Tensor Network approaches to Many-Body Localization

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Benasque, 27 February 2019

European Commission

Marie Skłodowska-Curie Actions

Thorsten B. Wahl, Arijeet Pal, and Steven H. Simon, Phys. Rev. X **7**, 021018 (2017)
Thorsten B. Wahl, Arijeet Pal, and Steven H. Simon, Nat. Phys. **15**, 164 (2019).
Thorsten B. Wahl, Phys. Rev. B **98**, 054204 (2018).
Amos Chan, and Thorsten B. Wahl, arXiv:1808.05656.

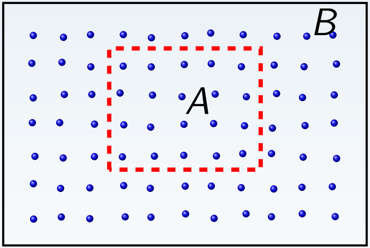
Thermalization in classical systems

$\longrightarrow$

Ergodicity in quantum systems

$$|\psi(t)\rangle = e^{iHt}|\psi(0)\rangle$$
$$H = H_A + H_B + H_{AB}$$

$$\rho_A \propto e^{-\beta H_A}$$



Eigenstate Thermalization Hypothesis (ETH)

J. M. Deutsch, Phys. Rev. A. **43**, 2046 (1991)
M. Srednicki, Phys. Rev. E **50**, 888 (1994)

Many-body localization in one dimension

Sufficiently strong disorder in 1D $\Rightarrow$ ergodicity breaking:

Many-body localization (MBL)

D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. **321**, 1126 (2006).

I. Gornyi, A. Mirlin, and D. Polyakov, Phys. Rev. Lett. **95**, 206603 (2005).

Rigorous proof: J. Z. Imbrie, J. Stat. Phys. **163**, 998 (2016)

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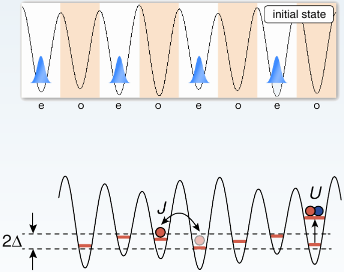
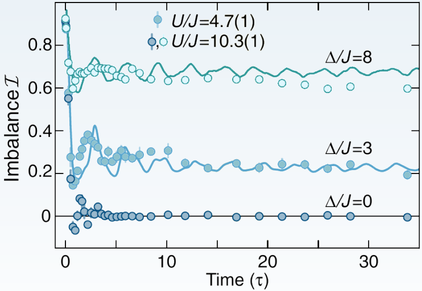




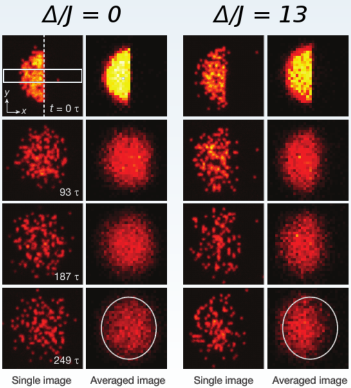
taken from: M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science **349**, 842 (2015)

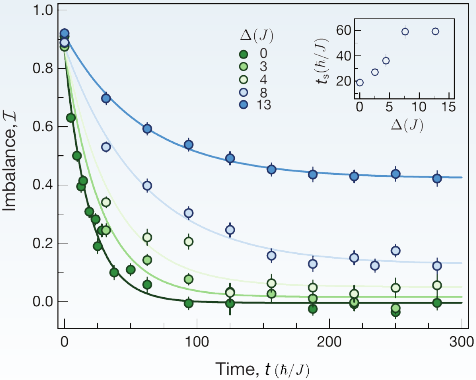
Many-body localization in higher dimensions?

Theralizing behavior in higher dimensions

W. De Roeck, J. Z. Imbrie, Phil. Trans. R. Soc. A 375, 20160422 (2017).

But:





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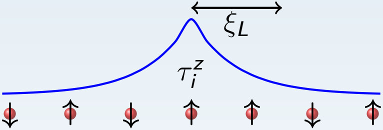
Many-body localization (MBL)

Disordered Heisenberg antiferromagnet: MBL for $W > W_c \approx 3.5$

$$H = \sum_{i=1}^{N} (J\,\mathbf{S}_i \cdot \mathbf{S}_{i+1} + h_i S_i^z), \qquad h_i \in [-W, W]$$



Local integrals of motion (LIOM):

$$H = U H_{\mathrm{diag}} U^\dagger$$

$$\tau_i^z = U \sigma_i^z U^\dagger$$

$$[H, \tau_i^z] = [\tau_i^z, \tau_j^z] = 0$$

M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013)

D. A. Huse, and V. Oganesyan, Phys. Rev. B **90**, 174202 (2014)

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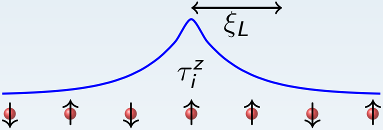

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$$H|\psi_{i_1 \ldots i_N}\rangle = E_{i_1 \ldots i_N}|\psi_{i_1 \ldots i_N}\rangle$$

$$\tau_1^z|\psi_{\uparrow i_2 \ldots i_N}\rangle = |\psi_{\uparrow i_2 \ldots i_N}\rangle$$
$$\tau_1^z|\psi_{\downarrow i_2 \ldots i_N}\rangle = -|\psi_{\downarrow i_2 \ldots i_N}\rangle \quad \text{etc.}$$

M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013)

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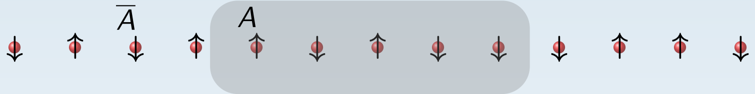
Area-law entangled eigenstates

$|\psi_{i_1 \ldots i_N}\rangle$:



$\rho_A = \mathrm{tr}_{\overline{A}}(|\psi_{i_1 \ldots i_N}\rangle\langle\psi_{i_1 \ldots i_N}|)$, entanglement entropy $S(\rho_A) \leq \mathrm{const.}$

M. Friesdorf, A. H. Werner, W. Brown, V. B. Scholz, and J. Eisert, Phys. Rev. Lett. **114**, 170505 (2015).

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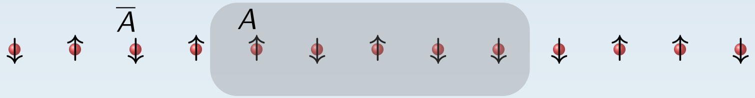

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Approximation by Tensor Network States

- DMRG-X

  V. Khemani, F. Pollmann, and S. L. Sondhi, Phys. Rev. Lett. **116**, 247204 (2016)

- spectral tensor networks

  F. Pollmann, V. Khemani, J. I. Cirac, and S. L. Sondhi, Phys. Rev. B **94**, 041116(R) (2016)

  T. B. Wahl, A. Pal, and S. H. Simon, Phys. Rev. X **7**, 021018 (2017)

Area-law entangled eigenstates

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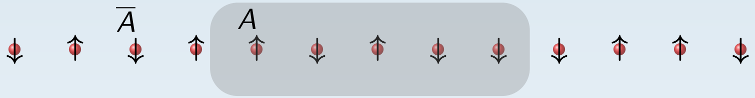

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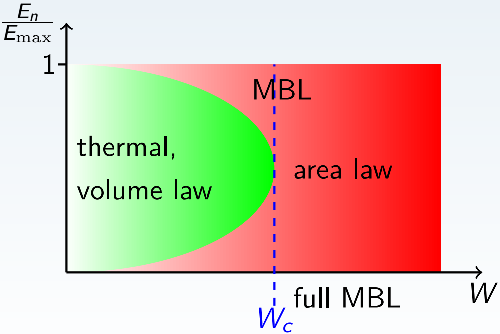

D. J. Luitz, N. Laflorencie, and F. Alet, Phys. Rev. B **91**, 081103 (2015)
However: W. De Roeck, F. Huveneers, M. Müller, and M. Schiulaz, Phys. Rev. B **93**, 014203 (2016)

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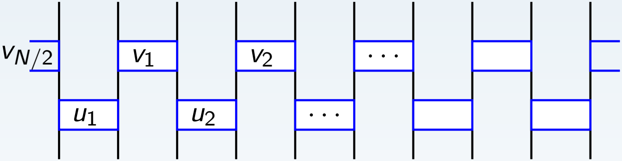
Spectral Tensor Networks

Goal

$\tilde{U} H \tilde{U}^\dagger \approx$ diagonal matrix

$\tilde{U} =$ 

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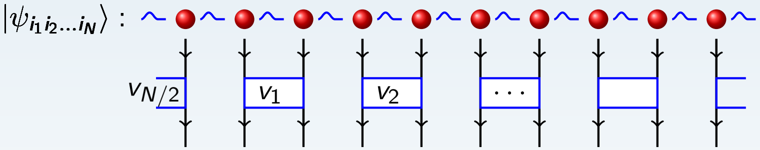

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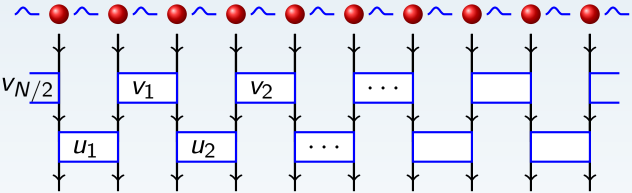
$|\psi_{i_1 i_2 \ldots i_N}\rangle$:



$$\approx |\uparrow \downarrow \uparrow \uparrow \downarrow \downarrow \downarrow \uparrow \downarrow \uparrow\rangle$$

Spectral Tensor Networks

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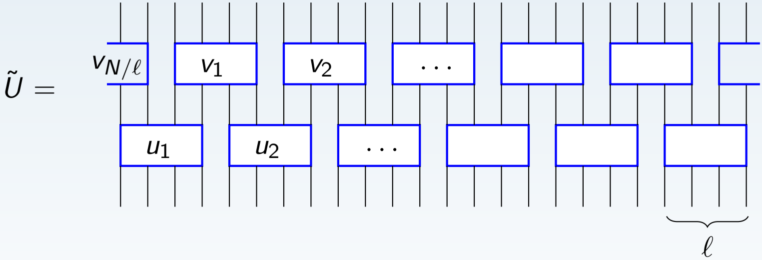
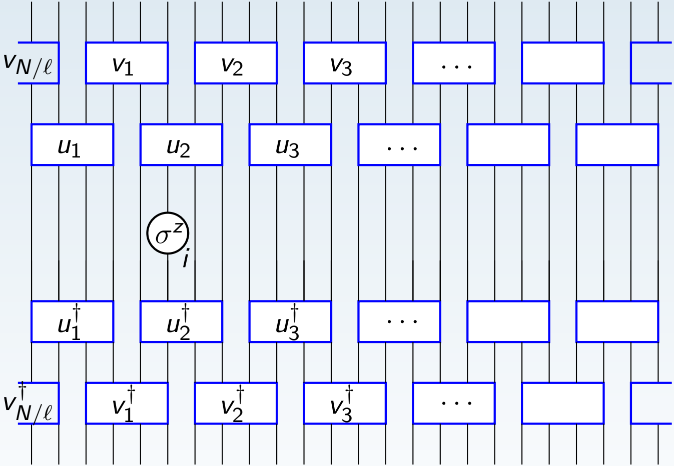
$\tilde{U} H \tilde{U}^\dagger \approx$ diagonal matrix



$$\tilde{\tau}_i^z = \tilde{U} \sigma_i^z \tilde{U}^\dagger \quad \Rightarrow \quad [\tilde{\tau}_i^z, \tilde{\tau}_j^z] = 0$$

Approximate local integrals of motion

$\tilde{\tau}_i^z = \tilde{U}\sigma_i^z\tilde{U}^\dagger =$



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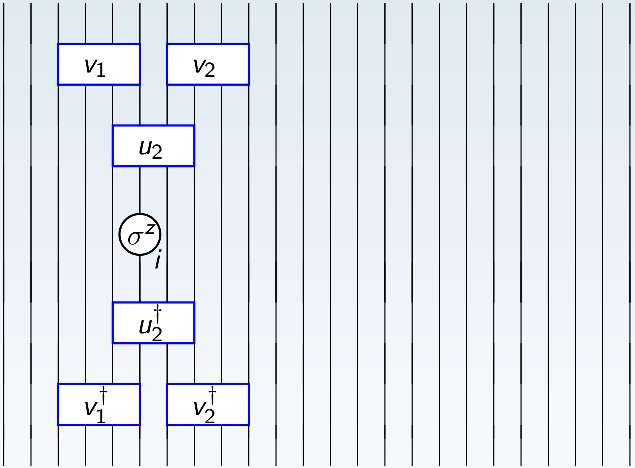

$$\tilde{U}^\dagger H \tilde{U} = \text{diagonal} + \mathcal{O}(e^{-\frac{\ell}{\xi_L}})$$

Figure of merit

Minimize $[H, \tilde{\tau}_i^z]$:

$$f = \frac{1}{2^N} \sum_{i=1}^{N} \mathrm{tr}\left([H, \tilde{\tau}_i^z] [H, \tilde{\tau}_i^z]^\dagger \right) = \sum_i f_i$$

Figure of merit

Minimize $[H, \tilde{\tau}_i^z]$:

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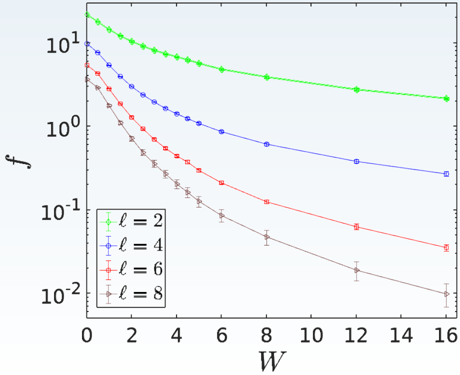
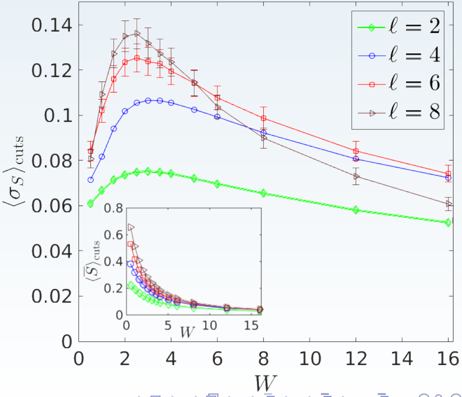
Heisenberg model, $N = 72$:

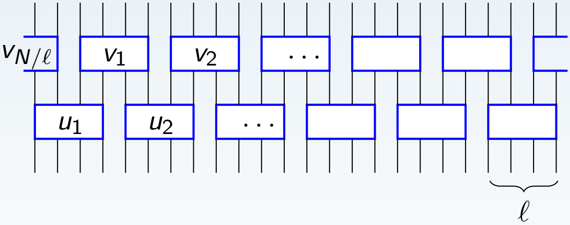




Summary (one dimension)

Full MBL regime

- local integrals of motion: τ_i^z
- all eigenstates fulfill the area law $\rightarrow$ spectral tensor networks: error $\propto exp\left(-\frac{\ell}{\xi_L}\right)$

$\tilde{U} =$ 

Numerical analysis

- minimize $[H, \tilde{\tau}_i^z]$
- phase transition: maximum of σ_S

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Delocalization in two dimensions

For any set of local τ_i^z: $[H, \tau_i^z] \neq 0$ for some i

W. De Roeck, and J. Z. Imbrie, Phil. Trans. R. Soc. A **375**, 20160422 (2017)

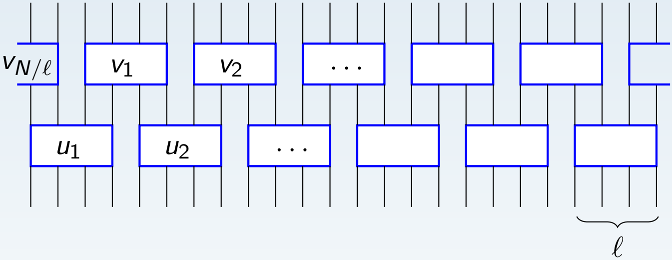
However: $\|[H, \tau_i^z]\|_{\mathrm{op}} \lll 1$

Relaxation time: $\tau \geq \frac{1}{\max_i \|[H,\tau_i^z]\|_{\mathrm{op}}} \ggg 1$

A. Chandran, A. Pal, C.R. Laumann, and A. Scardicchio, Phys. Rev. B **94**, 144203 (2016)

In two dimensions

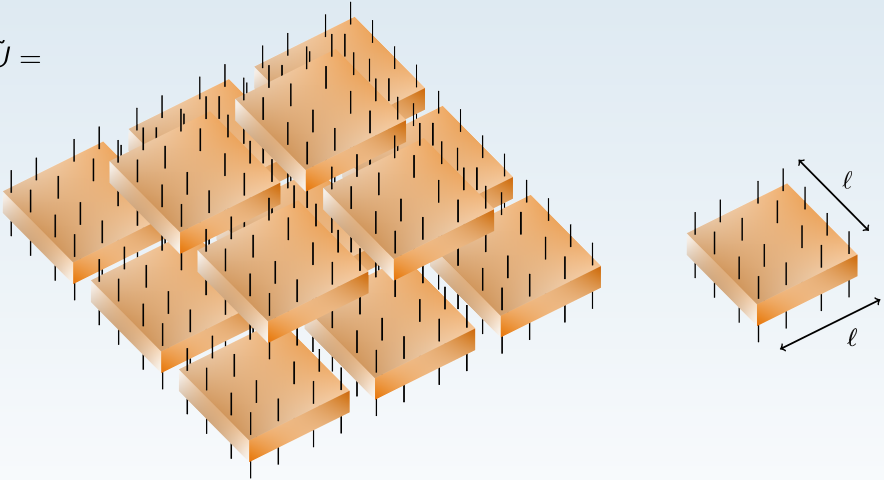
$\tilde{U} =$



becomes ...

In two dimensions

$\tilde{U} =$



$N \times N$ lattice:

$$f = \frac{1}{2^{N^2}} \sum_{i=1}^{N^2} \mathrm{tr}\left([H, \tilde{\tau}_i^z] \, [H, \tilde{\tau}_i^z]^\dagger \right) = \sum_i f_i$$

Bose-Hubbard model in 2D

$$H = -\sum_{\langle i,j \rangle} \left(a_i^\dagger a_j + a_i a_j^\dagger \right) + \frac{U'}{2} \sum_i n_i(n_i - 1) + \delta_i n_i$$

- $U' = 24.4$, δ_i from Gaussian distribution (half maximum width Δ)
- $n_{\mathrm{max}} = 1$, and $n_{\mathrm{max}} = 2$

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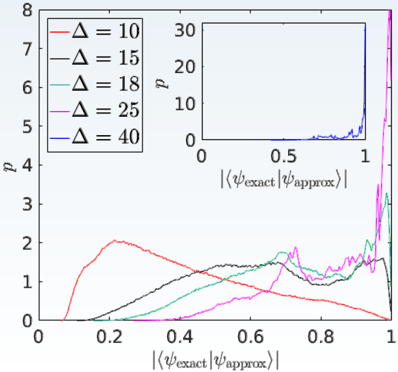
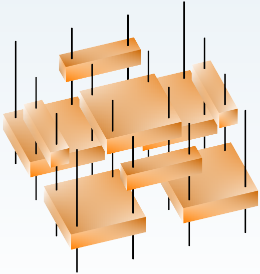
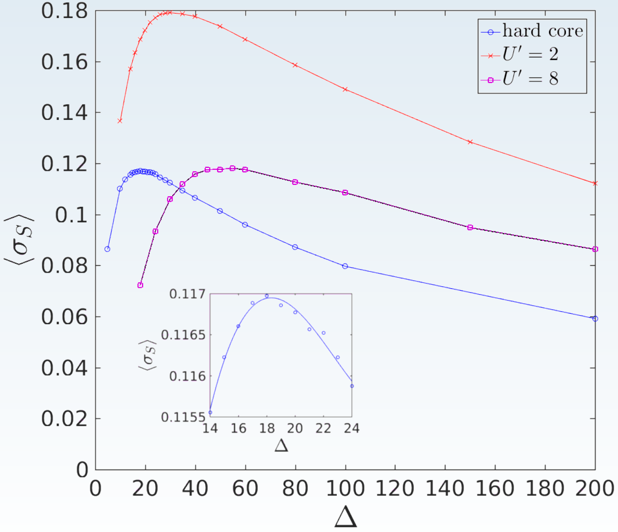
$n_{\max} = 1$,
4×4 lattice:





Evidence for MBL in two dimensions

10×10 lattice, $\ell \times \ell = 2 \times 2$



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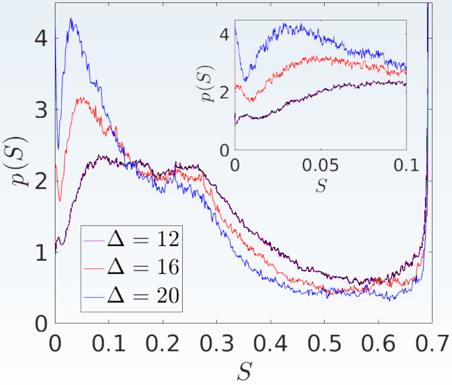

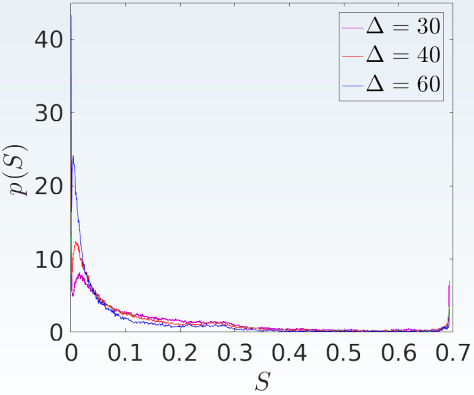

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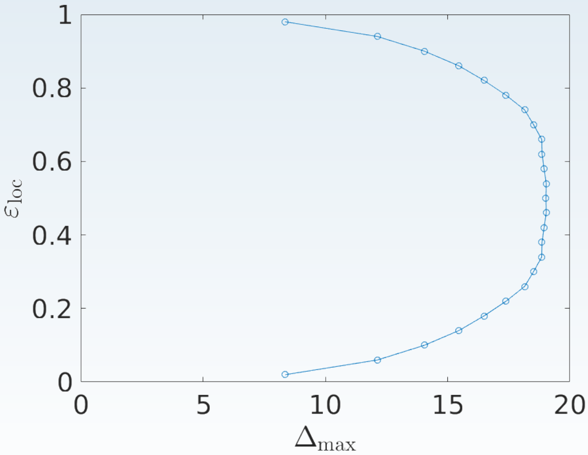
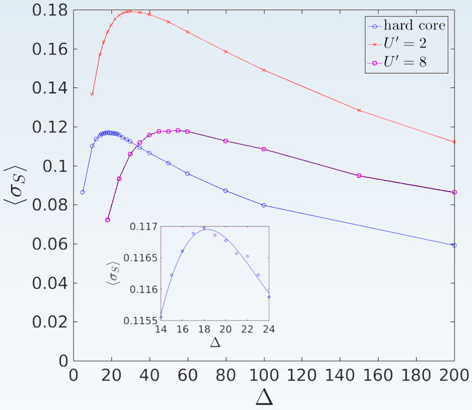


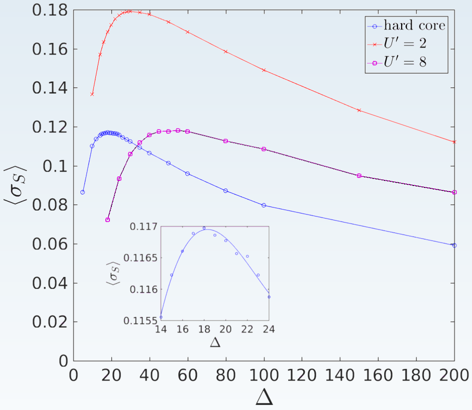
More recently: D. M. Kennes, arXiv:1811.04126.
H. Théveniaut, Z. Lan, and F. Alet, arXiv:1902.04091.

Comparison to the experiment



- $n_{\max} = 1$: $\Delta_c = 18.3$
- $n_{\max} = 2$:
  $\Delta_c(U' = 2) \approx 30$
  $\Delta_c(U' = 8) \approx 50$

Comparison to the experiment



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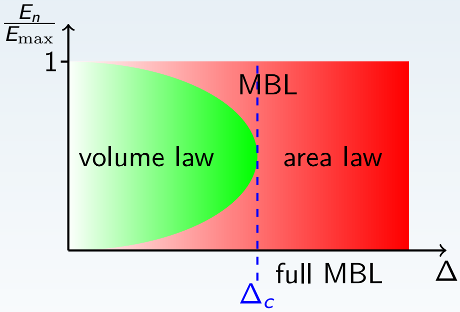
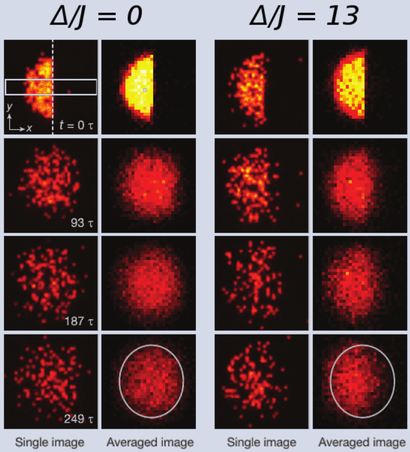


Experimentally: only $\sim 7\%$ doublons

J.-y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science **352**, 1547 (2016).

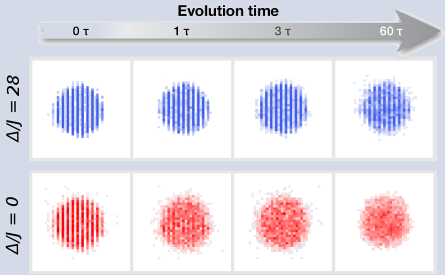
Experimental transition points

half-moon initialization: $\Delta_c \approx 5.3$



taken from: J.-y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science **352**, 1547 (2016).

charge-density wave: $\Delta_c \gg 5.3$?



taken from: A. Rubio-Abadal, J.-y. Choi, J. Zeiher, S. Hollerith, J. Rui, I. Bloch, and C. Gross, arXiv:1805.00056.

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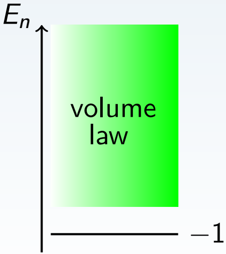
Topological Many-body Localized Phases

Cluster model:

Clean system

$$H = \sum_{j=1}^{N} \sigma_x^{j-1} \sigma_z^{j} \sigma_x^{j+1}$$

topological index: $ww^* = \pm\mathbb{1}$

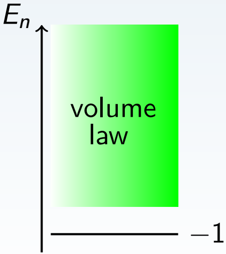


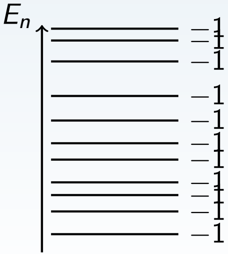
Y. Bahri, R. Vosk, E. Altman and A. Vishwanath, Nat. Commun. 6, 7341 (2015)

Topological Many-body Localized Phases

Cluster model:

Clean system	Disordered system
$$H=\sum_{j=1}^{N}\sigma_x^{j-1}\sigma_z^{j}\sigma_x^{j+1}$$	$$H=\sum_{j=1}^{N}\lambda_j\sigma_x^{j-1}\sigma_z^{j}\sigma_x^{j+1}$$
topological index: $ww^*=\pm\mathbb{1}$	topological index: $ww^*=\pm\mathbb{1}$





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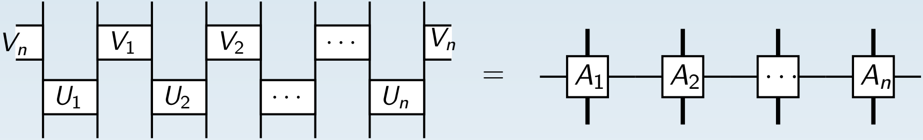
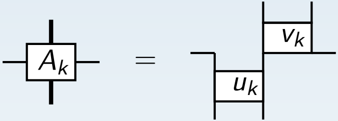
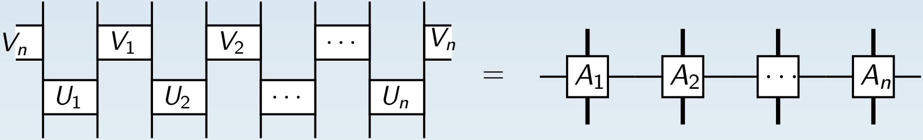
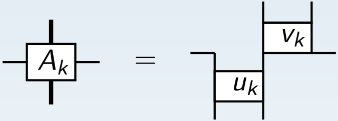
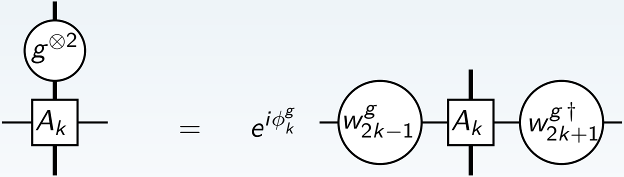

V_n
V_1
V_2
$\cdots$
V_n
U_1
U_2
$\cdots$
U_n
$=$
A_1
A_2
$\cdots$
A_n


A_k
$=$
v_k
u_k





on-site symmetry with symmetry group $G \ni g$:



- same topological index $\in \mathcal{H}^2(G, U(1))$ for all eigenstates
- topological stability to small local perturbations

Symmetry-protected MBL phases in one dimension

	Ground states (clean)	All eigenstates (MBL)
spin, TRS	$\mathbb{Z}_2$	$\mathbb{Z}_2$
spin, G	$\mathcal{H}^2(G, U(1))$	$\mathcal{H}^2(G, U(1))$
fermionic, TRS	$\mathbb{Z}_8$	$\mathbb{Z}_4$
fermionic, G	$\mathcal{H}^2(G \times \mathbb{Z}_2, U(1)) \times \mathbb{Z}_2$	$\mathcal{H}^2(G \times \mathbb{Z}_2, U(1))$

Summary and Outlook

Summary:

- experimentally observed MBL in 2D is "short"-time phenomenon
- experiment: $\Delta_c = 5.3$, theory: $\Delta_c = 18.3$
- analytical tool to classify topological MBL phases

Thorsten B. Wahl, Arijeet Pal, and Steven H. Simon, Phys. Rev. X **7**, 021018 (2017)
Thorsten B. Wahl, Arijeet Pal, and Steven H. Simon, Nat. Phys. **15**, 164 (2019).
Thorsten B. Wahl, Phys. Rev. B **98**, 054204 (2018).
Amos Chan, and Thorsten B. Wahl, arXiv:1808.05656.

Summary and Outlook

Summary:

- experimentally observed MBL in 2D is "short"-time phenomenon
- experiment: $\Delta_c = 5.3$, theory: $\Delta_c = 18.3$
- analytical tool to classify topological MBL phases

Outlook:

- experiment: charge density wave / larger filling
- theory: $\ell \times \ell = 3 \times 3$ simulations
- 2D topological MBL phases

Thorsten B. Wahl, Arijeet Pal, and Steven H. Simon, Phys. Rev. X **7**, 021018 (2017)
Thorsten B. Wahl, Arijeet Pal, and Steven H. Simon, Nat. Phys. **15**, 164 (2019).
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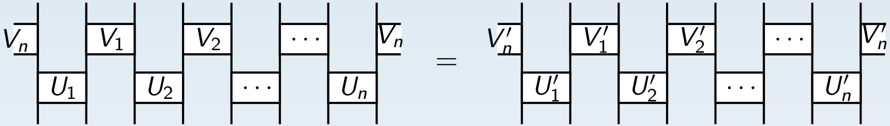
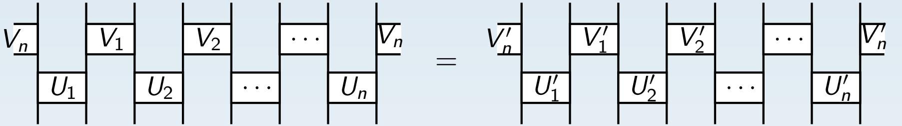
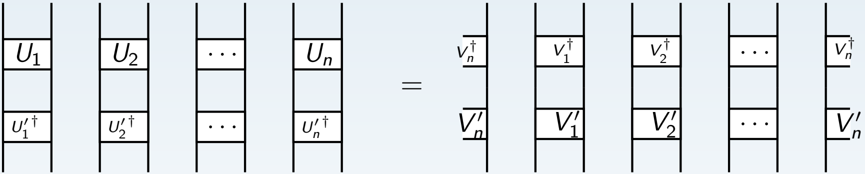
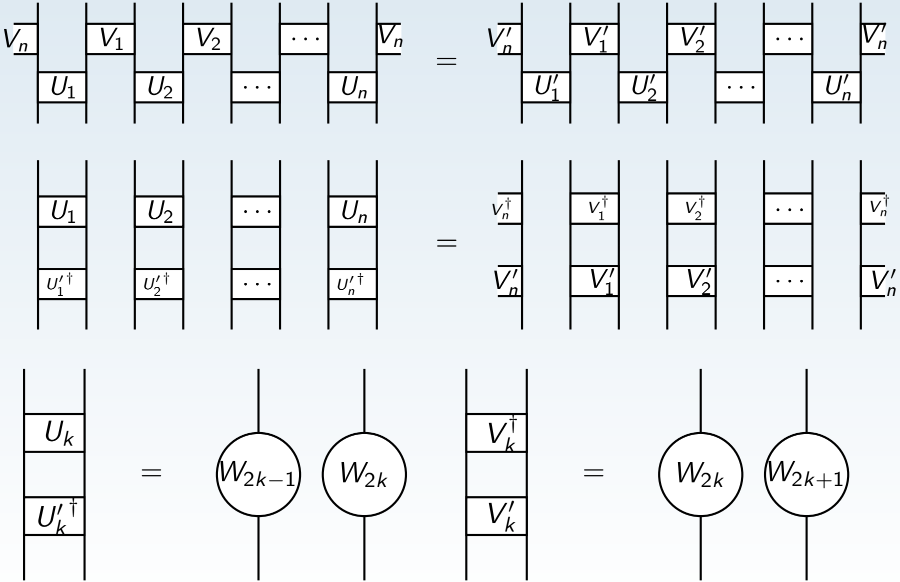

V_n
V_1
V_2
$\cdots$
V_n
U_1
U_2
$\cdots$
U_n
$=$
V'_n
V'_1
V'_2
$\cdots$
V'_n
U'_1
U'_2
$\cdots$
U'_n

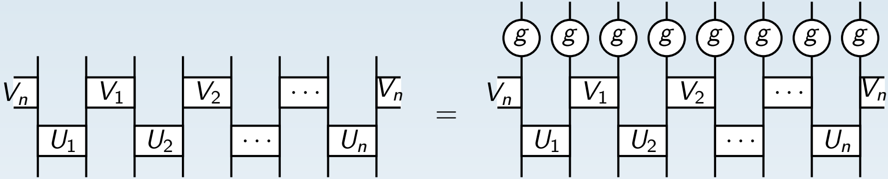


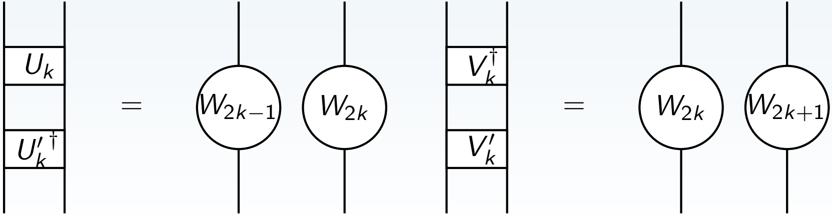
V_n
V_1
V_2
$\cdots$
V_n
U_1
U_2
$\cdots$
U_n
$=$
V'_n
V'_1
V'_2
$\cdots$
V'_n
U'_1
U'_2
$\cdots$
U'_n



U_1
U_2
$\cdots$
U_n
$U_1'^\dagger$
$U_2'^\dagger$
$\cdots$
$U_n'^\dagger$
$=$
$V_n^\dagger$
$V_1^\dagger$
$V_2^\dagger$
$\cdots$
$V_n^\dagger$
V'_n
V'_1
V'_2
$\cdots$
V'_n

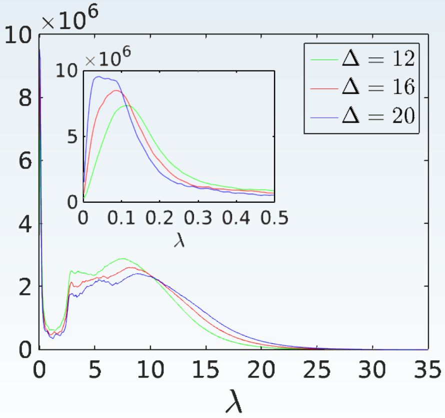

V_n V_1 V_2 $\cdots$ V_n
U_1 U_2 $\cdots$ U_n
=
V_n' V_1' V_2' $\cdots$ V_n'
U_1' U_2' $\cdots$ U_n'
U_1 U_2 $\cdots$ U_n
$U_1'^\dagger$ $U_2'^\dagger$ $\cdots$ $U_n'^\dagger$
=
$V_n^\dagger$ $V_1^\dagger$ $V_2^\dagger$ $\cdots$ $V_n^\dagger$
V_n' V_1' V_2' $\cdots$ V_n'
U_k
$U_k'^\dagger$
=
W_{2k-1} W_{2k}
$V_k^\dagger$
V_k'
=
W_{2k} W_{2k+1}

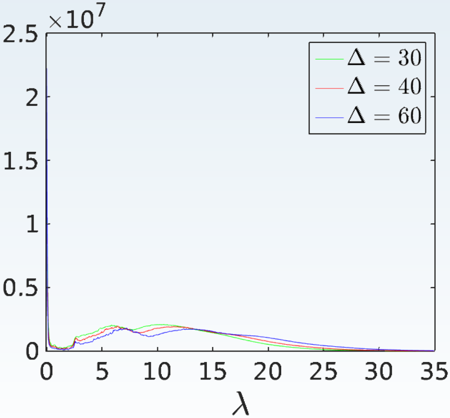

g g g g g g g g
V_n V_1 V_2 ⋯ V_n
U_1 U_2 ⋯ U_n
=
V_n V_1 V_2 ⋯ V_n
U_1 U_2 ⋯ U_n


U_k
$U_k'^{\dagger}$
=
W_{2k-1}
W_{2k}
$V_k^{\dagger}$
V_k'
=
W_{2k}
W_{2k+1}

Entanglement energies

2×2 reduced density matrix: $\rho_A = e^{-H_{\text{ent}}}$





Overlaps in 1D for $N = 16$

$\ell = 2$ $\ell = 4$ $\ell = 8$