolution in this projective space.

As an example of application of the theory of quasi-Lie schemes to a quantum case let us consider the quantum version of the anharmonic oscillator model proposed by Perelomov: a t-dependent Hamiltonian Hermitian operator in $\mathcal{L}^2(\mathbb{R}^n)$ of the form

$$\widehat{H}(t) := \frac{1}{2} \sum_{i=1}^{n} \left(\widehat{p}_i^2 + \omega^2(t) \widehat{x}_i^2 \right) + c(t) U(\widehat{x}_1, \ldots, \widehat{x}_n),$$

where $c(t) \neq 0$ and $\omega^2(t)$ are two real functions, $\widehat{x}_i$ and $\widehat{p}_i$ are the usual operators

$$\widehat{x}_i\psi(x) = x_i\,\psi(x), \qquad \widehat{p}_i\psi(x) = -i\frac{\partial\psi}{\partial x_i},$$

and $U(\widehat{x}_1,\ldots,\widehat{x}_n)$ is a quantum potential determined by an homogeneous polynomial of order k depending on the position operators $\widehat{x}_1,\ldots,\widehat{x}_n$, i.e. $U(\lambda\widehat{x}_1,\ldots,\lambda\widehat{x}_n) = \lambda^k U(\widehat{x}_1,\ldots,\widehat{x}_n)$ for every $\lambda \in \mathbb{R}$.

Perelomov left as an open problem to look for a quantum analogue of his results in the classical case.

We show next how to use a quasi-Lie scheme in this quantum case of a t-dependent Hamiltonian operator $-\mathrm{i}\widehat{H}(t)$.

We consider the linear spaces $V := \langle \mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_4\rangle$ and $W := \langle \mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3\rangle$, with

$$\widehat{H}_1 := \sum_{i=1}^n \frac{\widehat{p}_i^2}{2}, \quad \widehat{H}_2 := \frac{1}{4}\sum_{i=1}^n(\widehat{x}_i\widehat{p}_i+\widehat{p}_i\widehat{x}_i), \quad \widehat{H}_3 := \sum_{i=1}^n \frac{\widehat{x}_i^2}{2}, \quad \widehat{H}_4 := U(\widehat{x}_1,\ldots,\widehat{x}_n).$$

It is easy to check that W and V are finite-dimensional linear spaces of skew-Hermitian operators such that:

• $[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3] = \mathrm{i}\widehat{H}_3$, and then the space W is a real Lie algebra of skew-Hermitian operators.

• The following commutation relations are true:

$$[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_1] = -\mathrm{i}\widehat{H}_1, \qquad [\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_1] = -2\mathrm{i}\widehat{H}_2, \qquad [\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_4] = 0,$$

and it is also possible to check that because the assumed homogenity property of V $[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_4] \subset V$, and therefore the pair$(W, V)$ defines a quantum quasi-Lie scheme $S(W, V)$ such that $-\mathrm{i}\widehat{H}(t) \in V(\mathbb{R})$. In fact, as $[\widehat{x}_j, \widehat{p}_j] = \mathrm{i}\, I$ for $j = 1, \ldots, n$, then $\widehat{p}_j\widehat{x}_j = \widehat{x}_j\widehat{p}_j - \mathrm{i}I$ and

$$\sum_{j=1}^{n}(\widehat{x}_j\widehat{p}_j + \widehat{p}_j\widehat{x}_j) = 2\sum_{j=1}^{n}\widehat{x}_j\widehat{p}_j - \mathrm{i}\, n\, I.$$

Consequently,

$$[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_4] = -\frac{1}{2}\left[\sum_{j=1}^{n}\widehat{x}_j\widehat{p}_j, U(\widehat{x}_1, \ldots, \widehat{x}_n)\right] = \frac{\mathrm{i}}{2}\sum_{j=1}^{n}\widehat{x}^j\frac{\partial U}{\partial\widehat{x}^j}(\widehat{x}_1, \ldots, \widehat{x}_n),$$

and therefore $[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_4] = \frac{\mathrm{i}}{2}k\widehat{H}_4$, because of the *Euler's homogeneous function theorem*.

Lie Systems and Schrödinger equation

A linear SODE in normal form $\phi'' = b_1(x)\phi + b_2(x)\phi'$ can be written in the form of a system of two first-order differential equations in the variables (v_ϕ, ϕ):

$$\left\{ \begin{array}{lcl} v'_\phi & = & b_2(x)v_\phi + b_1(x)\phi \\ \phi' & = & v_\phi \end{array} \right. .$$

Identifying $\mathbb{R}^2$ with $T\mathbb{R}$, (v_ϕ, ϕ) are bundle coordinates, the preceding system determines the integral curves of the x-dependent vector field

$$X = v_\phi \frac{\partial}{\partial\phi} + (b_1(x)\phi + b_2(x)v_\phi) \frac{\partial}{\partial v_\phi},$$

which is said to be a SODE vector field because of the coefficient of $\partial/\partial\phi$.

The linear system determining its integral curves is

$$\begin{pmatrix} v'_\phi \\ \phi' \end{pmatrix} = \begin{pmatrix} b_2(x) & b_1(x) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_\phi \\ \phi \end{pmatrix} .$$

The projection onto $\mathbb{R}$ of such curves are solutions of the differential equation

$$\phi'' = b_2(x)\phi' + b_1(x)\phi.$$

We are mainly interested in equations of Schrödinger type, those with $b_2(x) \equiv 0$.

The corresponding vector field is a linear combination $X = b_1(x)X_1 - X_3$ where

$$X_1 = \phi \frac{\partial}{\partial v_\phi}\,, \qquad X_3 = -v_\phi \frac{\partial}{\partial \phi},$$

which together with

$$X_2 = \frac{1}{2}\left(v_\phi \frac{\partial}{\partial v_\phi} - \phi \frac{\partial}{\partial \phi}\right),$$

close on a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$:

$$[X_1, X_3] = 2\,X_2, \qquad [X_1, X_2] = X_1, \qquad [X_3, X_2] = -X_3\,.$$

Therefore Schrödinger type equations and the corresponding linear systems are Lie systems with Vessiot-Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

Vector fields X_1, X_2 and X_3 are fundamental vector fields corresponding to the linear action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$.

The map $F : \mathbb{R}^2_* \to \mathbb{R}$ defined by $F(x,y) = x/y$ is equivariant with respect to the the restriction of the linear action Φ of $SL(2,\mathbb{R})$ on $\mathbb{R}^2_*$ and the action Ψ of $SL(2,\mathbb{R})$

on $\mathbb{R}$, or even better on the real projective line $\mathbb{R}P^1 = \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, by linear fractional transformations, i.e. $\Psi : SL(2, \mathbb{R}) \times \mathbb{R}P^1 \to \mathbb{R}P^1$ is defined by

$$\begin{aligned} \Psi(A\,,u) &= \frac{\alpha\,u+\beta}{\gamma\,u+\delta}\,, \qquad \text{if} \;\; u \neq -\frac{\delta}{\gamma}\,, \\ \Psi(A\,,\infty) &= \frac{\alpha}{\gamma}\,, \qquad \Psi\left(A\,,-\frac{\delta}{\gamma}\right) = \infty\,, \\ A &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R})\,. \end{aligned}$$

Equivariance means that $F \circ \Phi_A = \Psi_A \circ F$. The corresponding fundamental vector fields of the action Ψ are now

$$\overline{X}_1 = \frac{\partial}{\partial u}\,, \quad \overline{X}_2 = u\frac{\partial}{\partial u}\,, \quad \overline{X}_3 = u^2\frac{\partial}{\partial u}\,,$$

and as F is equivariant, the fundamental vector fields associated to Φ and Ψ are F-related, i.e. $\overline{X}_i = F_*(X_i)$, $i = 1, 2, 3$, and then a system defined by the vector fields $\overline{X}_i$ is a Lie system corresponding to a Riccati equation.

The image under F of an integral curve of the x-dependent vector field $X = b_1(x)\,X_1 + b_2(x)\,X_2 + b_3(x)\,X_3$, which is a linear system, is an integral curve of $\bar{X} = \bar{b}_1(x)\,X_1 + \bar{b}_2(x)\,X_2 + \bar{b}_3(x)\,X_3$, i.e. a solution of the corresponding Riccati equation.

We can also consider a new vector field

$$X_4 = \frac{1}{2}\left(v_\phi \frac{\partial}{\partial v_\phi} + \phi \frac{\partial}{\partial \phi}\right),$$

which commutes with X_1, X_2 and X_3 and they generate the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$.

The integral curves of a generic Lie system with such a Vessiot-Lie algebra, namely $X = b_1(x)\, X_1 + b_2(x)\, X_2 + b_3(x)\, X_3 + b_4(x)\, X_4$, are determined by the system

$$\begin{pmatrix} v'_\phi \\ \phi' \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(b_2(x) + b_4(x)) & b_1(x) \\ -b_3(x) & \frac{1}{2}(b_2(x) - b_4(x)) \end{pmatrix} \begin{pmatrix} v_\phi \\ \phi \end{pmatrix}.$$

Note also that $\overline{X}_4 = F_*(X_4) = 0$. A SODE vector field is a particular case corresponding to the choice $b_3 = -1$ and $b_2 = b_4$.

We can use that the group of curves in a Lie group G acts on the set Lie systems with associated Vessiot-Lie algebra $\mathfrak{g}$ to reduce a given Lie system to another one of the same type. In our case, to relate Schrödinger type equations with different potentials by means of curves in $GL(2,\mathbb{R})$. This is the essence of Darboux transformation method.

The advantage to see Schrödinger equations as Lie systems with Vessiot-Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ instead of $\mathfrak{sl}(2,\mathbb{R})$ is that we can transform by curves which are in $GL(2,\mathbb{R})$ but not in $SL(2,\mathbb{R})$.

Remark that if we are interested in taking into account the tangent bundle character of $T\mathbb{R}$, we should consider transformations induced from those of the base manifold.

So, given a strictly positive function φ_0 we can associate the function ϕ with the new function $\bar\phi$ by means of $\phi = \varphi_0\,\bar\phi$. This induces a transformation

$$\begin{pmatrix} v_\phi \\ \phi \end{pmatrix} = \begin{pmatrix} \varphi_0(x) & \varphi_0'(x) \\ 0 & \varphi_0(x) \end{pmatrix} \begin{pmatrix} v_{\bar\phi} \\ \bar\phi \end{pmatrix} .$$

If ϕ is a solution of $\phi'' = b_1(x)\phi + b_2(x)\phi'$, then $\bar\phi$ satisfies

$$\varphi_0(x)\bar\phi'' + (2\varphi_0'(x) - b_2(x)\varphi_0(x))\bar\phi' + (\varphi_0'' - b_1(x)\varphi_0 - b_2(x)\varphi_0')\bar\phi = 0,$$

and then when $\varphi_0(x)$ is a particular solution of the given equation, the transformed equation reduces to $\varphi_0(x)\,\bar\phi'' + (2\,\varphi_0'(x) - b_2\,\varphi_0(x))\bar\phi' = 0$, in which the dependent variable $\bar\phi$ is absent, which is quickly integrated by order reduction.

This is an explicit example of reduction procedure for Lie system when a particular solution is known.

The transformation $\phi = \varphi_0\,\bar\phi$ is very useful in factorization problems, the differential operator $\partial/\partial x$ becoming a ladder-like operator $\partial/\partial x - \varphi_0'/\varphi_0$. This allows to introduce factorisable Hamiltonians, their partners leading to supersymmetric quantum mechanics and interesting relations among their spectra, in particular special methods for introducing or removing eigenvalues and eigenstates.

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THANKS FOR YOUR ATTENTION !!!

CONGRATULATIONS, MIKHAIL!!!

and

THANKS FOR ALL