Topological phases in the context of non-Hermitian physics

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February 26, 2019

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Knut och Alice Wallenbergs Stiftelse

Dissipation

Non-Hermitian Hamiltonians $H \neq H^\dagger$ describe dissipation

Relevant for, e.g., optical systems with gain and loss

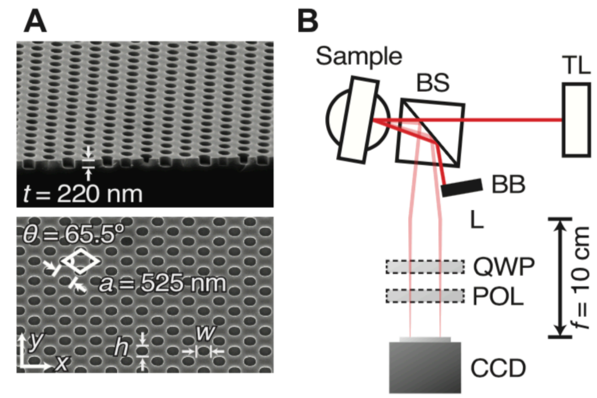
PT symmetry: real spectrum in the PT-unbroken phase Bender and Boettcher, PRL **80**, 5243 (1998).

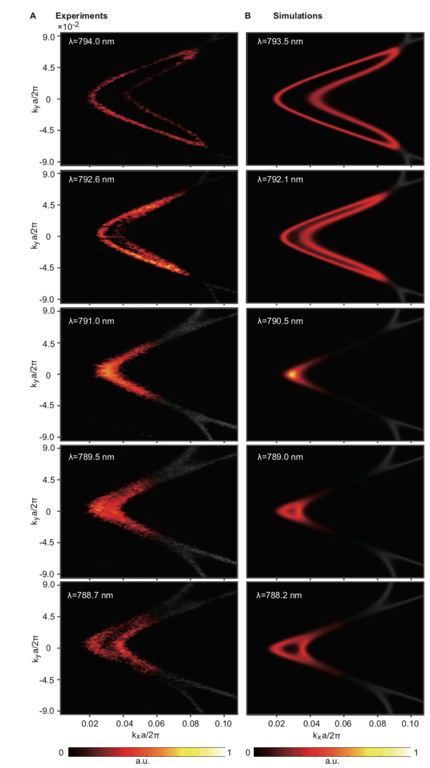
Toy alternative of the Lindblad master equation

Review article: Martinez Alvarez, Barrios Vargas, Berdakin, and Foa Torres, Eur. Phys. J. Spec. Top. **227**, 1295 (2018).

Experimental observation of Fermi arcs in two dimensions

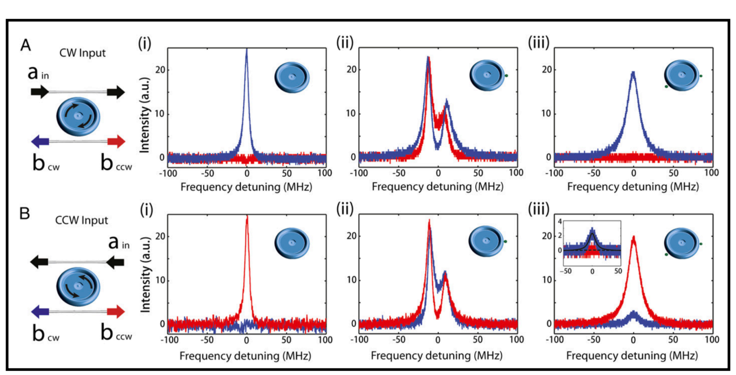
Zhou et al., Science **359**, eaap9859 (2018).





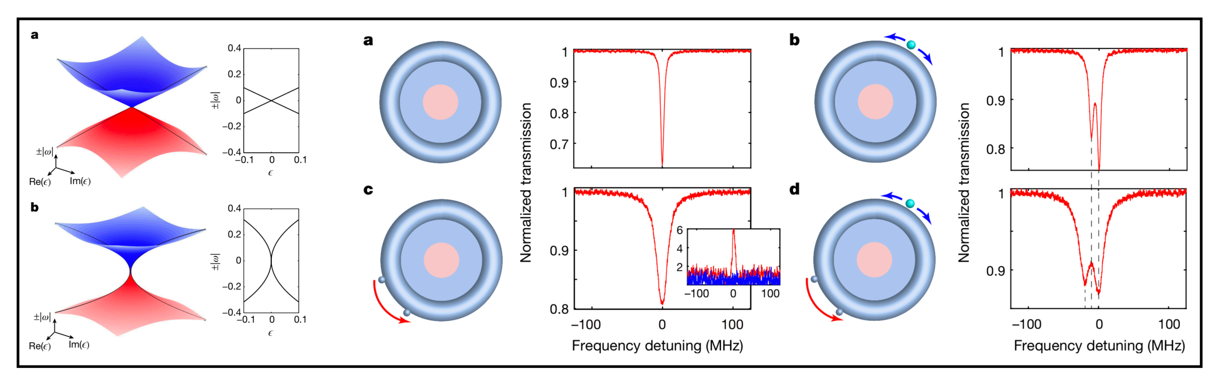
Spectrum features exceptional points, which behave vastly different from their Hermitian counterparts

Unidirectional transmission



Peng et al., PNAS **113**, 6845 (2016).

Enhanced sensitivity



Chen et al., Nature **548**, 192 (2017).

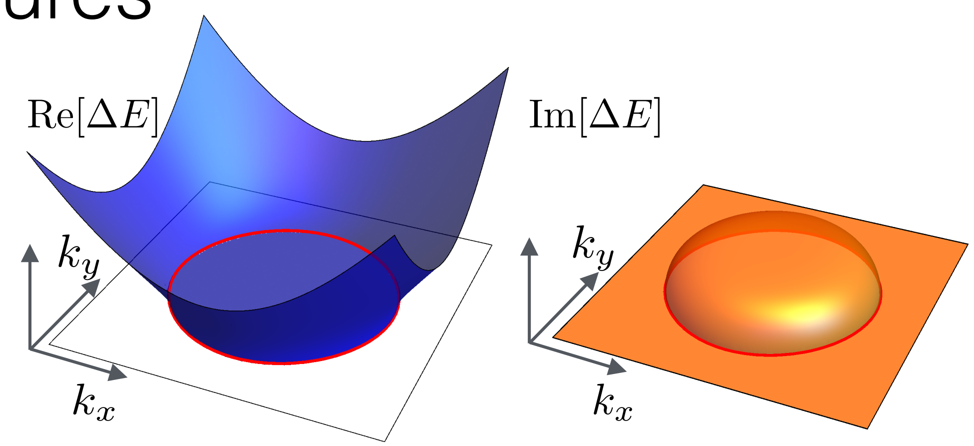
In this talk

Discuss basics of non-Hermitian matrices

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Generic appearance of exceptional structures

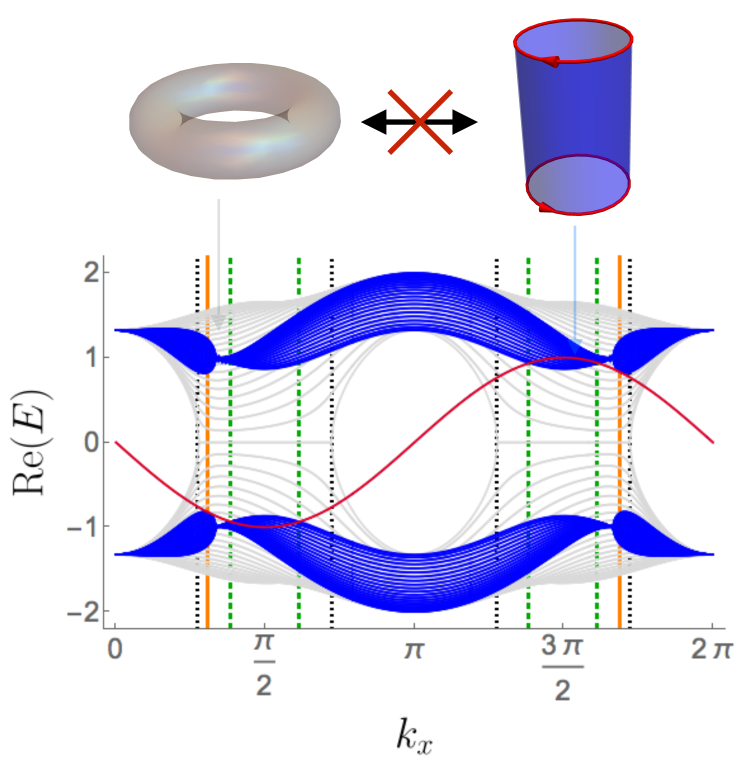
In the presence of symmetries



Jan Carl Budich, Johan Carlström, **Flore K. Kunst**, and Emil J. Bergholtz, Phys. Rev. B **99**, 041406(R) (2019). *Editors' Suggestion*

Biorthogonal bulk-boundary correspondence

In first-order topological systems



Flore K. Kunst, Elisabet Edvardsson, Jan Carl Budich, and Emil J. Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018). *Editors' Suggestion*

And in higher-order models

Elisabet Edvardsson, **Flore K. Kunst**, and Emil J. Bergholtz, arXiv:1812.09060. *Phys. Rev. B Rapid, in press.*

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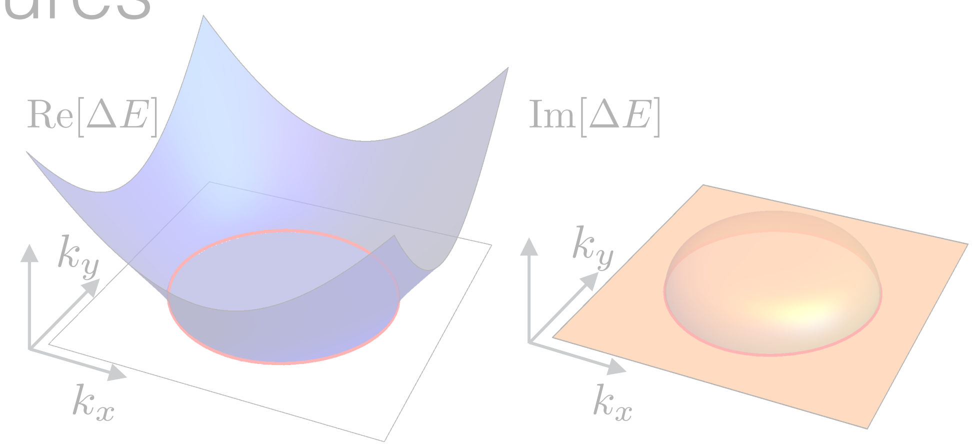

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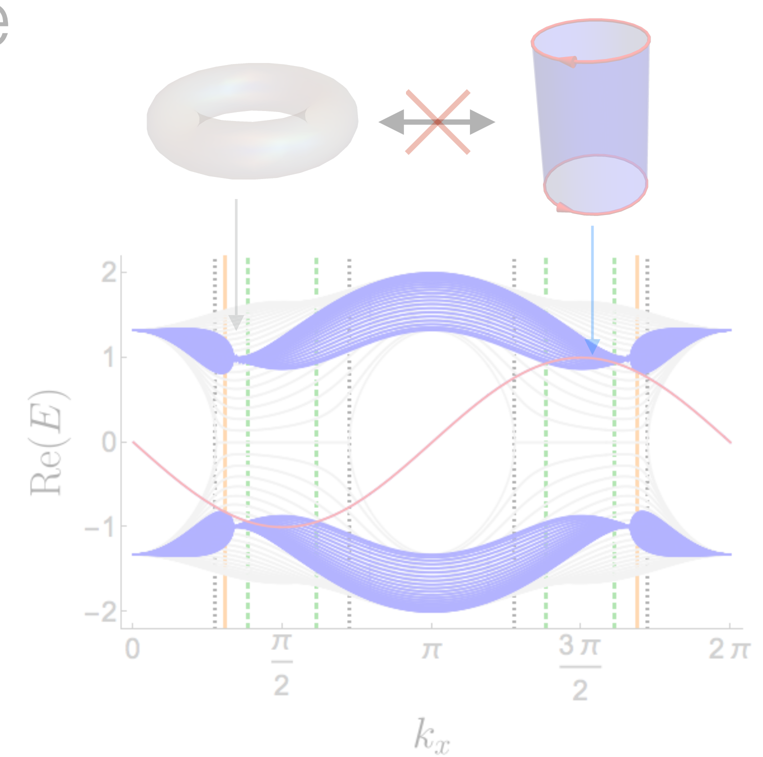

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Basics

Simple 2x2 non-Hermitian model

$$H \neq H^\dagger$$

$$H = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \qquad \alpha \neq \beta^*$$

Eigenvalues are generally complex $E_\pm = \pm\sqrt{\alpha\beta}$

The imaginary part is associated with a lifetime $\mathrm{Im}(E) \sim 1/\tau$

Non-orthonormal eigenvectors

$$H\left|\Psi_\pm^R\right\rangle = E_\pm\left|\Psi_\pm^R\right\rangle \qquad \left\langle\Psi_\pm^L\right| H = E_\pm\left\langle\Psi_\pm^L\right|$$

$$\Psi_\pm^R \propto \begin{pmatrix} \sqrt{\alpha} \\ \pm\sqrt{\beta} \end{pmatrix}$$

Right and left eigenvectors are inequivalent

$$\Psi_\pm^L \propto \begin{pmatrix} \sqrt{\beta} & \pm\sqrt{\alpha} \end{pmatrix}$$

$$\left(\Psi_i^L\right)^\dagger \neq \Psi_i^R$$

Choose to form a biorthogonal set when $\alpha\beta \neq 0$

$$\langle\Psi_i^L|\Psi_j^R\rangle = \delta_{ij}$$

Degeneracy

The energy is doubly degenerate for $\beta = 0$

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Only one eigenvector remains $\Psi^R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\Psi^L = \begin{pmatrix} 0 & 1 \end{pmatrix}$

The Hamiltonian is defective

The eigenvectors coalesce, i.e., the geometric multiplicity is not equal to the algebraic multiplicity

Such degeneracies are called exceptional points

Heiss, J. Phys. A: Math. Theor. **45**, 444016 (2012).

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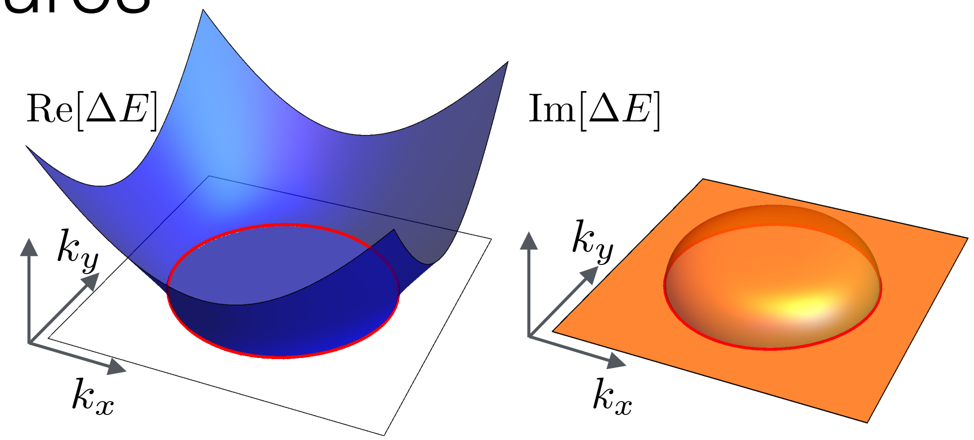

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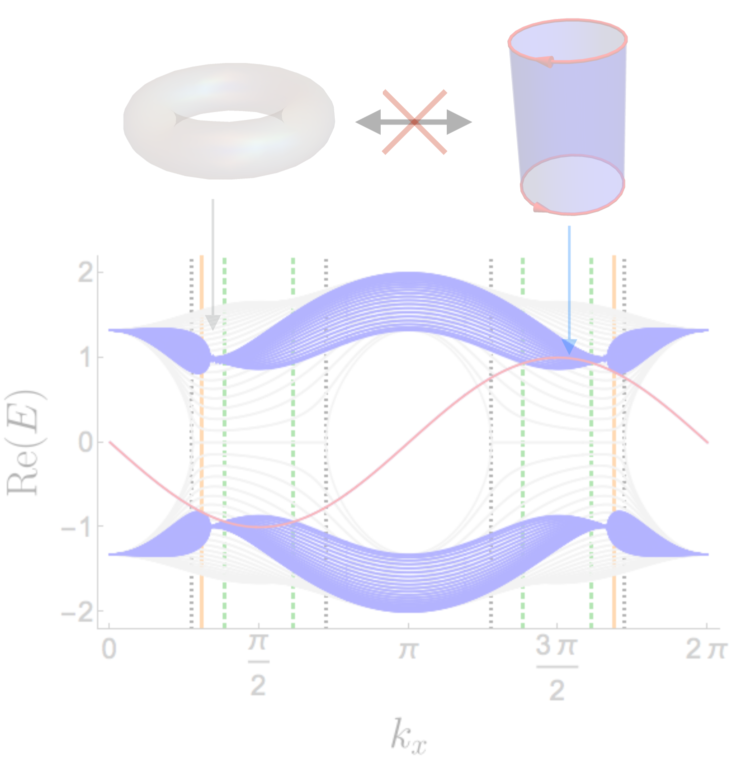


And in higher-order models

Elisabet Edvardsson, **Flore K. Kunst**, and Emil J. Bergholtz, arXiv:1812.09060. *Phys. Rev. B Rapid, in press.*

Hermitian case

In the Hermitian case, two bands cross generically in three dimensions

Generic two-band model

$$H(\mathbf{k}) = \begin{pmatrix} d_0(\mathbf{k}) + d_z(\mathbf{k}) & d_x(\mathbf{k}) - i d_y(\mathbf{k}) \\ d_x(\mathbf{k}) + i d_y(\mathbf{k}) & d_0(\mathbf{k}) - d_z(\mathbf{k}) \end{pmatrix}$$
$$= \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma} + d_0(\mathbf{k})\sigma_0$$

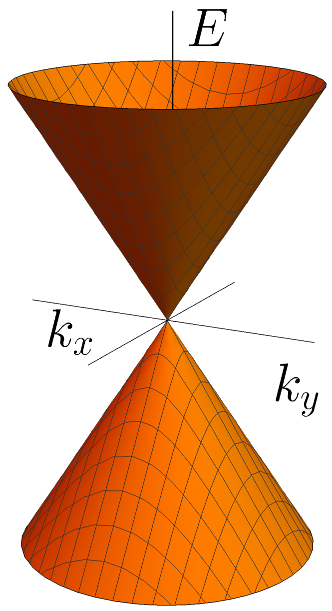
Eigenvalues $E_\pm(\mathbf{k}) = d_0(\mathbf{k}) \pm \sqrt{d_x^2(\mathbf{k}) + d_y^2(\mathbf{k}) + d_z^2(\mathbf{k})}$

Need to tune three parameters to find a degeneracy

Generic in three dimensions, but requires fine tuning in two dimensions

Example: Weyl node

$$H(\mathbf{k}) = v\mathbf{k} \cdot \boldsymbol{\sigma}$$



Non-Hermitian case

In the non-Hermitian case, two bands cross generically in two dimensions

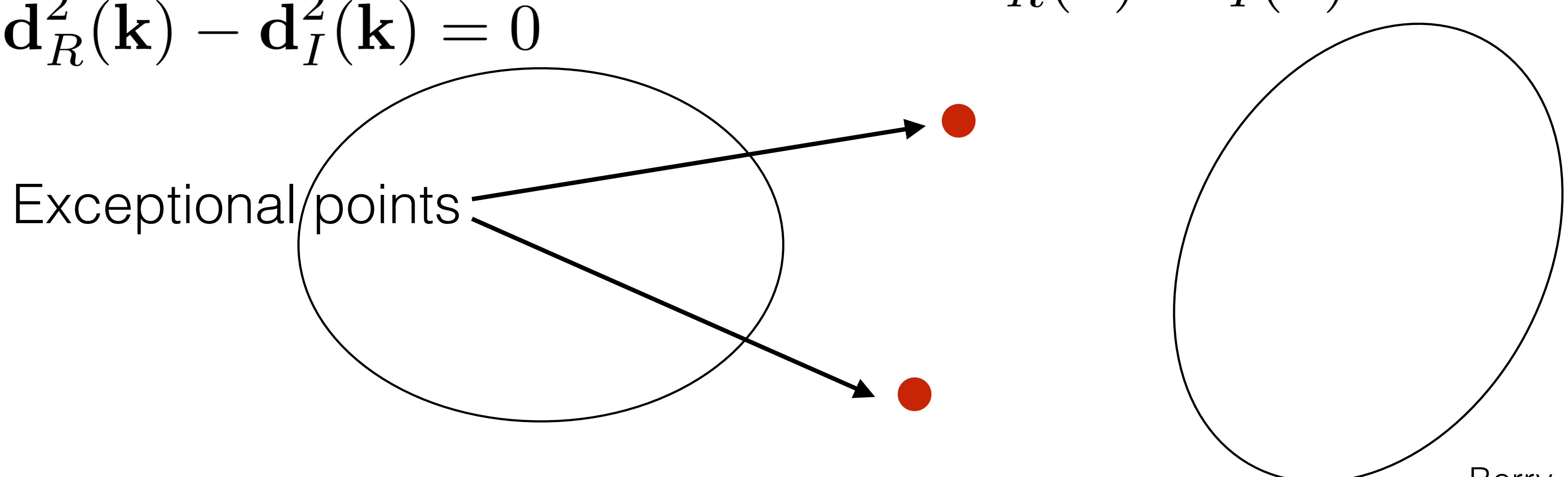
Generic two-band model $H(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}$ $\quad \mathbf{d}(\mathbf{k}) = \mathbf{d}_R(\mathbf{k}) + i\mathbf{d}_I(\mathbf{k})$

Eigenvalues $E_{\pm}(\mathbf{k}) = \pm\sqrt{\mathbf{d}_R^2(\mathbf{k}) - \mathbf{d}_I^2(\mathbf{k}) + 2i\mathbf{d}_R(\mathbf{k}) \cdot \mathbf{d}_I(\mathbf{k})}$

Need to tune two parameters to find a degeneracy

Generic in two dimensions

$\mathbf{d}_R^2(\mathbf{k}) - \mathbf{d}_I^2(\mathbf{k}) = 0$ $\quad \mathbf{d}_R(\mathbf{k}) \cdot \mathbf{d}_I(\mathbf{k}) = 0$

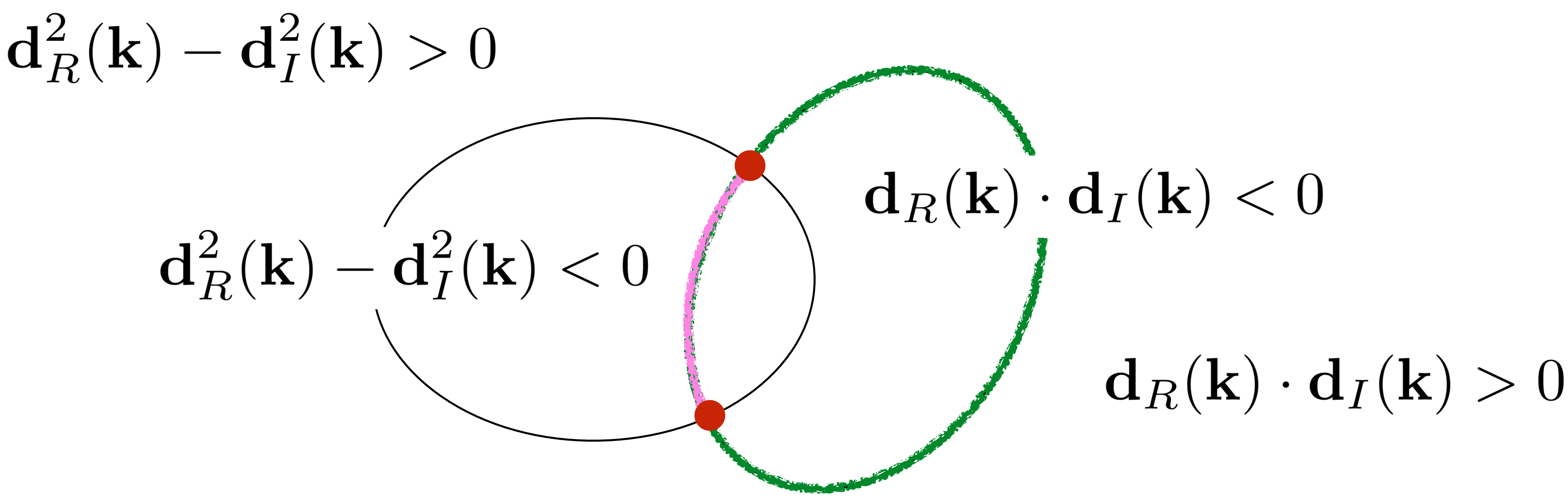


Berry, Czech. J. Phys. **54**, 1039 (2004).
Carlström and Bergholtz, Phys. Rev. A **98**, 042114 (2018).

Generic Fermi arcs

Generic Fermi arcs appear

$$E_{\pm}(\mathbf{k}) = \pm\sqrt{\mathbf{d}_R^2(\mathbf{k}) - \mathbf{d}_I^2(\mathbf{k}) + 2i\mathbf{d}_R(\mathbf{k})\cdot\mathbf{d}_I(\mathbf{k})}$$



Fermi arc: $\mathrm{Re}(E) = 0$ $\quad$ $\mathbf{d}_R(\mathbf{k})\cdot\mathbf{d}_I(\mathbf{k}) = 0$ $\quad$ $\mathbf{d}_R^2(\mathbf{k}) - \mathbf{d}_I^2(\mathbf{k}) < 0$

i-Fermi arc: $\mathrm{Im}(E) = 0$ $\quad$ $\mathbf{d}_R(\mathbf{k})\cdot\mathbf{d}_I(\mathbf{k}) = 0$ $\quad$ $\mathbf{d}_R^2(\mathbf{k}) - \mathbf{d}_I^2(\mathbf{k}) > 0$

Kozii and Fu, arXiv:1708.05841.
Carlström and Bergholtz, Phys. Rev. A **98**, 042114 (2018).

Exceptional points

Exceptional points are non-analytical

Explicit example

$$H(\mathbf{k}) = k_x\sigma_x + k_y\sigma_y + i\gamma\sigma_x$$

$$E_\pm(\mathbf{k}) = \pm\sqrt{k_x^2 + k_y^2 - \gamma^2 + 2i\gamma k_x}$$

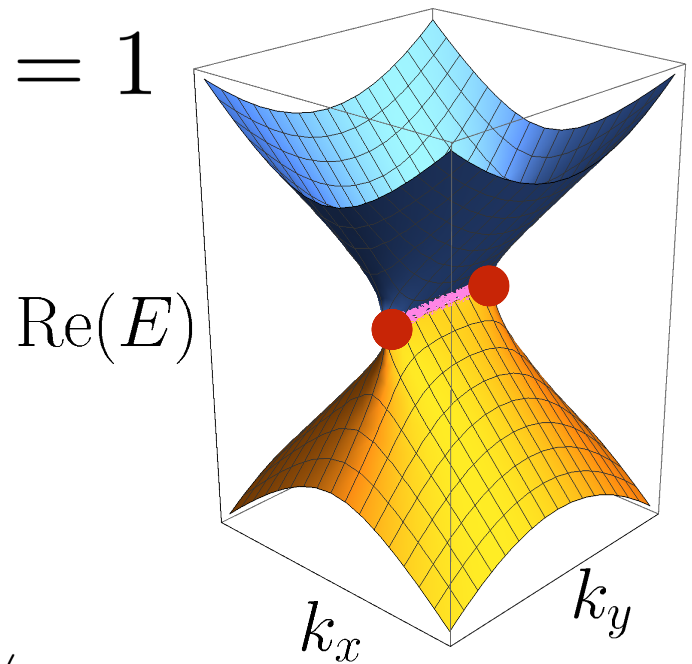
Exceptional points appear when $k_x = 0$ and $k_y = \pm\gamma$

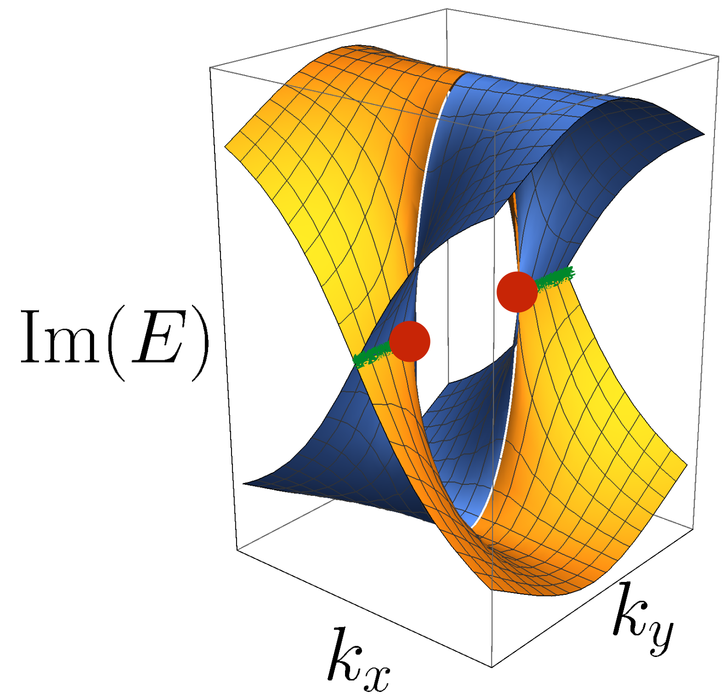
They are connected by Fermi arcs

Expanding around the exceptional point yields

$$|E| \sim \sqrt{|\mathbf{k} - \mathbf{k}_{\mathrm{EP}}|}$$

$\gamma = 1$





Symmetries

43 symmetry classes exist for non-Hermitian systems

Bernard and LeClair, arXiv:cond-mat/0110649 (2001).

An example: $H = qH^\dagger q^{-1}$ $\qquad q^\dagger q^{-1} = qq^\dagger = \mathbb{I}$

This is a trivial, unitary transformation in the Hermitian case

Generic two-band model: $H = \mathbf{d} \cdot \boldsymbol{\sigma} + d_0 \, \sigma_0$

Choice one: $q = \sigma_0$ $\longrightarrow$ Makes the Hamiltonian Hermitian

Choice two: $q = \sigma_x$ $\longrightarrow$ $d_x, d_0 \in \mathbb{R} \qquad d_y, d_z \in i\mathbb{R}$

$\longrightarrow \mathbf{d}_R \cdot \mathbf{d}_I = 0$

$$\longrightarrow E_\pm = \pm\sqrt{\mathbf{d}_R^2 - \mathbf{d}_I^2}$$

Budich, Carlström, **Kunst**, and Bergholtz, Phys. Rev. B **99**, 041406(R) (2019).

Generic exceptional surfaces

We only need to tune one parameter to find a degeneracy

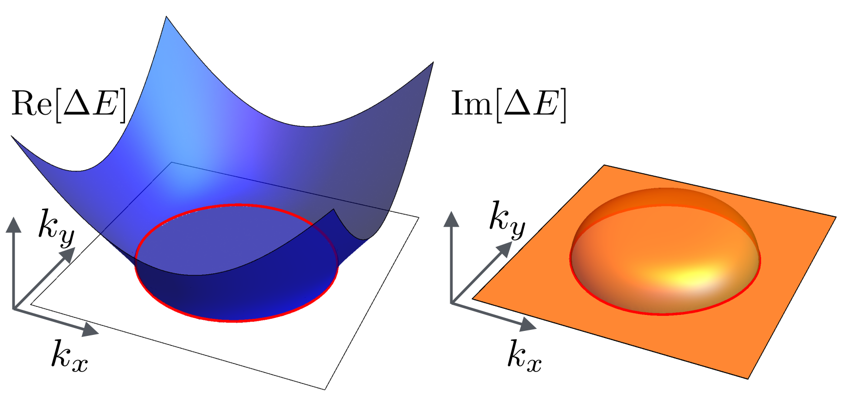
1D: generic exceptional points

2D: generic exceptional lines

3D: generic exceptional surfaces

In the presence of symmetries, exceptional structures of $d-1$ dimensions appear generically in d -dimensional systems

⟶ They form boundaries for d -dimensional Fermi volumes



$$d_x(\mathbf{k}) = (2 - \cos k_x - \cos k_y) \qquad d_z(\mathbf{k}) = i/4$$

This is drastically different from the Hermitian case

Budich, Carlström, **Kunst**, and Bergholtz, Phys. Rev. B **99**, 041406(R) (2019).

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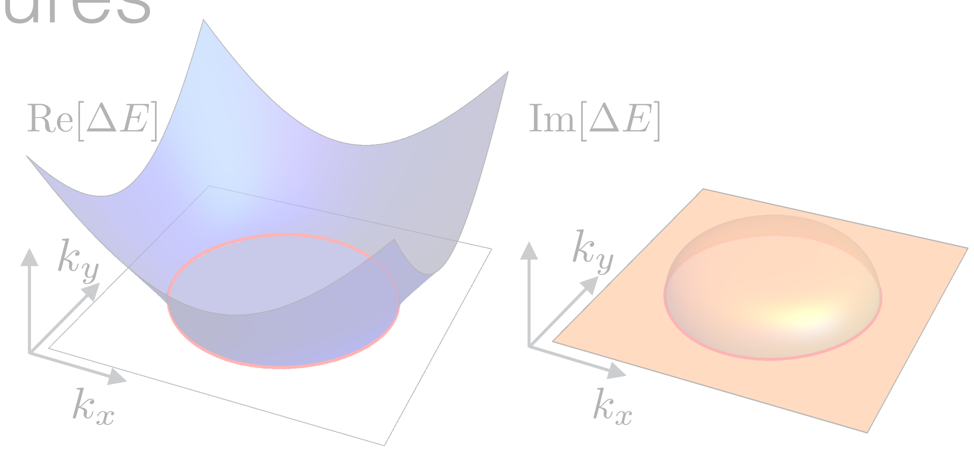

Biorthogonal bulk-boundary correspondence

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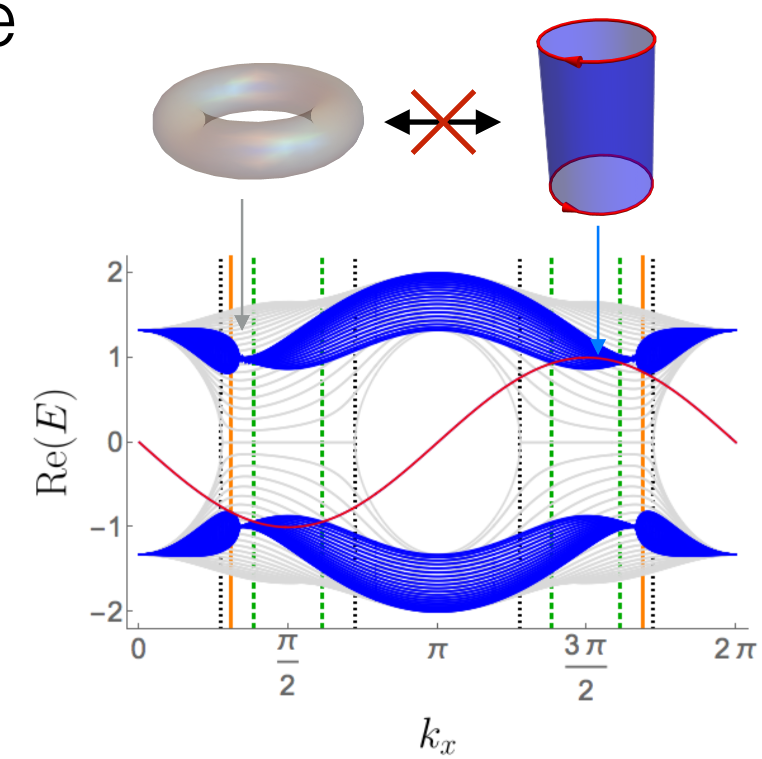
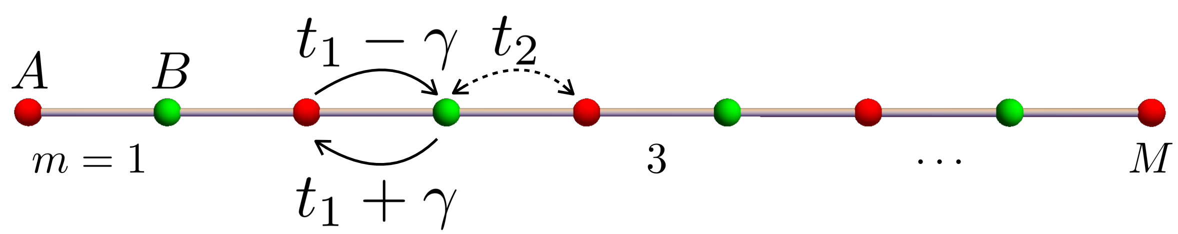


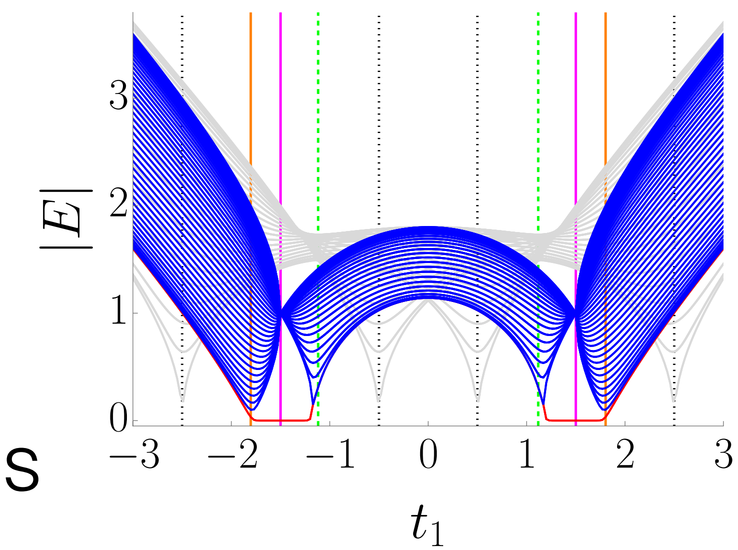
And in higher-order models

Elisabet Edvardsson, **Flore K. Kunst**, and Emil J. Bergholtz, arXiv:1812.09060. *Phys. Rev. B Rapid, in press.*

Breakdown of bulk-boundary correspondence

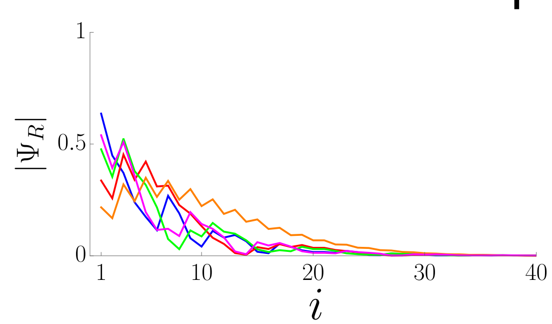
We study a non-Hermitian SSH chain

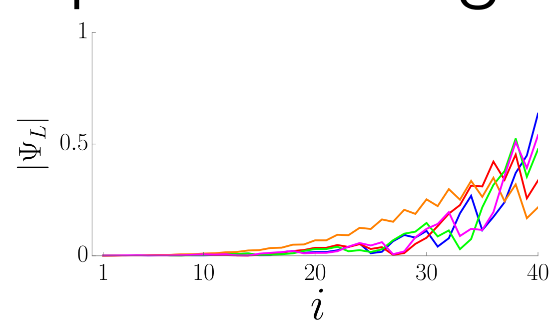




The spectrum is sensitive to boundary conditions

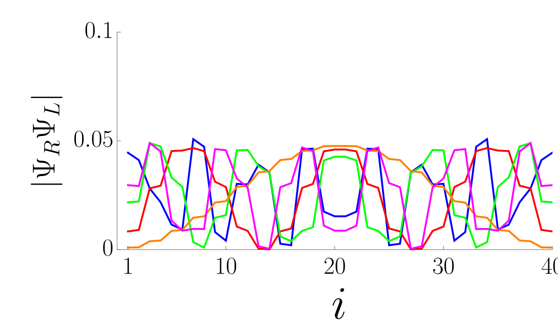
“Bulk” states pile up at the edges





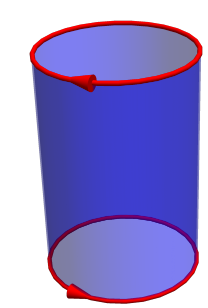
→ Non-Hermitian skin effect

But the biorthogonal product does not



Topological invariants fail to describe the physics

Need to consider a model with open boundary conditions from the start



Biorthogonal quantum mechanics

The eigensystem of a non-Hermitian Hamiltonian reads

$$H\left|\Psi_{R,n}\right\rangle = E_n\left|\Psi_{R,n}\right\rangle \quad H^\dagger\left|\Psi_{L,n}\right\rangle = E_n^*\left|\Psi_{L,n}\right\rangle$$

Away from the exceptional points, a complete biorthonormal basis can be formed $\left\langle\Psi_{L,n}|\Psi_{R,m}\right\rangle = \delta_{n,m}\left\langle\Psi_{L,n}|\Psi_{R,n}\right\rangle = \delta_{n,m}$

The biorthogonal expectation value is used to find the eigenvalues $\left\langle\Psi_{L,n}|H|\Psi_{R,n}\right\rangle = E_n \quad E_n \in \mathbb{C}$

The biorthogonal projection operator reads

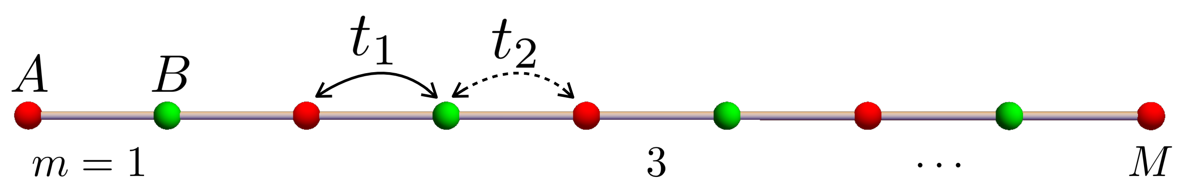
$$\Pi_n = \left|\Psi_{R,n}\right\rangle\left\langle\Psi_{L,n}\right| \quad \sum_n \Pi_n = \mathbb{1}$$

Biorthogonal quantum mechanics can be seen as a generalization of ordinary quantum mechanics

Brody, J. Phys. A: Math. Theor. **47**, 035305 (2013).

Hermitian SSH chain

Consider



SSH, Phys. Rev. B **22**, 2099 (1980).

Real-space Hamiltonian with open boundary conditions

$$H^M = \begin{pmatrix} 0 & t_1 & 0 & 0 & 0 & 0 \\ t_1 & 0 & t_2 & 0 & 0 & 0 \\ 0 & t_2 & 0 & t_1 & 0 & 0 \\ 0 & 0 & t_1 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \vdots & \ddots \end{pmatrix} \qquad \Psi \sim \begin{pmatrix} 1 \\ 0 \\ -t_1/t_2 \\ 0 \\ (-t_1/t_2)^2 \\ \vdots \end{pmatrix}$$

This is an exact wave-function solutions with energy $E = 0$

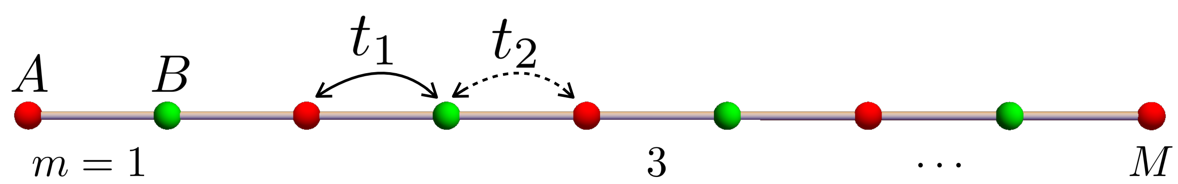
End state: localized to the left when $|t_1/t_2| < 1$

localized to the right when $|t_1/t_2| > 1$

Bulk state: $|t_1/t_2| = 1$

Hermitian SSH chain

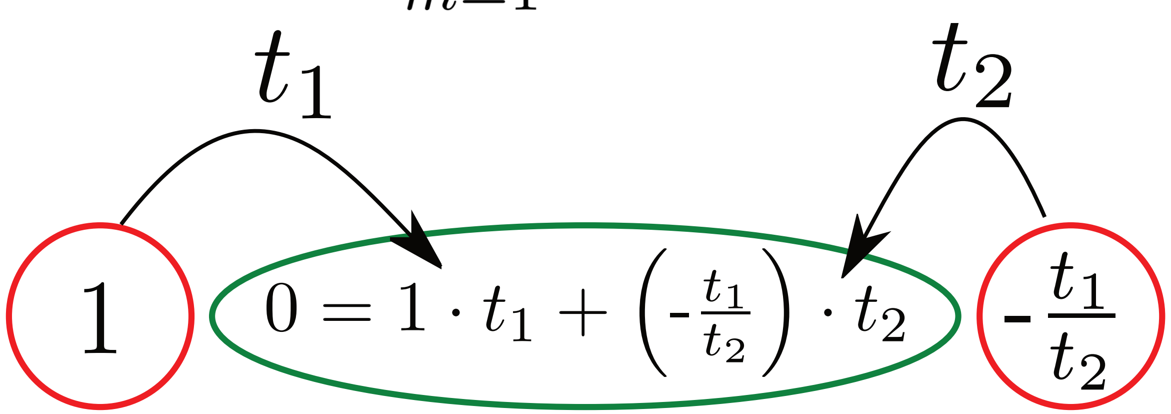
Consider



SSH, Phys. Rev. B **22**, 2099 (1980).

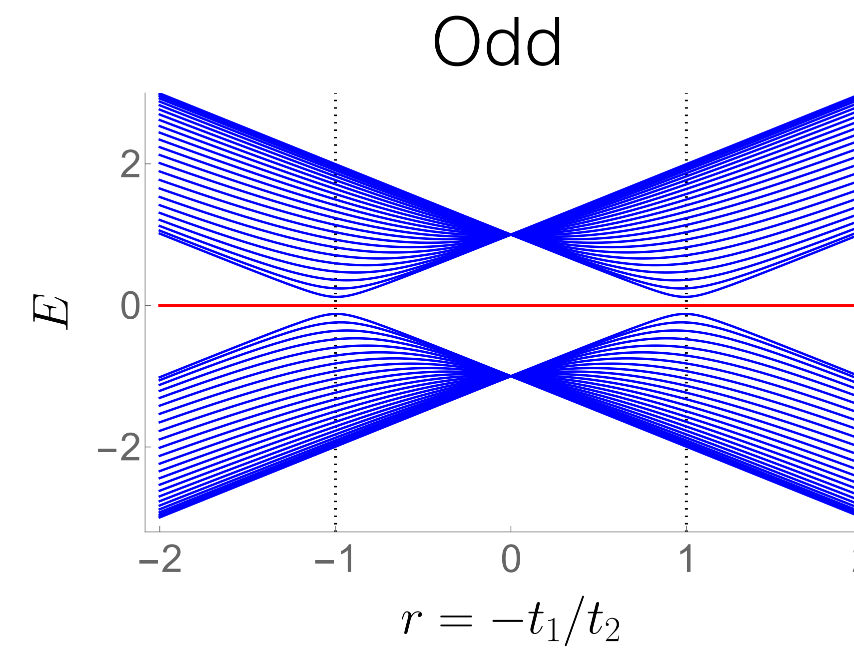
Destructive interference on *B* yields

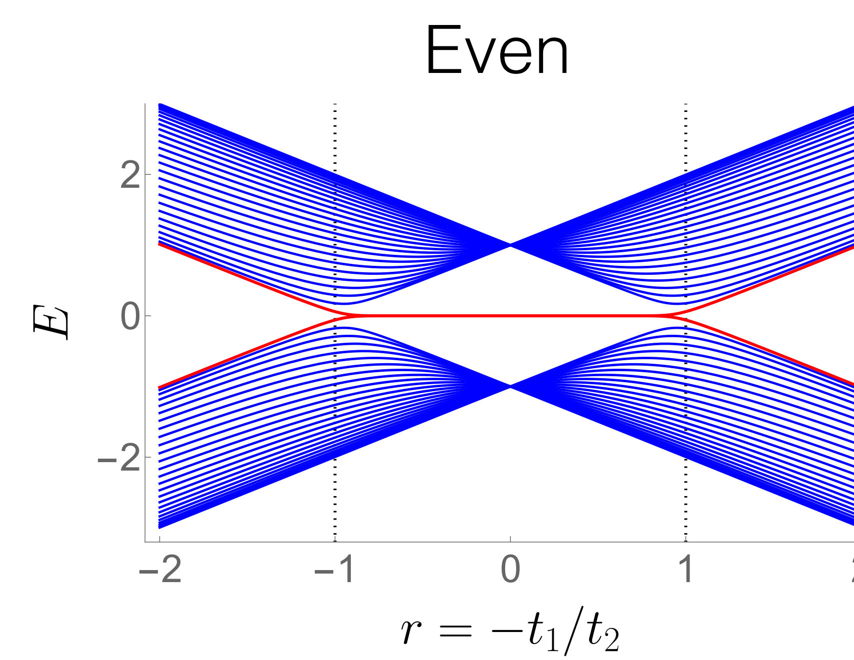
$$|\psi_{\text{end}}\rangle = \mathcal{N} \sum_{m=1}^{M} \left(-\frac{t_1}{t_2}\right)^m c^{\dagger}_{A,m}\,|0\rangle$$



Exact solution with $E=0$ broken unit cell

Approximate solution without broken unit cell

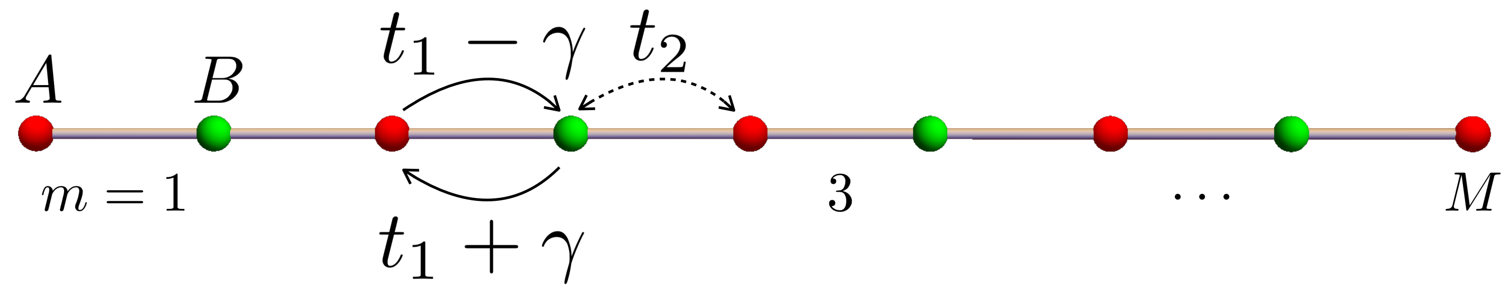




Kunst, Trescher, and Bergholtz, Phys. Rev. B **96**, 085443 (2017).
Kunst, van Miert, and Bergholtz, Phys. Rev. B **97**, 241405(R) (2018).
Kunst, van Miert, and Bergholtz, Phys. Rev. B **99**, 085426 (2019).
Kunst, van Miert, and Bergholtz, Phys. Rev. B **99**, 085427 (2019).

Non-Hermitian SSH chain

Consider



Look for eigenstates of the form $|\Psi_R\rangle = \mathcal{N} \sum_{m=1}^{M} r_R^m c_{A,m}^\dagger |0\rangle$ $\quad |\Psi_L\rangle = \mathcal{N} \sum_{m=1}^{M} r_L^m c_{A,m}^\dagger |0\rangle$

Exact destructive interference on B yields

$\longrightarrow \quad r_R = -\frac{t_1 - \gamma}{t_2} \neq r_L = -\frac{t_1 + \gamma}{t_2}$

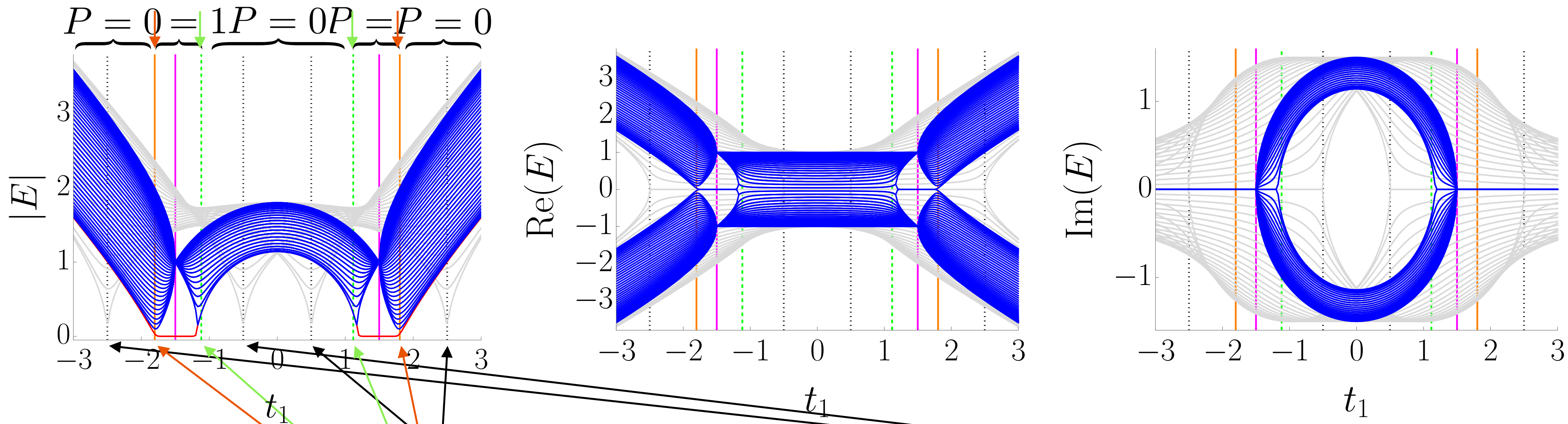
Weight of the wave function on each A motif $\quad \langle \Pi_m \rangle_{\alpha,\alpha'} \equiv \langle \Psi_\alpha | \Pi_m | \Psi_{\alpha'} \rangle$

$$\Pi_m = |e_{A,m}\rangle \langle e_{A,m}| + |e_{B,m}\rangle \langle e_{B,m}|$$

$\langle \Pi_m \rangle_{RR} \sim |r_R|^{2m}$ $\qquad \langle \Pi_m \rangle_{LL} \sim |r_L|^{2m}$ $\qquad \boxed{\langle \Pi_m \rangle_{LR} \sim (r_L^* r_R)^m}$

Kunst, Edvardsson, Budich, and Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).

Non-Hermitian SSH chain



$$r_R = -\frac{t_1 - \gamma}{t_2}$$

$$r_L = -\frac{t_1 + \gamma}{t_2}$$

$|r_R| = 1 \longrightarrow t_1 = \pm t_2 + \gamma$ $\quad$ $|r_L| = 1 \longrightarrow t_1 = \pm t_2 - \gamma$

$|r_L^* r_R| = 1 \longrightarrow t_1 = \pm\sqrt{\gamma^2 + t_2^2}, \pm\sqrt{\gamma^2 - t_2^2}$

→ Biorthogonal bulk-boundary correspondence

Biorthogonal polarization

$$P \equiv 1 - \lim_{M\to\infty} \left\langle \psi_L \left| \frac{\sum_m m\,\Pi_m}{M} \right| \psi_R \right\rangle \longrightarrow$$

$P = 0$ when $|r_L^* r_R| > 1$

$P = 1$ when $|r_L^* r_R| < 1$

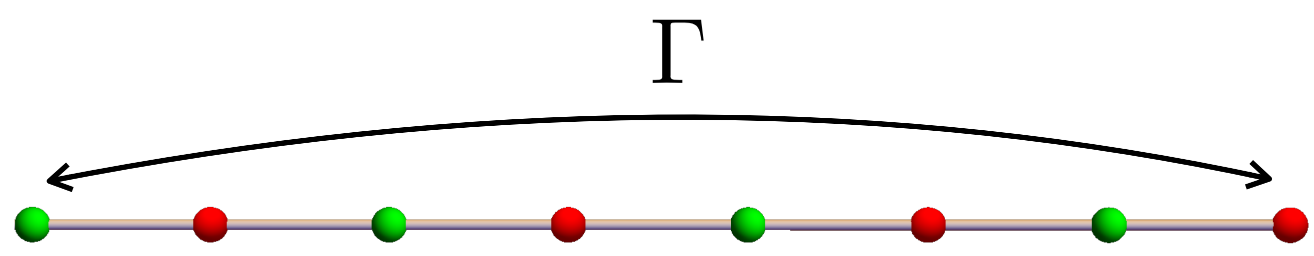
P jumps when $|r_L^* r_R| = 1$

$$t_1 = \pm\gamma$$

Kunst, Edvardsson, Budich, and Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).

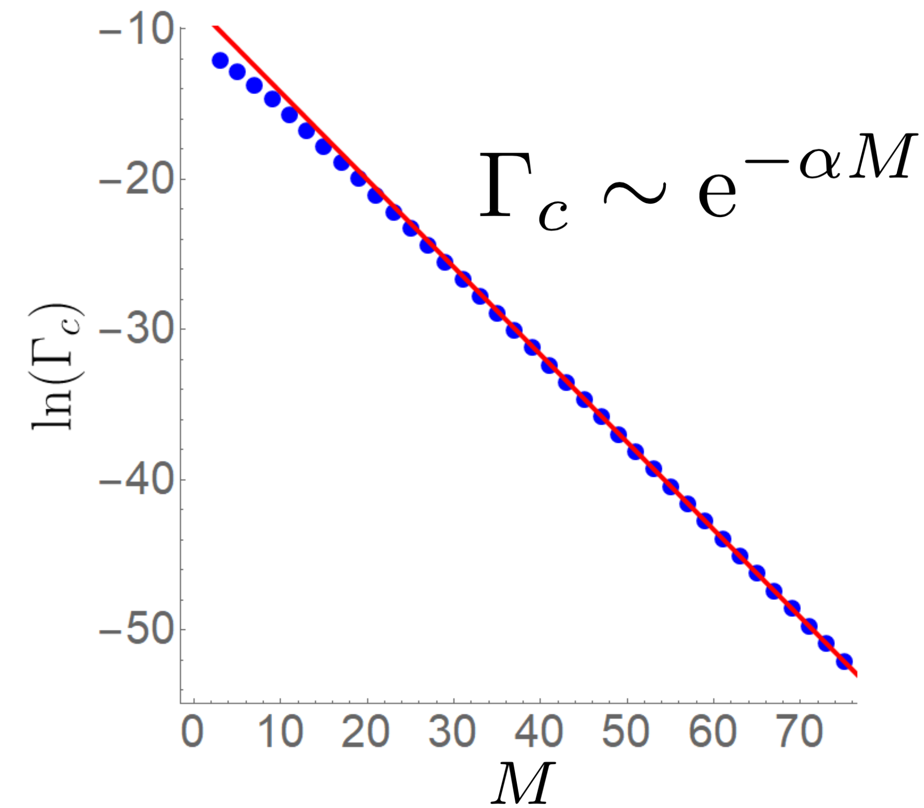
Boundary conditions

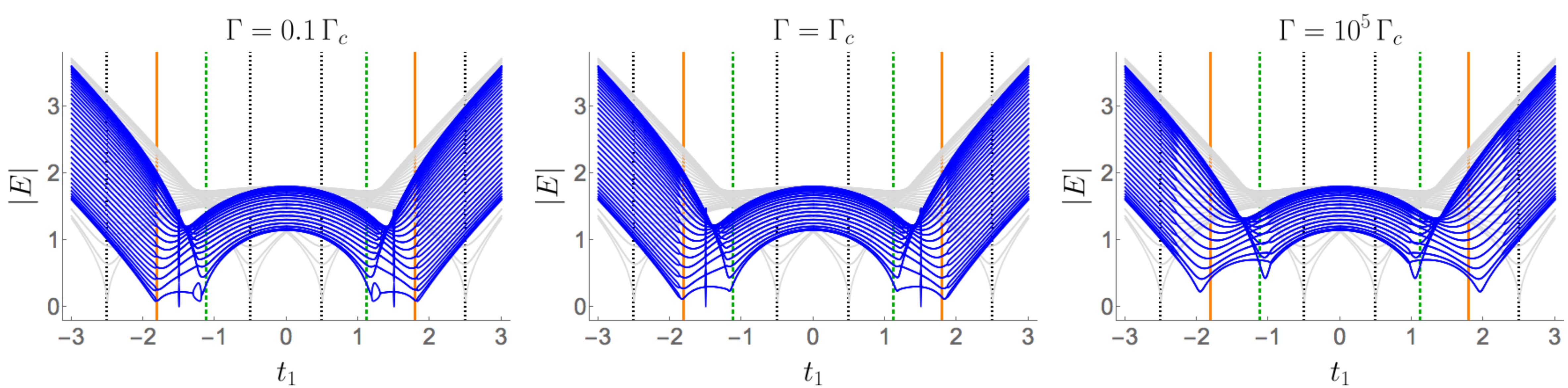
Couple the ends of the chain



Crossover at exponentially small Γ

Understood from non-Hermitian skin effect

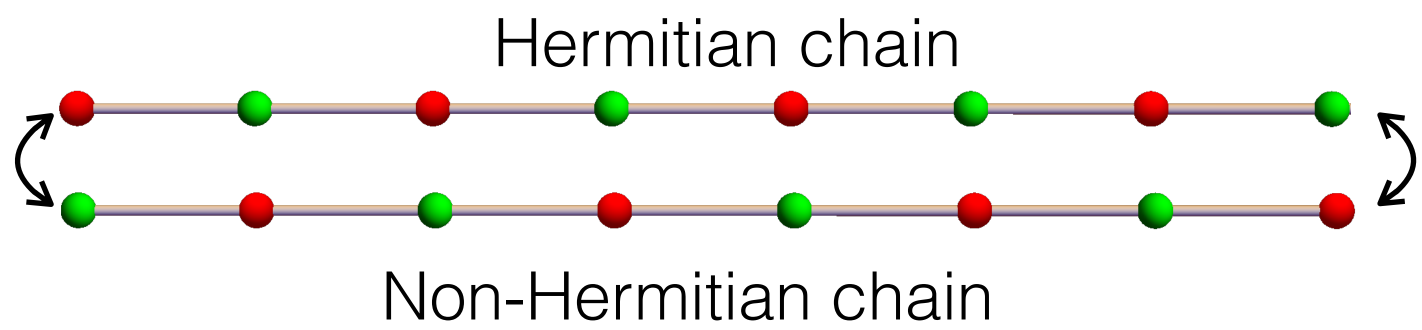


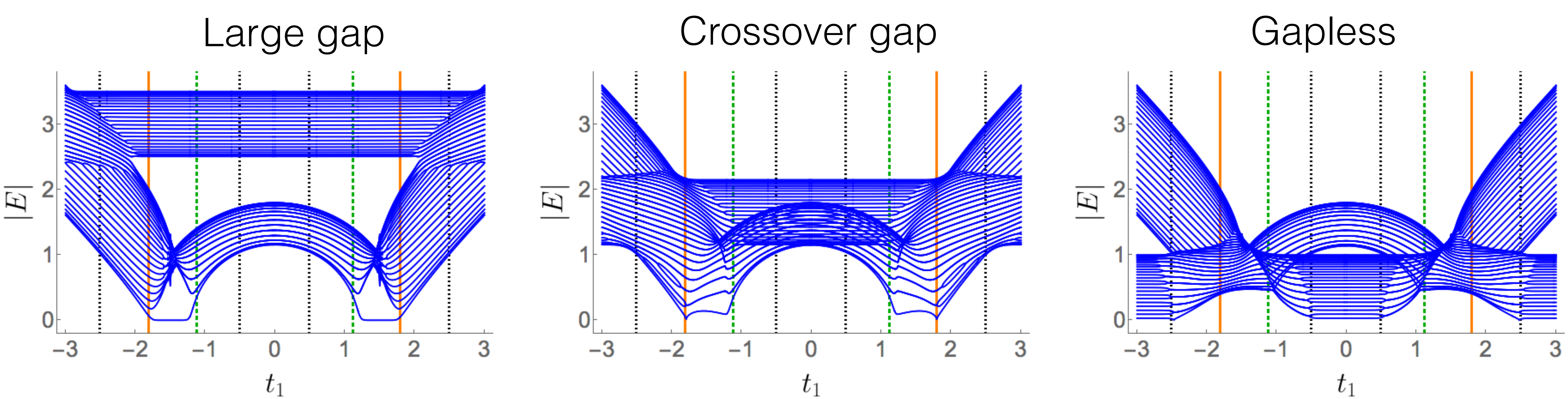


Kunst, Edvardsson, Budich, and Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).

Domain wall

Couple the ends via a Hermitian chain





Kunst, Edvardsson, Budich, and Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).

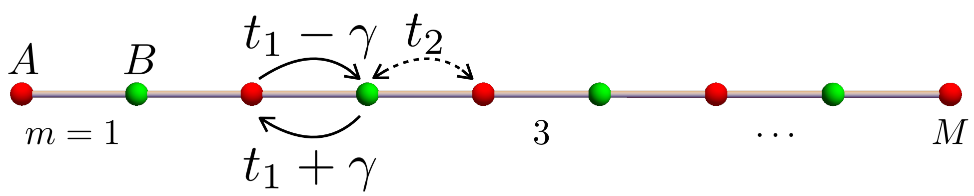
Non-Hermitian Chern insulator

Non-Hermitian Chern insulator

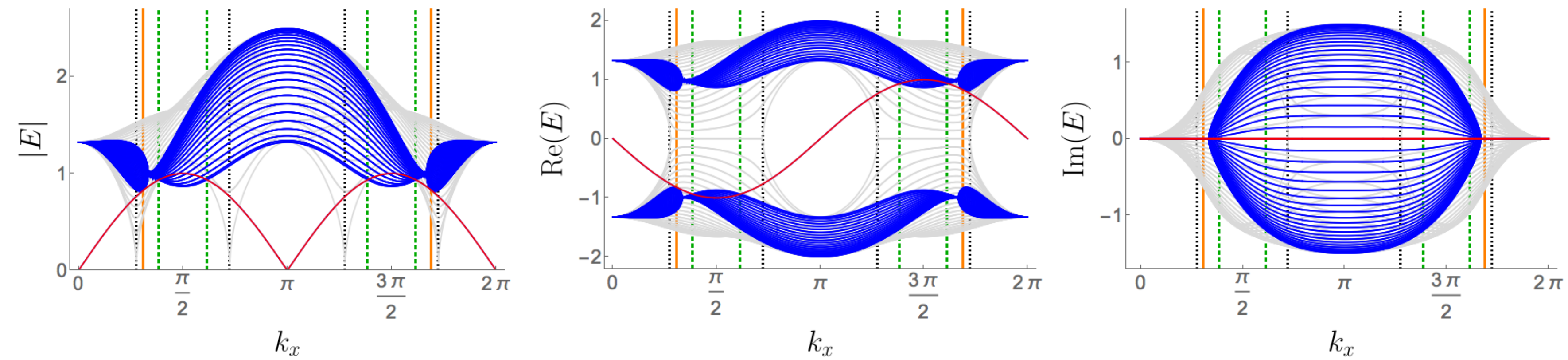
$t_1 = t + \delta\cos(k)$

$t_2 = t - \delta\cos(k)$

$d_z(k) = -\Delta\sin(k)$

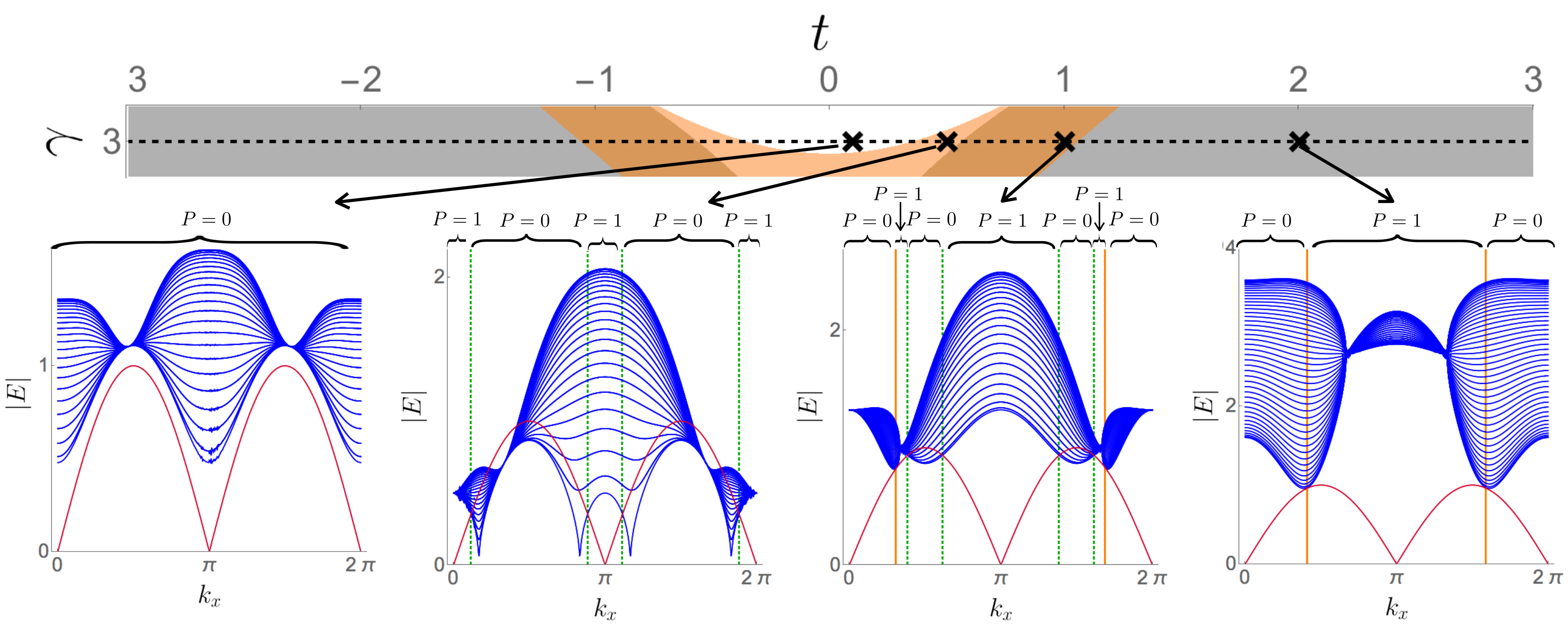


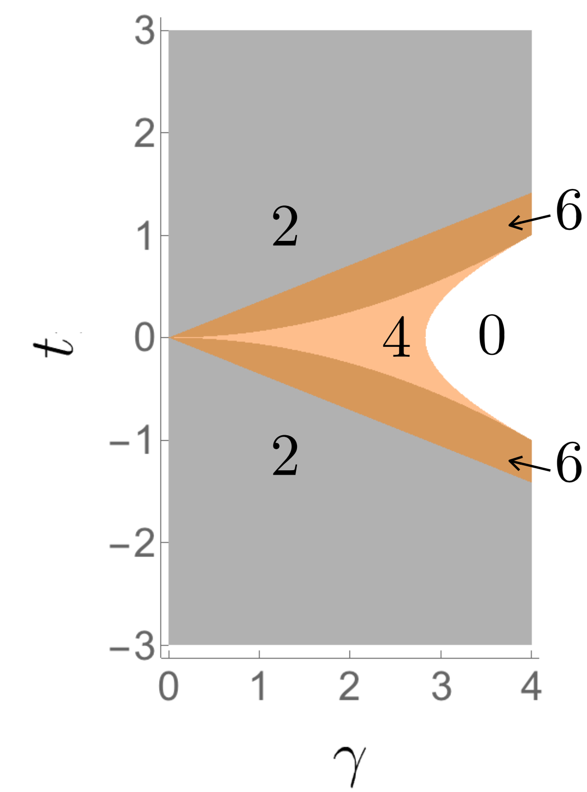
$r_R = -\frac{t_1 - \gamma}{t_2} \qquad r_L = -\frac{t_1 + \gamma}{t_2}$



From $|r_L^* r_R| = 1$

$$\longrightarrow \quad \cos(k) = \frac{\gamma^2}{16t\delta}, \pm\sqrt{\frac{\gamma^2/8 - t^2}{\delta^2}}$$



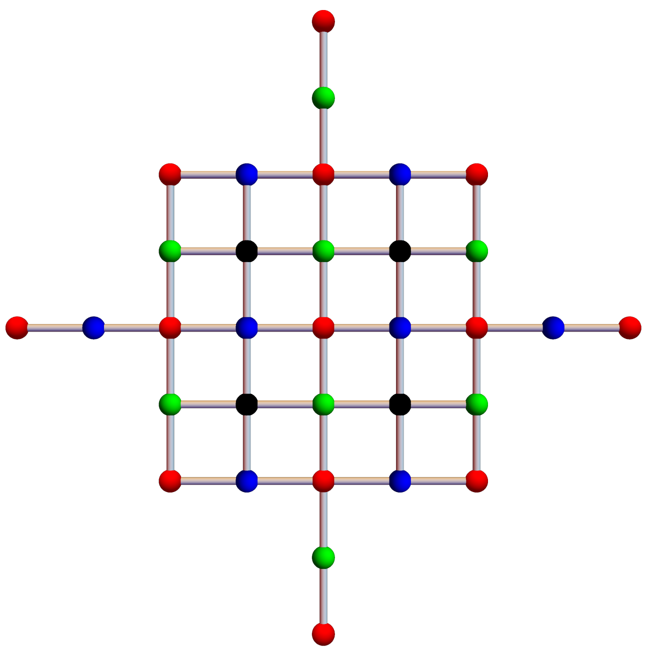


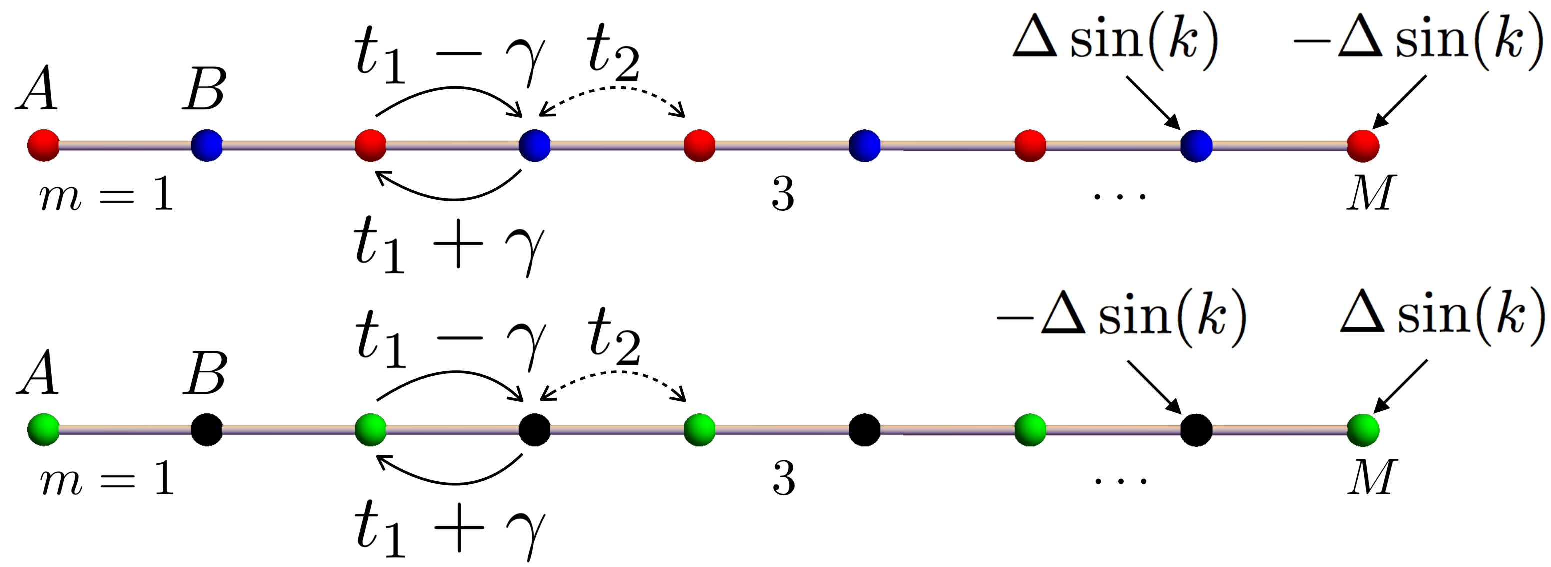
Kunst, Edvardsson, Budich, and Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).

Further generalization

Consider a mirror symmetric lattice

$$t_1 = t + \delta\cos(k)$$
$$t_2 = t - \delta\cos(k)$$

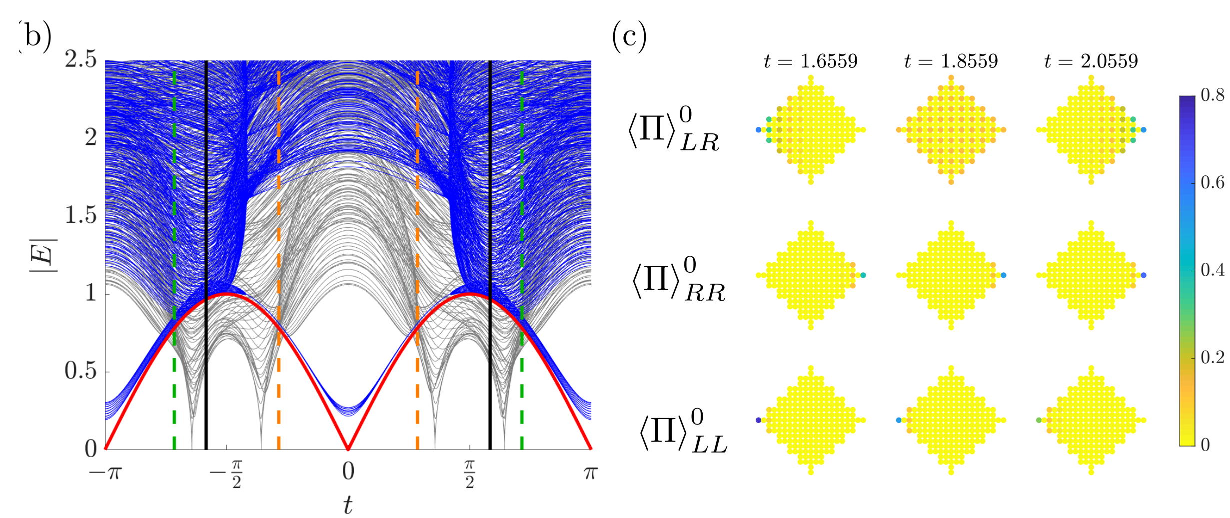




$$|\Psi_R(k)\rangle = \mathcal{N}_R \sum_{m,m'} [r_R(k)]^m (-1)^{m'} c^\dagger_{A,m,m'}(k)\,|0\rangle$$

$$|\Psi_L(k)\rangle = \mathcal{N}_L \sum_{m,m'} [r_L(k)]^m (-1)^{m'} c^\dagger_{A,m,m'}(k)\,|0\rangle$$

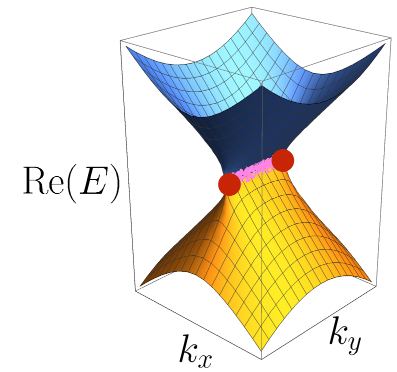
$$r_R = -\frac{t_1 - \gamma}{t_2} \qquad r_L = -\frac{t_1 + \gamma}{t_2}$$

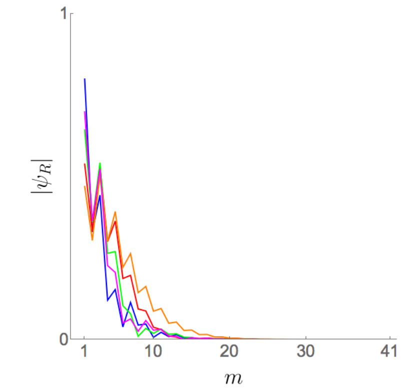


Kunst, van Miert, and Bergholtz, Phys. Rev. B **97**, 241405(R) (2018).
Kunst, van Miert, and Bergholtz, Phys. Rev. B **99**, 085426 (2019).
Edvardsson, **Kunst**, and Bergholtz, arXiv:1812.09060.

Conclusion

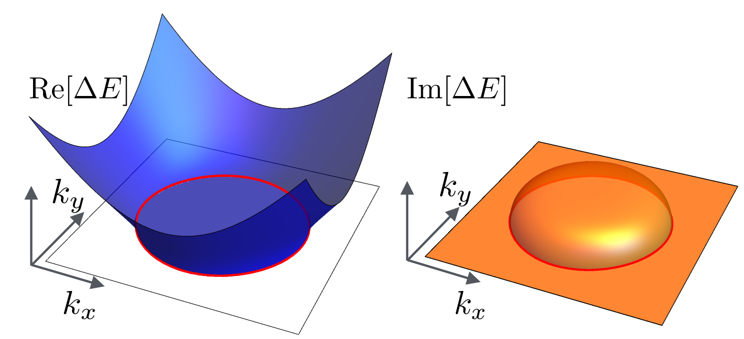
Non-Hermitian Hamiltonians possess many unexpected features, e.g., exceptional points, skin effect, …





Gap closings are generic in 1D in the presence of symmetries

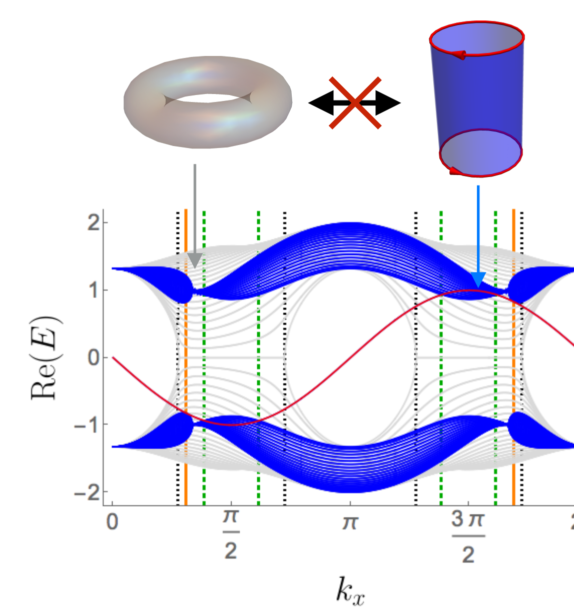
Jan Carl Budich, Johan Carlström, **Flore K. Kunst**, and Emil J. Bergholtz, Phys. Rev. B **99**, 041406(R) (2019). *Editors' Suggestion*



Biorthogonal bulk-boundary correspondence directly in systems with open boundaries

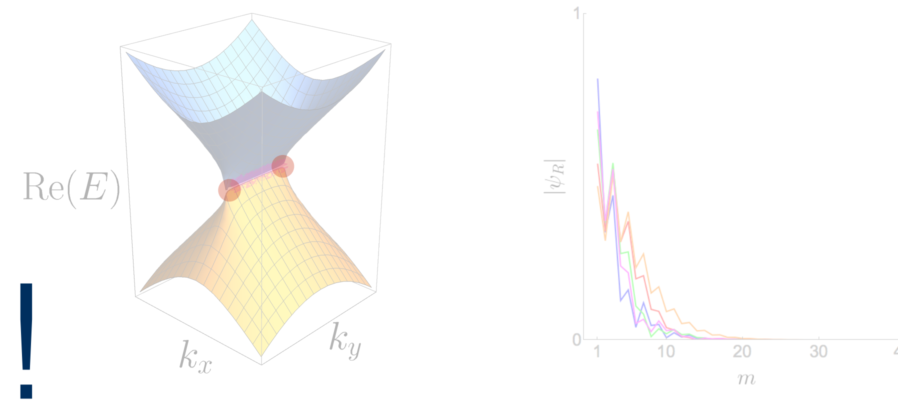
Flore K. Kunst, Elisabet Edvardsson, Jan Carl Budich and Emil J. Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018). *Editors' Suggestion*

Edvardsson, **Kunst**, and Bergholtz, arXiv:1812.09060. *Phys. Rev. B Rapid, in press.*



Conclusion

Non-Hermitian Hamiltonians possess many unexpected features, e.g., exceptional points, skin effect, …



Thank you!

Gap closings are generic in 1D in the presence of symmetries

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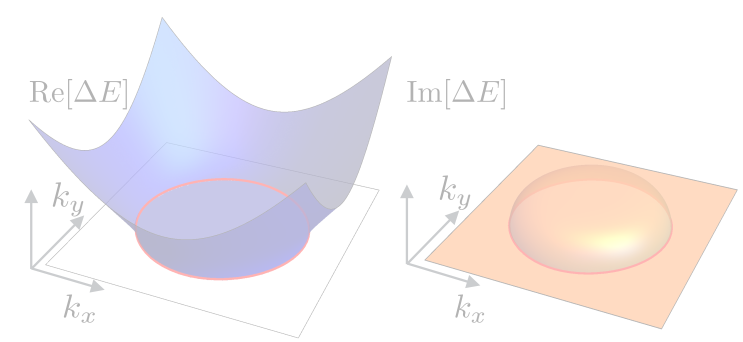

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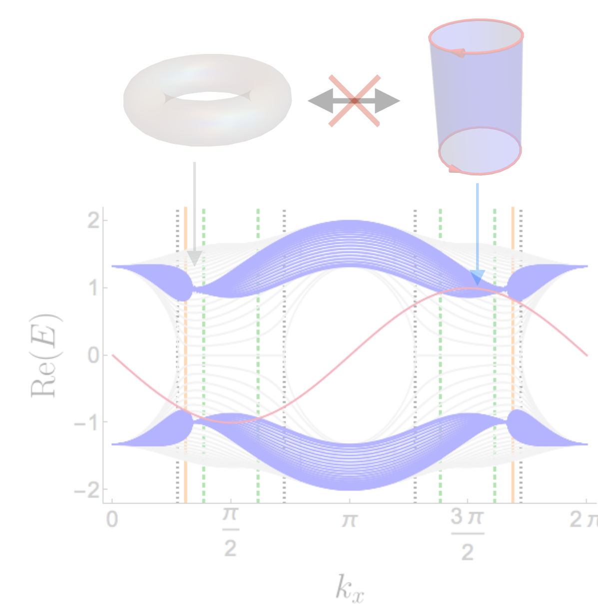

Extra slides

Bernard-LeClair Classification

Bernard and LeClair, arXiv:cond-mat/0110649 (2001).

Four symmetry relations

$P:$	$H = -pHp^{-1}, \quad p^2 = \mathbb{I}$	chiral symmetry
$C:$	$H = \epsilon_c c H^T c^{-1}, \quad c^T c^{-1} = \pm\mathbb{I}$	time-reversal symmetry → $\epsilon_i = \pm$
$K:$	$H = \epsilon_k k H^* k^{-1}, \quad k k^* = \pm\mathbb{I}$	particle-hole symmetry → $\epsilon_i = \pm$
$Q:$	$H = \epsilon_q q H^\dagger q^{-1}, \quad q^\dagger q^{-1} = \mathbb{I}$	pseudo Hermiticity

where $p,\ c,\ k,\ q$ are unitary transformations, and $\epsilon_i = \pm$