QCD
Taller de Altas Energías - TAE 2019

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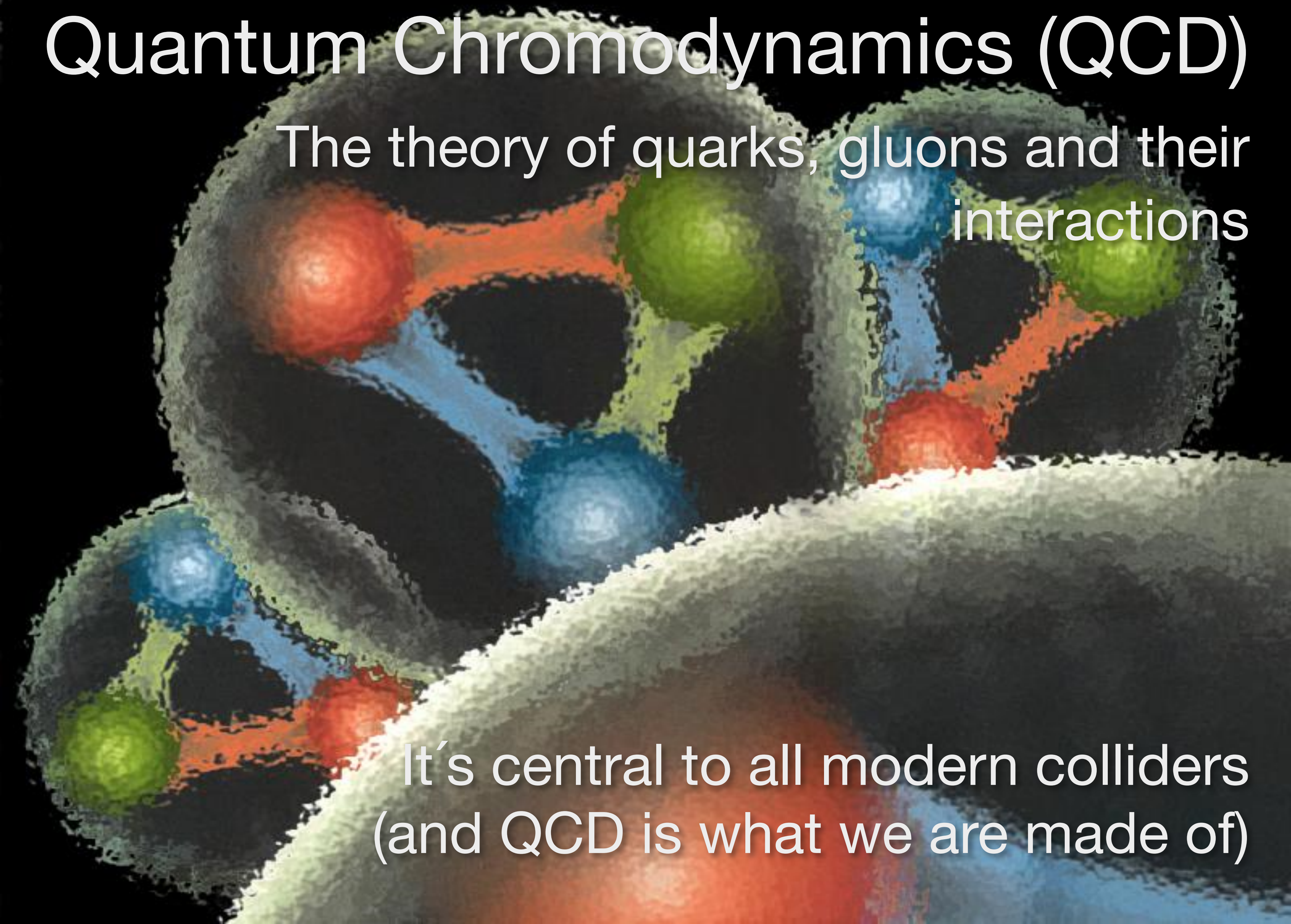
Lecture 1: basic concepts

Centro de Ciencias de Benasque Pedro Pascual

particleface · cost European Cooperation in Science & Technology

Quantum Chromodynamics (QCD)

The theory of quarks, gluons and their interactions



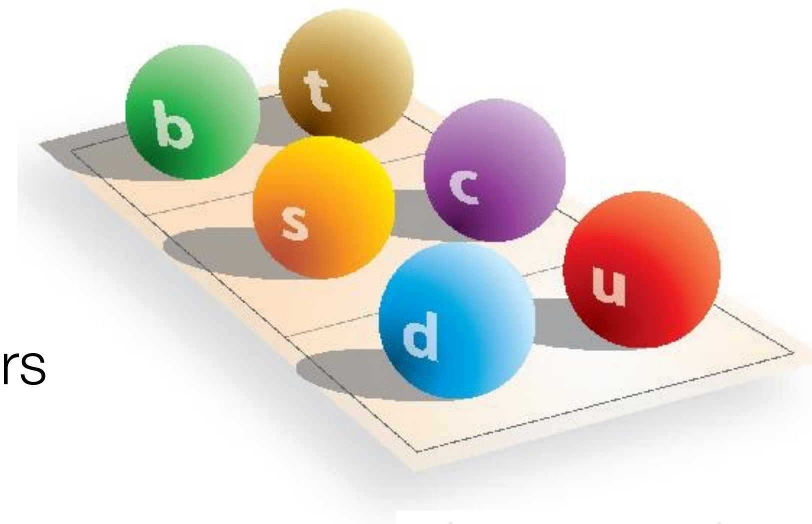
It´s central to all modern colliders
(and QCD is what we are made of)

Outline

1. Basic concepts: QCD Lagrangean, the strong coupling and IR divergences in e+e-
2. New methods in pQCD: helicity formalism, generalized unitarity, and the collinear limit of QCD
3. pQCD at hadron colliders, parton densities and jets

The ingredients of QCD

QCD is a gauge invariant QFT based on the local **SU(3)** symmetry group **(Yang–Mills theory)**



- **Quarks** (and antiquarks): six flavours
  - they come in 3 colours
- **Gluons**: massless gauge bosons
  - a bit like photons in QED
  - but there are 8 of them, and they are colour charged

- and the **strong coupling** $\alpha_S(\mu) = g_S^2(\mu)/(4\pi)$
  - that is not so small (strongest force)
  - and run fast: at the LHC, in the range 0.08 @ 5 TeV to O(1) at 0.5 GeV

QUANTUM THEORY OF GRAVITATION*

By R. P. Feynman

(Received July 3, 1963)

There is another theory, more well-known to meson physicists, called the Yang-Mills theory, and I take the one with zero mass; it is a special theory that has never been investigated in great detail. It is very analogous to gravitation; instead of the coordinate transformation group being the source of everything, it's the isotopic spin rotation group that's the source of everything. It is a non-linear theory, that's like the gravitation theory, and so forth. At the suggestion of Gell-Mann I looked at the theory of Yang-Mills with zero mass, which has a kind of gauge group and everything the same; and found exactly the same difficulty. And therefore in meson theory it was not strictly unknown difficulty, because it should have been noticed by meson physicists who had been fooling around the Yang-Mills theory. They had not noticed it because they're practical, and the Yang-Mills theory with zero mass obviously does not exist, because a zero mass field would be obvious; it would come out of nuclei right away. So they didn't take the case of zero mass and investigate it carefully.

quark Lagrangean + colour

The quark part of the Lagrangean

$$\mathcal{L}_q = \bar{\psi}_i \left(\delta_{ij} (\imath \not{\partial} - m) + g_S\, \mathbf{t}^a_{ij} \not{A}^a \right) \psi_j \,, \qquad \not{\partial} = \gamma^\mu \partial_\mu \qquad \not{A}^a = \gamma^\mu A^a_\mu$$

where quarks carry three colors $\psi_i = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$

- SU(3) local gauge symmetry: $8(=3^2-1)$ generators $\mathbf{t}^1_{ij}, \ldots, \mathbf{t}^8_{ij}$ with $\mathrm{Tr}(\mathbf{t}^a \cdot \mathbf{t}^b) = T_R\, \delta^{ab}$ | $T_R = 1/2$ corresponding to the 8 gluons $A^1_\mu, \ldots, A^8_\mu$
- In the fundamental representation: $\mathbf{t}^a = \lambda^a/2$ Gell-Mann matrices, traceless and Hermitian

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

Gluon Lagrangean

The gluonic part of the Lagrangean

$$\mathscr{L}_g = -\frac{1}{4} F^a_{\mu_1\mu_2} F^{\mu_1\mu_2 a} + \mathscr{L}_{\text{gauge fixing}}$$

where the field tensor is

$$F^a_{\mu_1\mu_2} = \partial_{\mu_1} A^a_{\mu_2} - \partial_{\mu_2} A^a_{\mu_1} + \imath g_S \left(-\imath f^{aa_1a_2}\right) A^{a_1}_{\mu_1} A^{a_2}_{\mu_2}$$

$f_{aa_1a_2}$ are the **structure constants** of SU(3): antisymmetric in all indices.
Needed for gauge invariance of the Lagrangean

$$[\mathbf{t}^{a_1}, \mathbf{t}^{a_2}] = \imath f_{a_1a_2a} \mathbf{t}^a$$

Color algebra [Catani, Seymour]

- It is useful to introduce a basis in color+spin

$$\mathscr{M}_N^{c_1,\ldots,c_N;s_1,\ldots,s_N}(p_1,\ldots,p_N) = \left(\langle c_1,\ldots | \otimes \langle s_1,\ldots |\right) | 1,\ldots,N\rangle_N$$

- Then associate a color-charge to each emission $\mathbf{T}_i \equiv T_i^c \,|\, c\rangle$, e.g. emission of one gluon of color c from each parton $\boldsymbol{i}$

- color conservation $\sum \mathbf{T}_i \,|\, 1,\ldots,N\rangle_N = 0$

for a gluon: $T^a_{bc} \equiv -\imath f_{abc}$

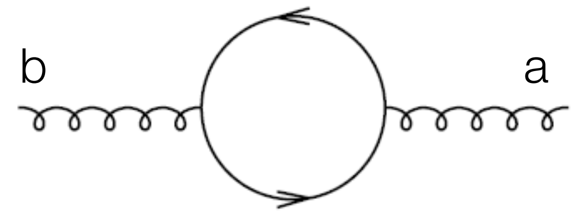
for a quark: $T^a_{\alpha\beta} = t^a_{\alpha\beta}$

for an anti-quark: $T^a_{\alpha\beta} = -\, t^a_{\beta\alpha}$

- The colour algebra is $\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i \;, \quad \mathbf{T}_i^2 = C_i$ Casimir operator

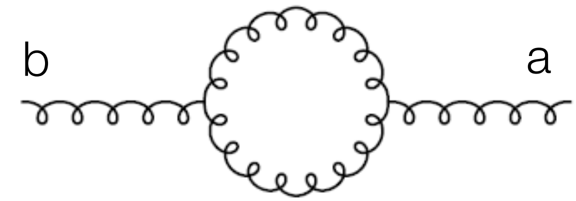
Normalisation, Casimir and Fierz

$$\mathrm{Tr}(\mathbf{t}^a \cdot \mathbf{t}^b) = T_R\, \delta^{ab} \qquad T_R = \frac{1}{2}$$

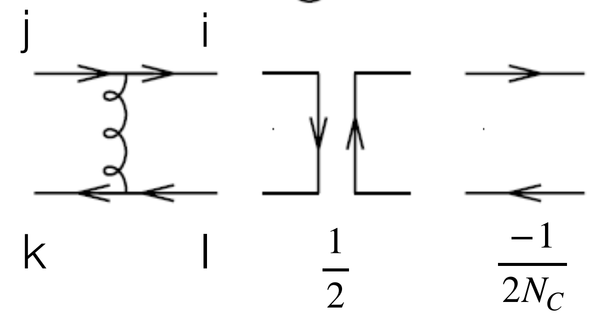


$$\mathbf{T}_q^2 = \sum_a t^a_{ik}\, t^a_{kj} = C_F\, \delta_{ij} \qquad C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3}$$

$$\mathbf{T}_g^2 = \sum_{c,d} f^{acd}\, f^{bcd} = C_A\, \delta^{ab} \qquad C_A = N_C = 3$$



$$\sum_a t^a_{ij}\, t^a_{kl} = \frac{1}{2}\left(\delta_{jk}\delta_{il} - \frac{1}{N_C}\delta_{ij}\delta_{kl}\right)$$



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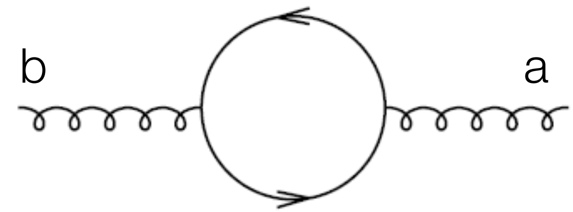

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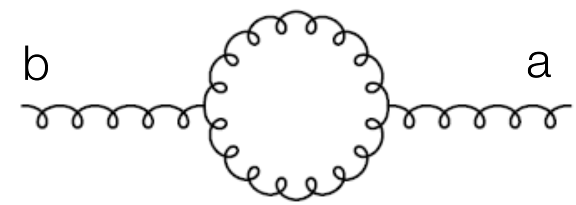

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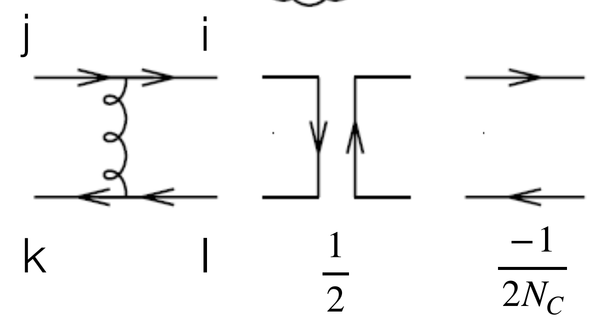
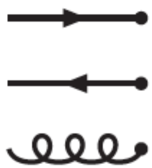
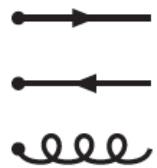


probability of a gluon to emit $q\bar{q}$ < quark to emit gluons < gluon to emit gluons

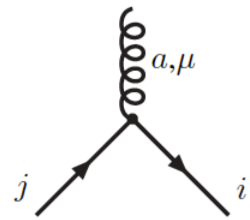
Feynman's rules

incoming quark $u(p)$   $\bar{u}(p)$ outgoing quark

incoming antiquark $\bar{v}(p)$ $v(p)$ outgoing antiquark

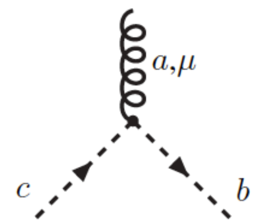
incoming gluon $\varepsilon^a_\mu(k)$ $\varepsilon^{a*}_\mu(k)$ outgoing gluon

$$\frac{\imath\,\delta_{ij}}{\not{p} - m + \imath 0}$$



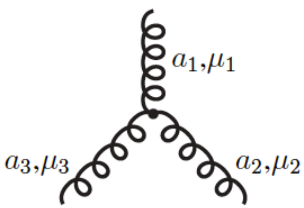
$$\imath\, g_S\, \mathbf{T}^a\, \gamma^\mu \quad | \quad \mathbf{T}^a = t^a_{ij}$$

$$\frac{\imath\,\delta_{ab}}{k^2 + \imath 0}$$



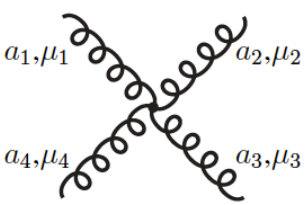
$$\imath\, g_S\, \mathbf{T}^a\, k^\mu \quad | \quad \mathbf{T}^a = -\,\imath f_{abc}$$

ghosts required in covariant gauges to preserve unitarity



$$\imath\, g_S\, \mathbf{T}^{a_1}_{a_2 a_3}\, g_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} + \text{cyclic perm.} \quad | \quad \mathbf{T}^{a_1}_{a_2 a_3} = -\,\imath f_{a_1 a_2 a_3}$$

outgoing momenta



$$\imath\, g_S^2\, \mathbf{T}^{a_1}_{a_2 a}\mathbf{T}^{a_3}_{a a_4}\left(g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}\right) + (234) \to (342) + (234) \to (423)$$

$$\frac{\imath\,\delta_{a_1 a_2}\, d_{\mu_1\mu_2}(k)}{k^2 + \imath 0}
\begin{cases}
d_{\mu_1\mu_2}(k,\xi) = -\,g_{\mu_1\mu_2} + (1-\xi)\dfrac{k^{\mu_1}k^{\mu_2}}{k^2 + \imath 0} \\[2ex]
d_{\mu_1\mu_2}(k,n) = -\,g_{\mu_1\mu_2} + \dfrac{k^{\mu_1}n^{\mu_2} + n^{\mu_1}k^{\mu_2}}{n\cdot k} \quad | \quad n^2 = 0
\end{cases}$$

covariant gauge (Feynman $\xi = 1$)

physical/axial no ghosts


QCD and
Collider Physics
R.K. ELLIS, W.J. STIRLING
AND B.R. WEBBER
CAMBRIDGE MONOGRAPHS
ON PARTICLE PHYSICS, NUCLEAR PHYSICS
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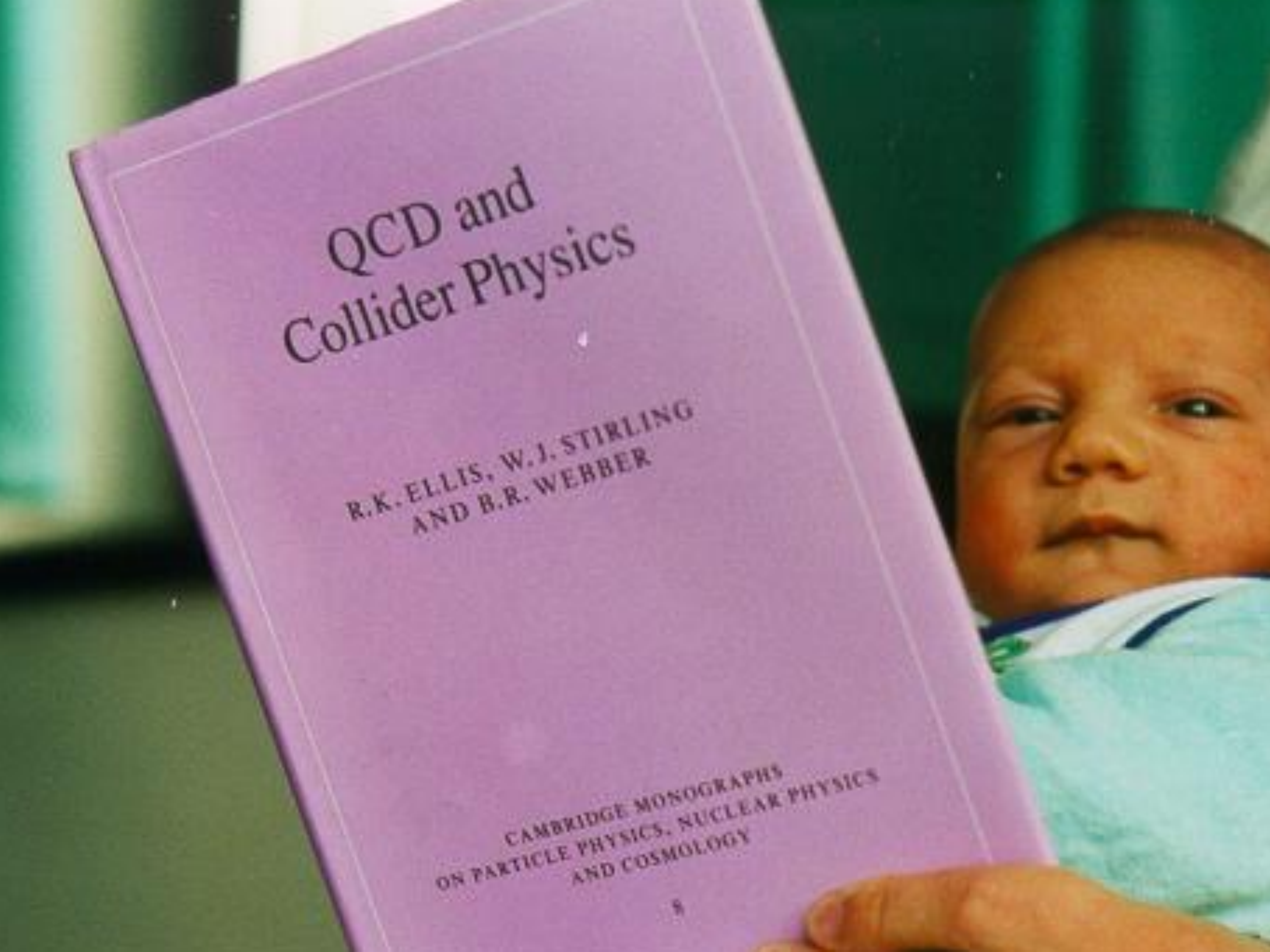
8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Perturbative Quantum Field Theory

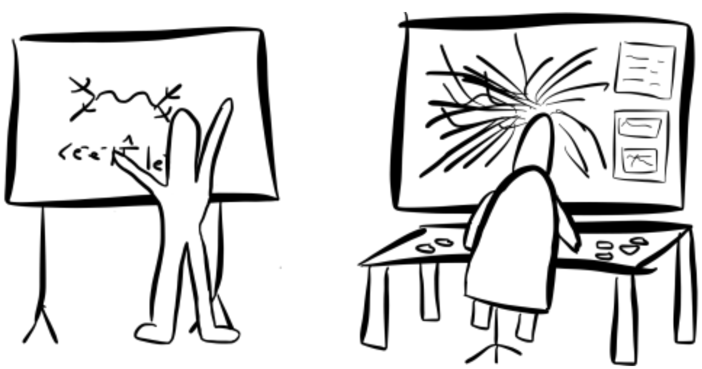
- Relies on the idea of order-by-order expansion in the small coupling $\alpha_S \ll 1$

$$\alpha_S + \alpha_S^2 + \alpha_S^3 + \ldots$$

small

smaller

negligible ?



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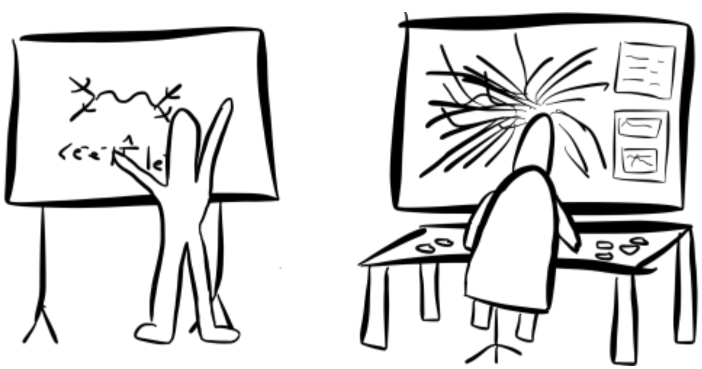
- consider order-by-order all possible **quantum fluctuations** | encoded through loop Feynman's diagrams, and Feynman diagrams with emission of extra radiation



How big is the coupling ?

All the SM couplings (including mass and Yukawa in the $\overline{MS}$ scheme) depend on the energy scale (obey **Renormalization Group Equations RGE**), the QCD coupling and the quark masses run fast

$$\frac{\partial\, a_S(\mu)}{\partial \log \mu^2} = \beta(a_S) = -\, a_S^2 \left(b_0 + a_S\, b_1 + a_S^2\, b_2 + \dots\right) \;, \qquad a_S = \frac{\alpha_S}{\pi}$$

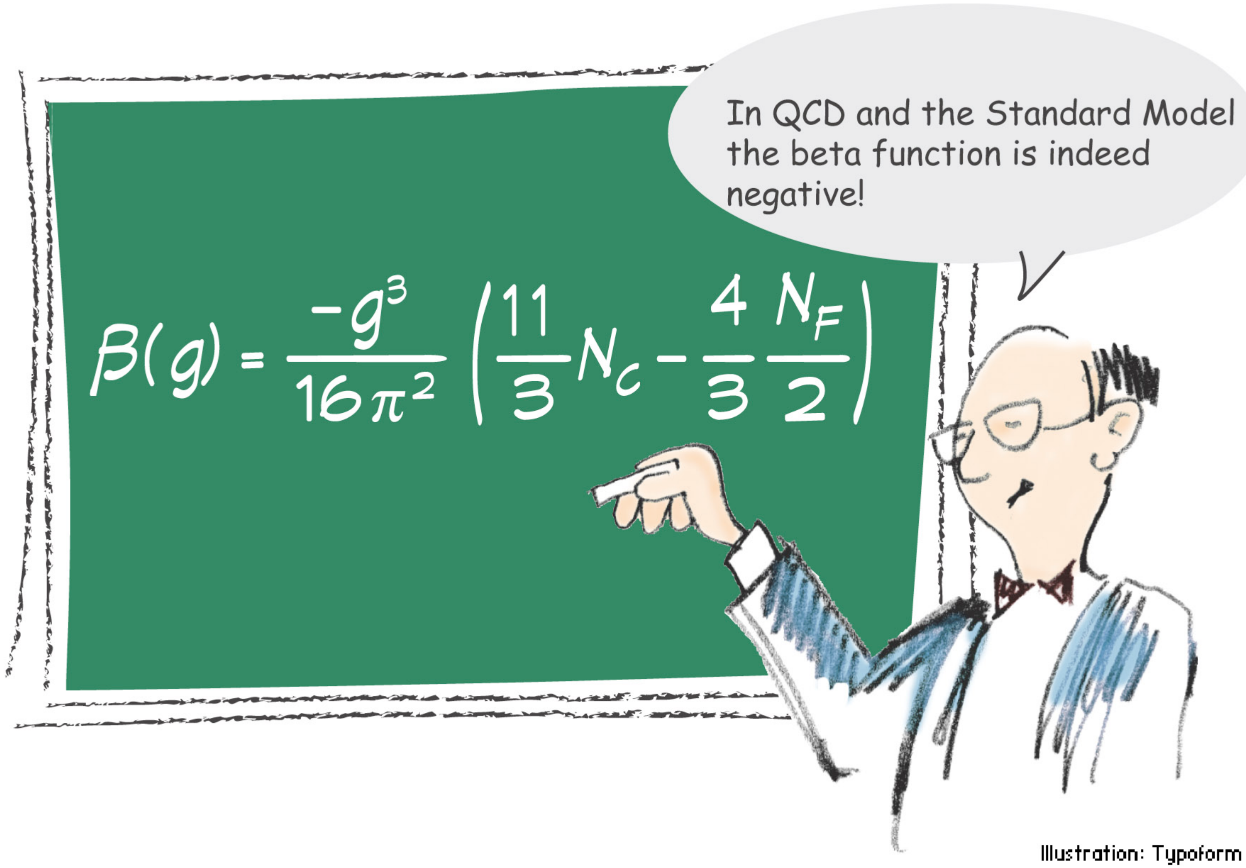
$$\frac{\partial \log m_q(\mu)}{\partial \log \mu^2} = \gamma_m(a_S) = -\, a_S \left(g_0 + a_S\, g_1 + a_S^2\, g_2 + \dots\right)$$

with coefficients

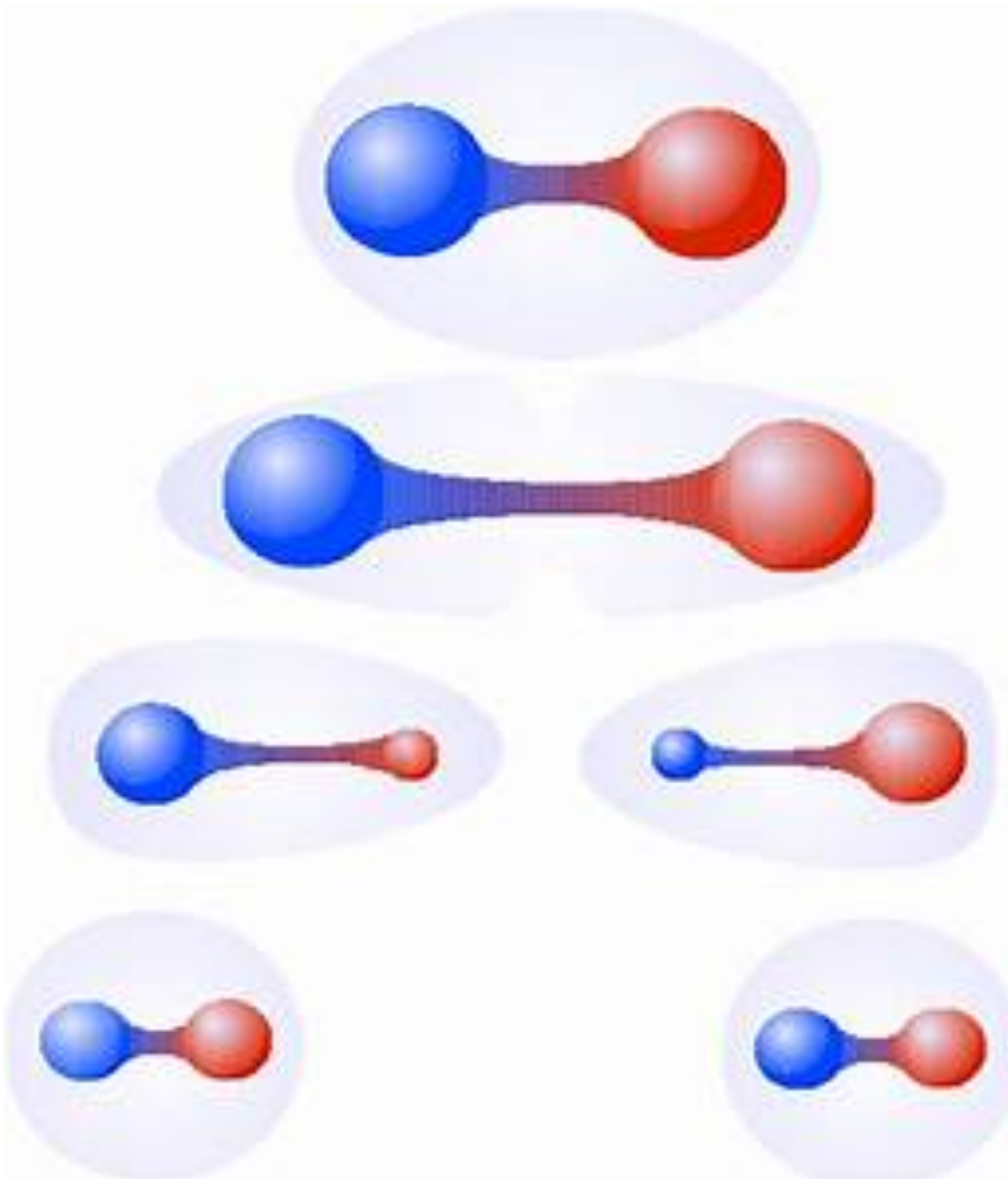
$$b_0 = \frac{1}{12}\left(11 C_A - 2 N_F\right) \;, \quad b_1 = \frac{1}{24}\left(17 C_A^2 - (5 C_A + 3 C_F)\, N_F\right) \;,$$

$$g_0 = 1 \;, \qquad g_1 = \frac{1}{16}\left(\frac{202}{3} - \frac{20}{9} N_F\right) \;.$$

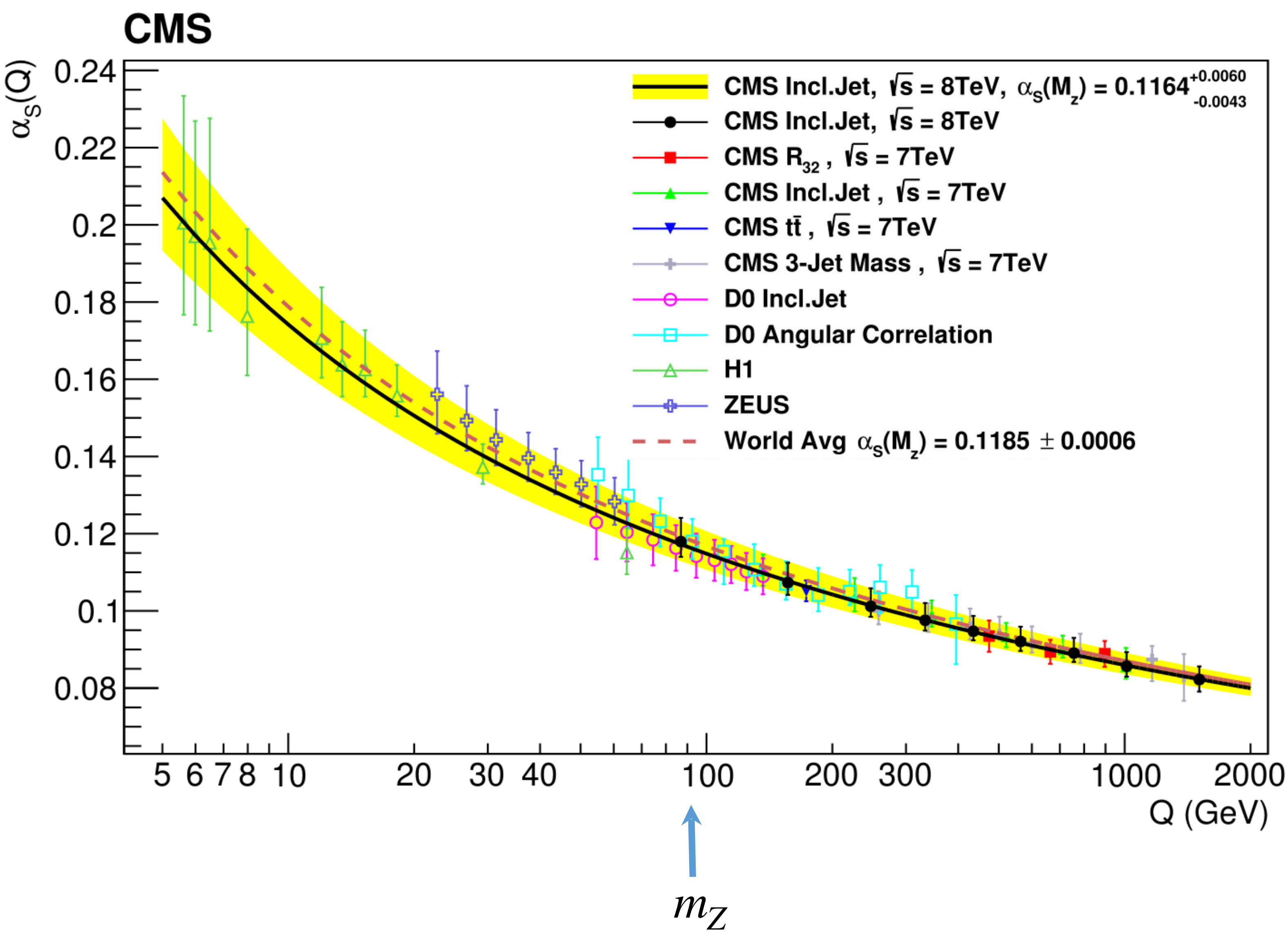
steep function in the number of quark flavors N_F



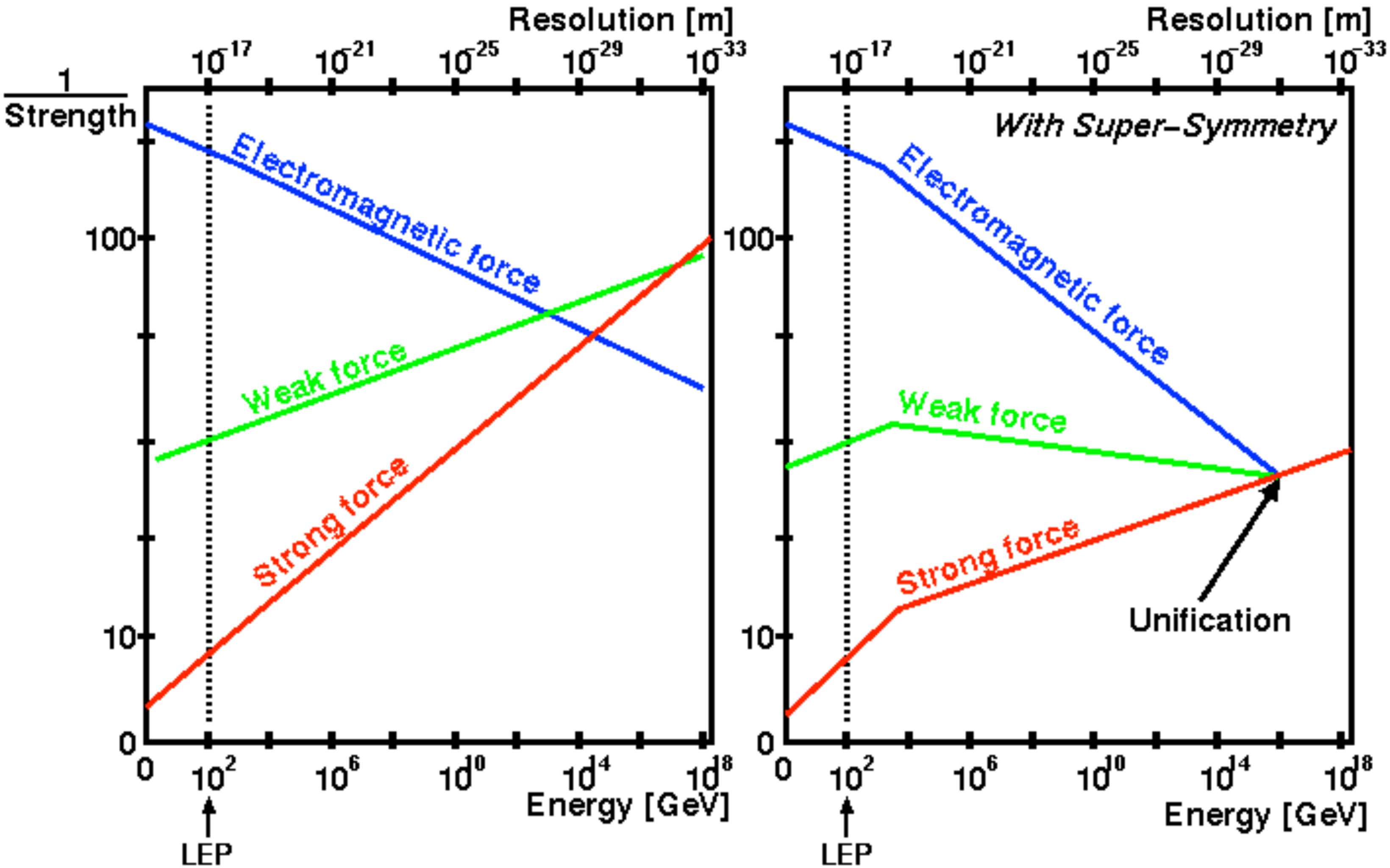
- Sign $\beta(\alpha_S) < 0$: **Asymptotic Freedom** due to gluon self-interactions
  First calculated by `t Hooft, published by Gross, Politzer, Wilczek (Nobel Prize 2004)
- **At high scales:** coupling becomes small, quarks and gluons are almost free, strong interactions are weak
- **At low scales:** coupling becomes large, quarks and gluons interact strongly, confined into hadrons, perturbation theory fails



The evolution of the strong coupling

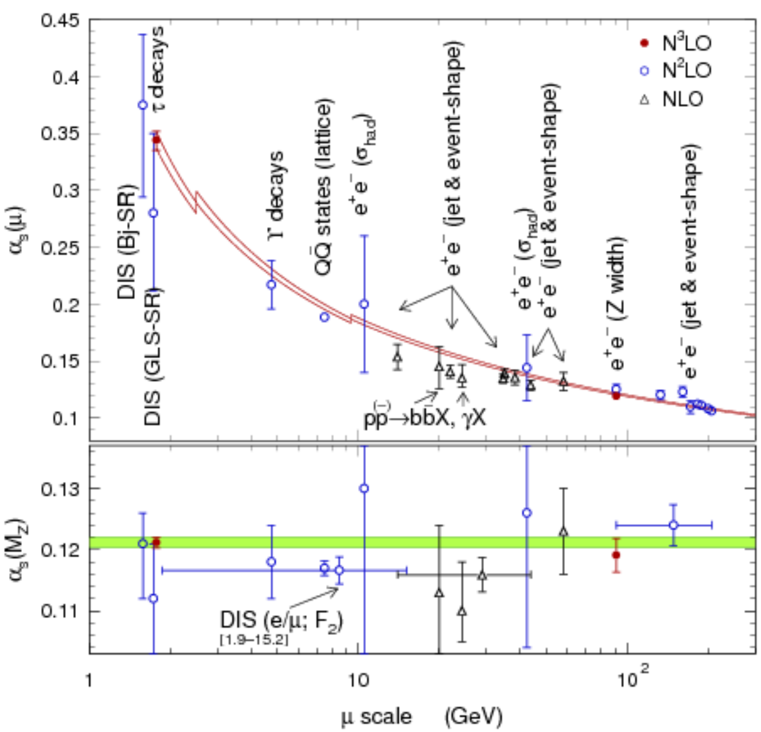


Unification of forces



Quark flavour thresholds

- $\beta(\alpha_S)$ and $\gamma_m(\alpha_S)$ are **steep** functions in the number of quark flavors N_F



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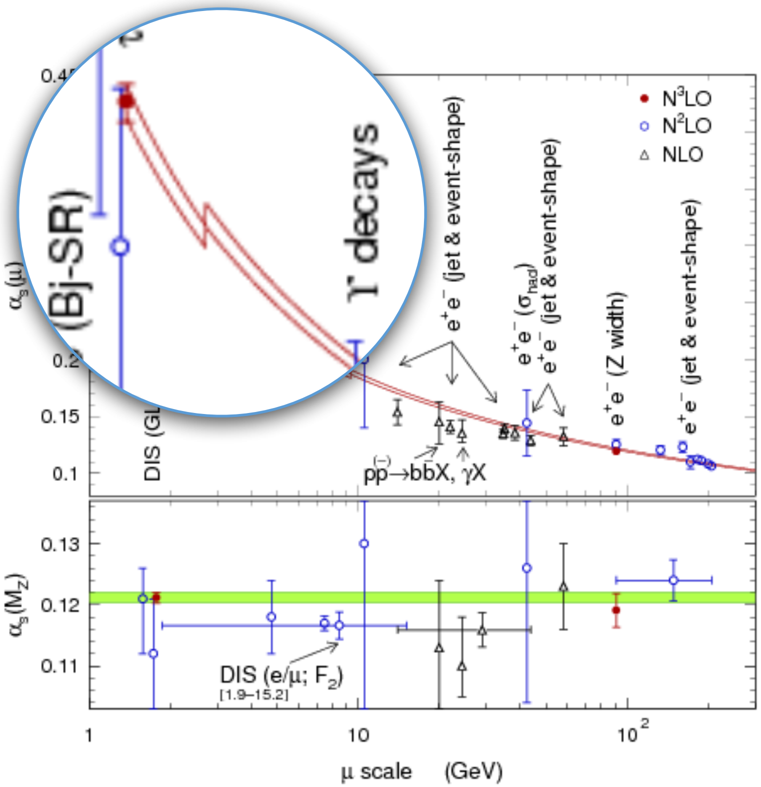
- $\beta(\alpha_S)$ and $\gamma_m(\alpha_S)$ are **steep** functions in the number of quark flavors N_F
- $\alpha_S(\mu)$ might become **discontinuous**, is that a problem ?



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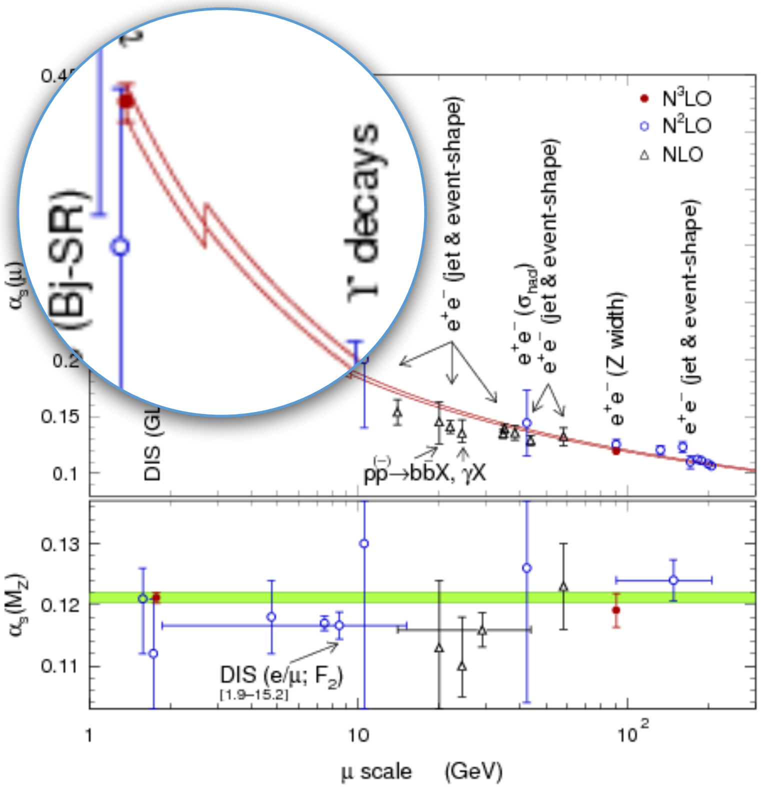
- Interpret it in the context of **Effective Theories** with different number of active flavours, and match the coupling at threshold



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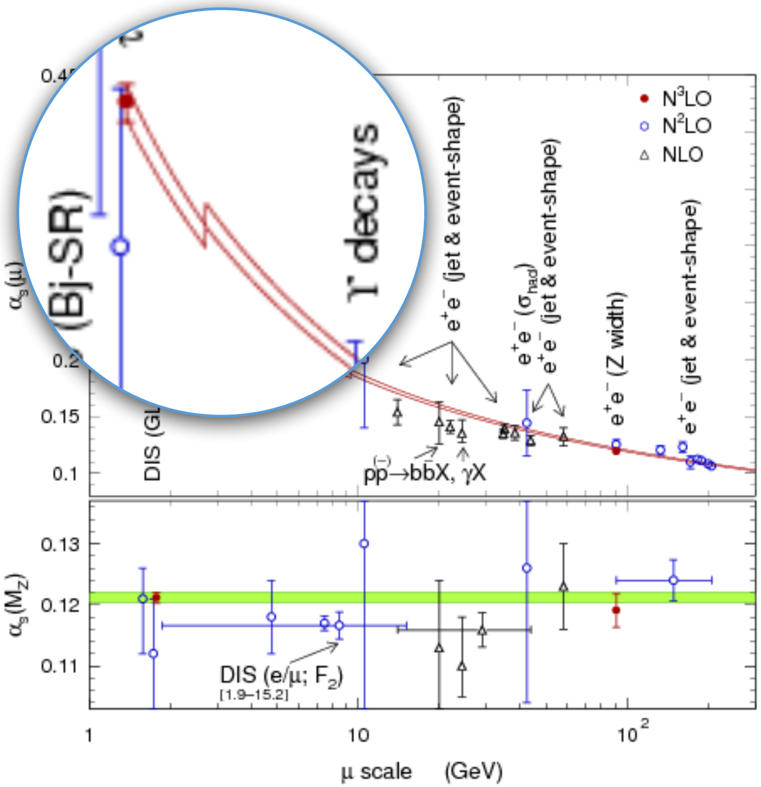

$$a_S^{(N_F)}(\mu_{th}) = a_S^{(N_F-1)}(\mu_{th}) \left[1 + \sum C_k(x) \left(a_S^{(N_F-1)}(\mu_{th}) \right)^k \right]$$

$$m_q^{(N_F)}(\mu_{th}) = m_q^{(N_F-1)}(\mu_{th}) \left[1 + \sum H_k(x) \left(a_S^{(N_F-1)}(\mu_{th}) \right)^k \right]$$

$$x = \log(\mu_{th}^2 / m_q^2)$$

$$C_1 = \frac{x}{6} , \qquad C_2 = -\frac{11}{72} + \frac{19}{24}x + \frac{x^2}{36}$$

$$H_1 = 0 , \qquad H_2 = -\frac{89}{432} + \frac{5}{36}x - \frac{x^2}{12}$$

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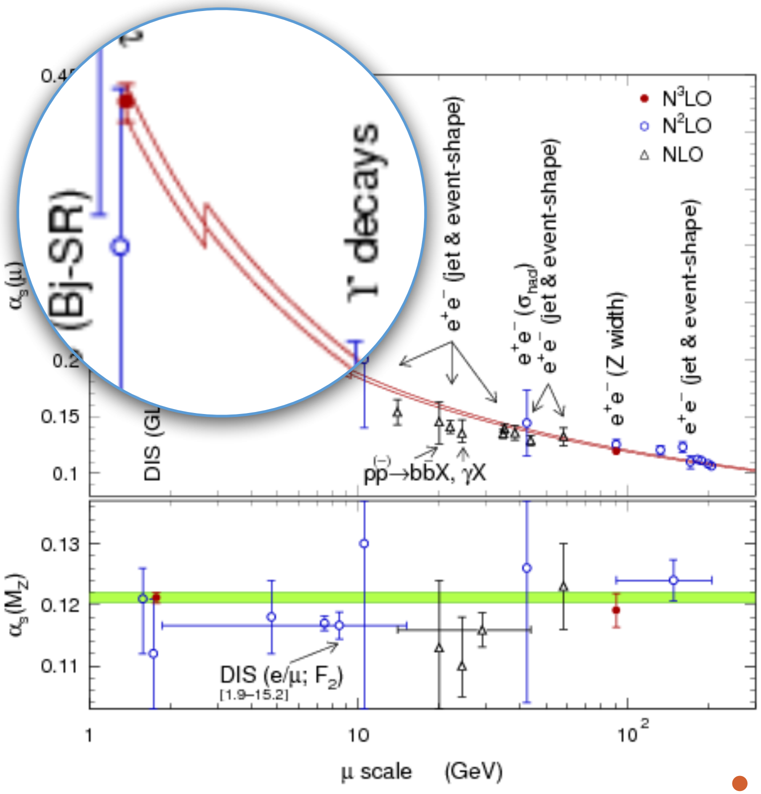

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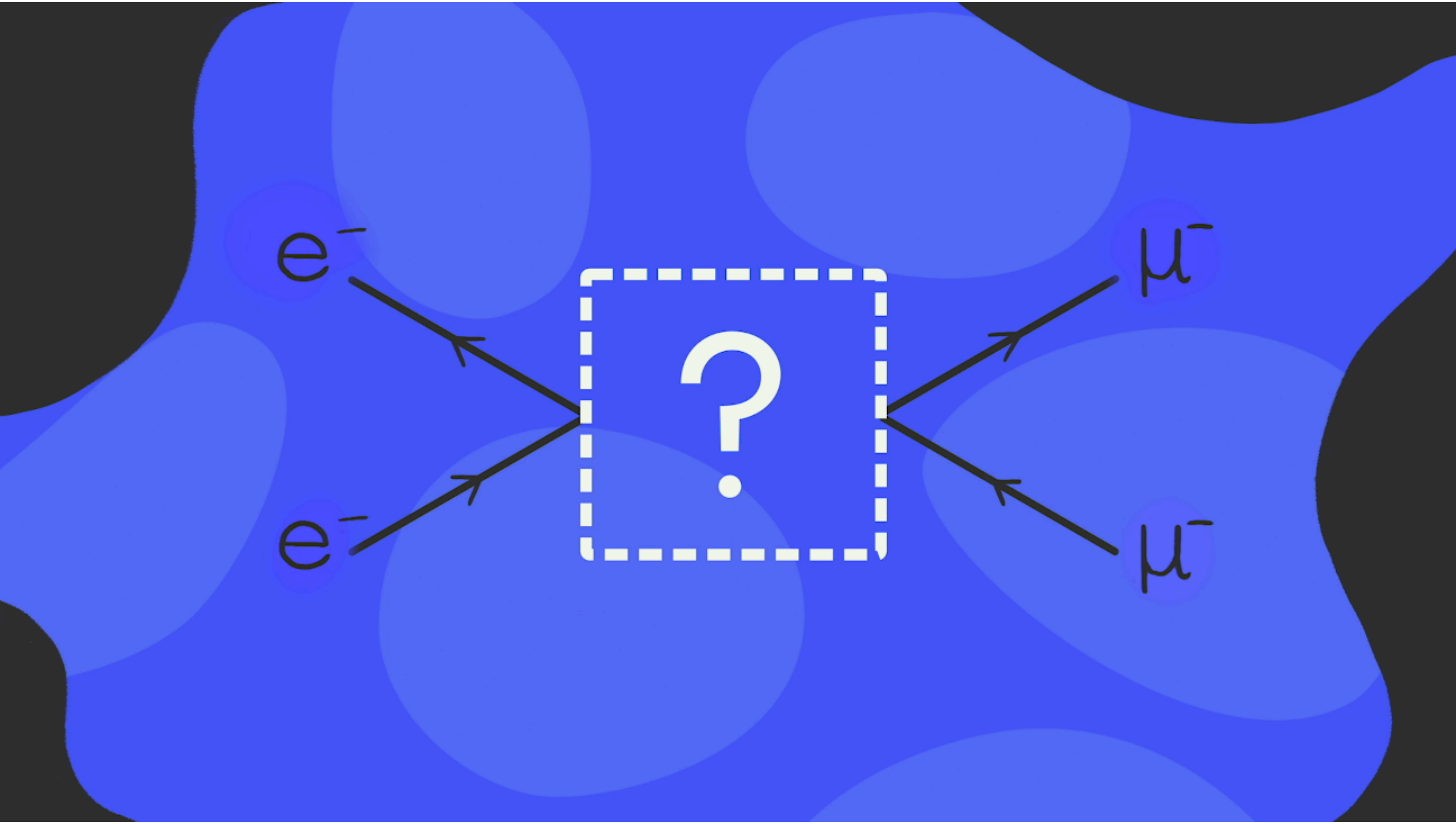
- Matching is independent of $\mu_{\rm th}$, up to higher orders
- Similar discussions for parton densities

Exercises:

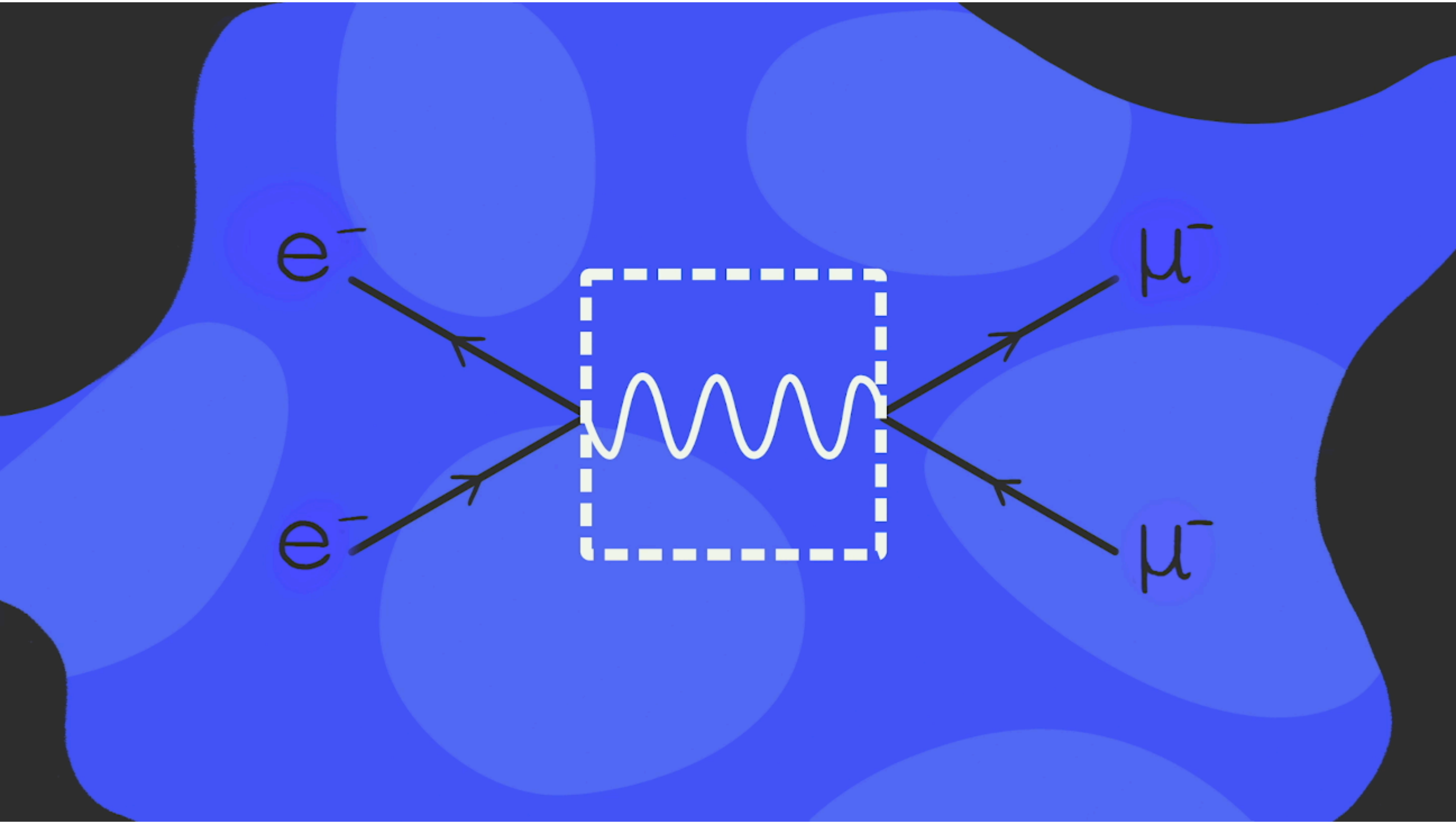
1. Integrate analytically the one-loop and two-loop RGE for the strong coupling, and one-loop for a quark mass
2. Calculate $\alpha_S(10\ \mathrm{GeV})$ and $\alpha_S(1\ \mathrm{TeV})$ from $\alpha_S(m_Z) = 0.1185 \pm 0.0006$
3. If $m_b(m_b) = 4.2 \pm 0.1\ \mathrm{GeV}$, what is $m_b(m_Z)$
4. Hint

$$a_S(\mu) = \frac{a_S(\mu_0)}{1 + b_0\, a_S(\mu_0)\ \log \frac{\mu^2}{\mu_0^2}} = \frac{1}{b_0\ \log \frac{\mu^2}{\Lambda^2_{\mathrm{QCD}}}}$$

Then calculate Λ_{QCD}, the “fundamental” scale of QCD, at which the coupling blows up (NB: it is not unambiguously defined at higher orders)



https://www.quantamagazine.org/how-feynman-diagrams-revolutionized-physics-20190514/



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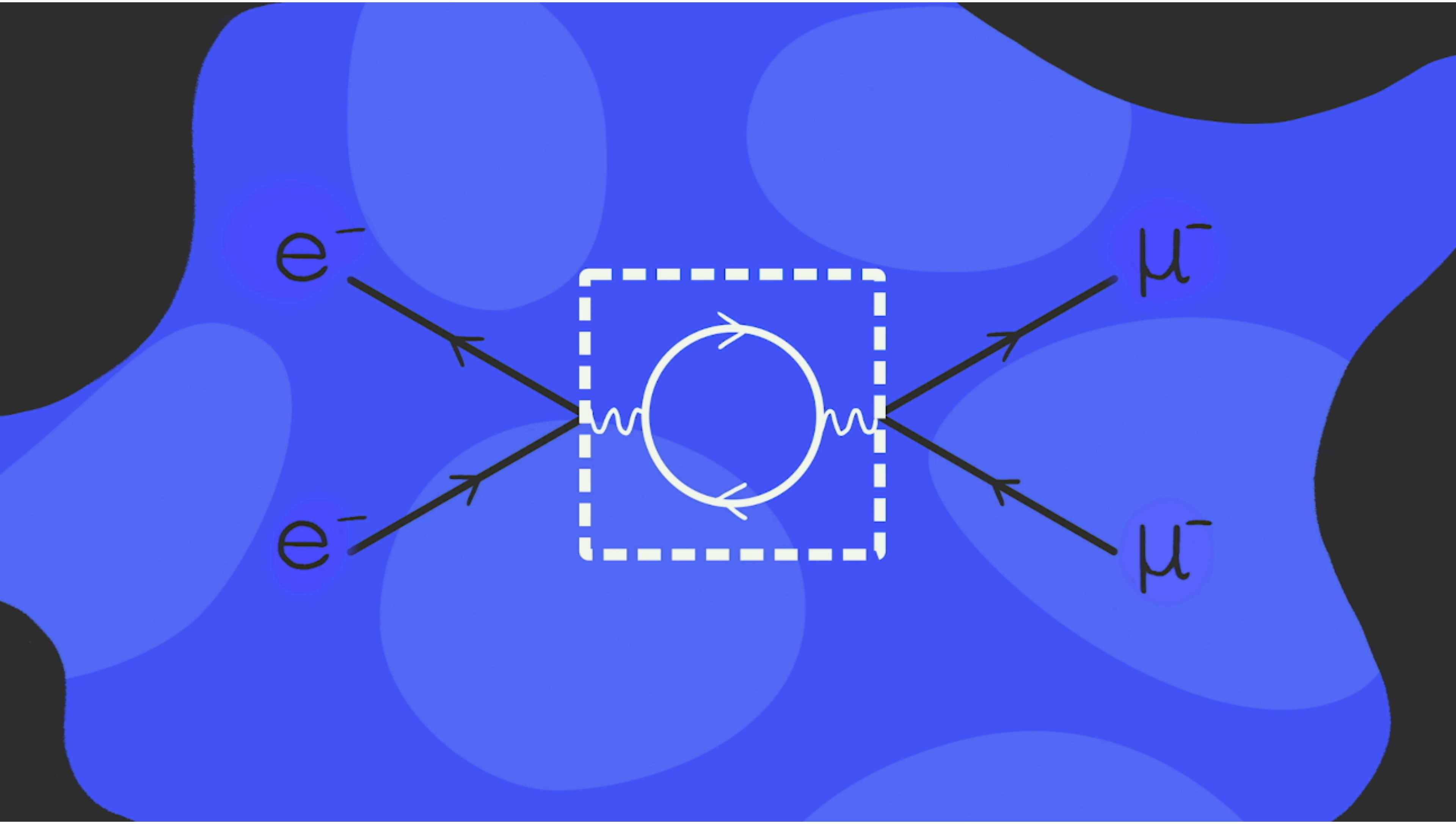

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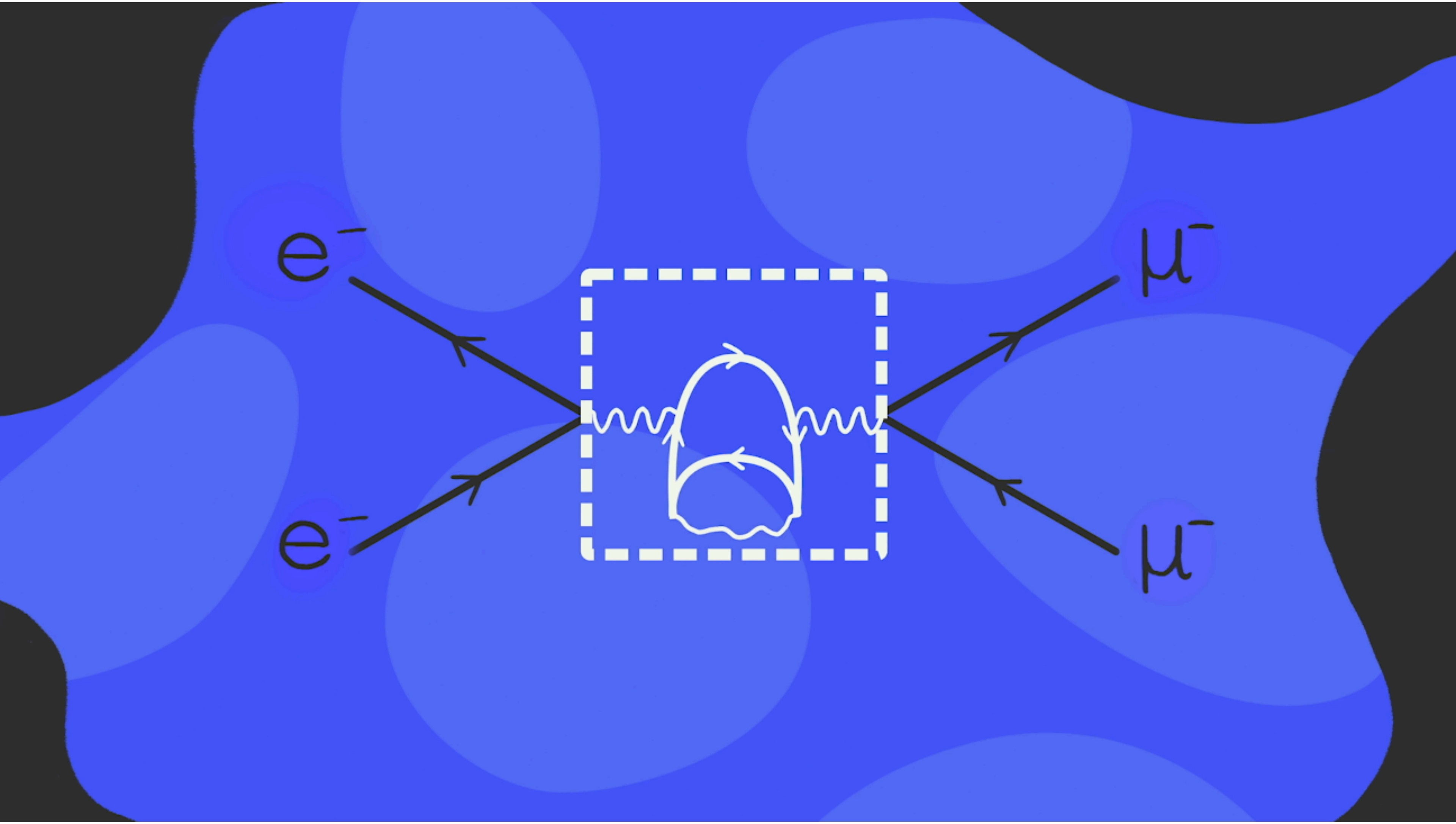

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QFT = QUANTUM MECHANICS + SPACE-TIME

- Loops encode quantum fluctuations at **infinite energy** (**zero distance**): **SM/BSM** extrapolated at energies $\gg M_{\rm Plank}$

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soft singularities (IR)

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soft singularities (IR)

collinear singularities (IR)

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soft singularities (IR)

collinear singularities (IR)

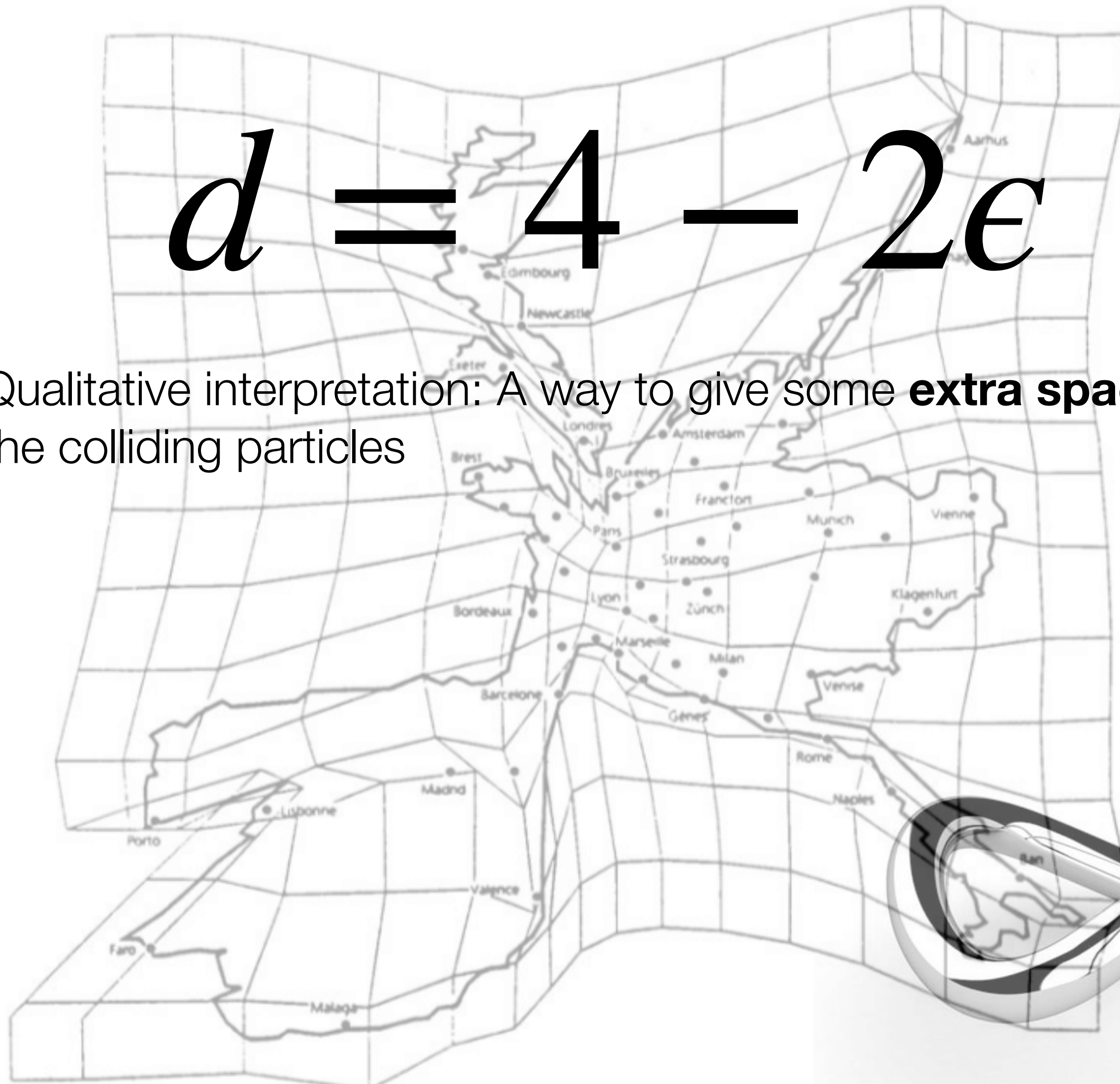
Ultraviolet singularities (UV)

and **threshold** singularities,
integrable but numerically unstable

in **four** space-time dimensions

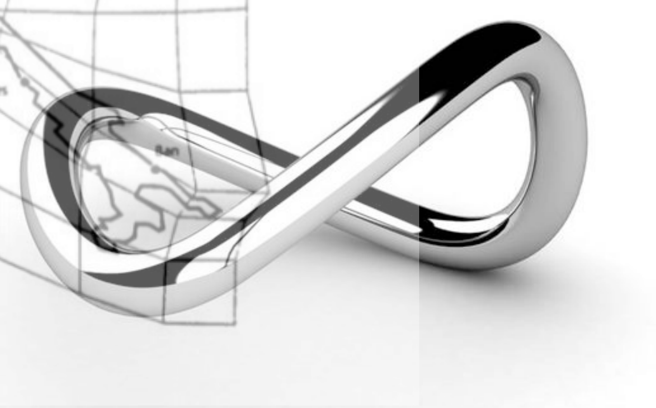
['t Hooft, Veltmann 1972]
[Giambiagi, Bollini 1972]

DIMENSIONAL REGULARISATION (DREG)



$$d = 4 - 2\epsilon$$

- Qualitative interpretation: A way to give some **extra space** to the colliding particles



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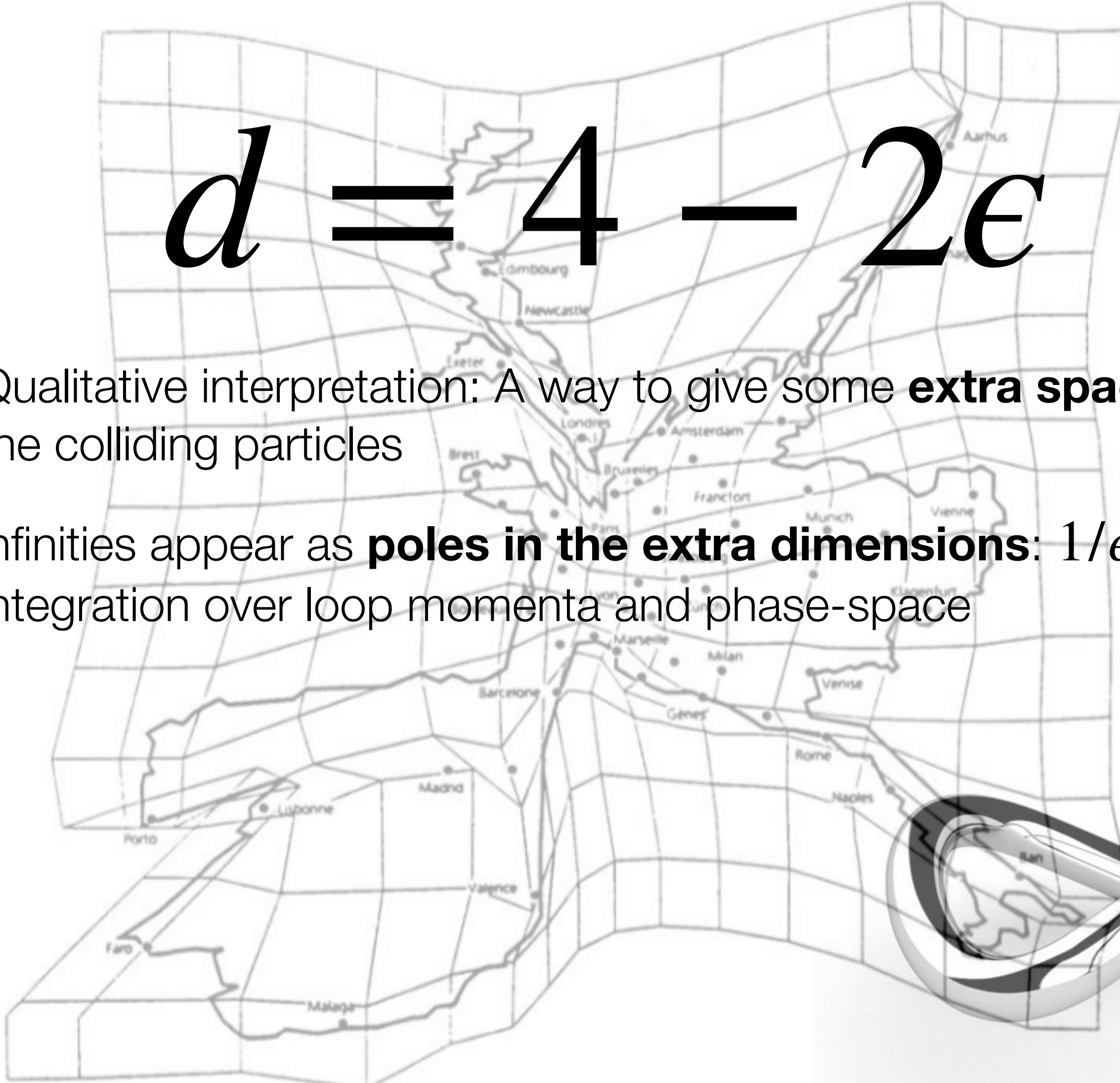

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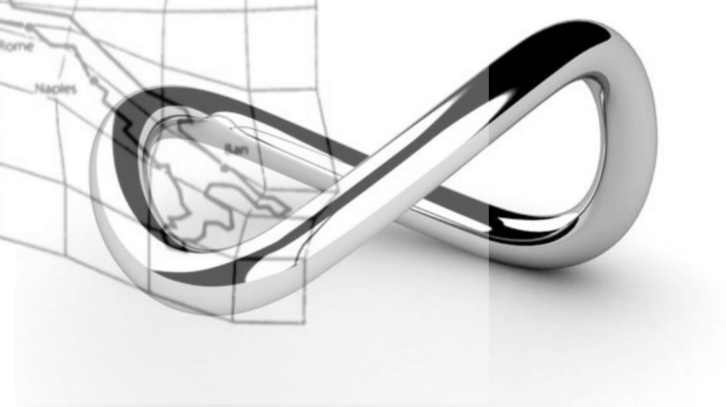

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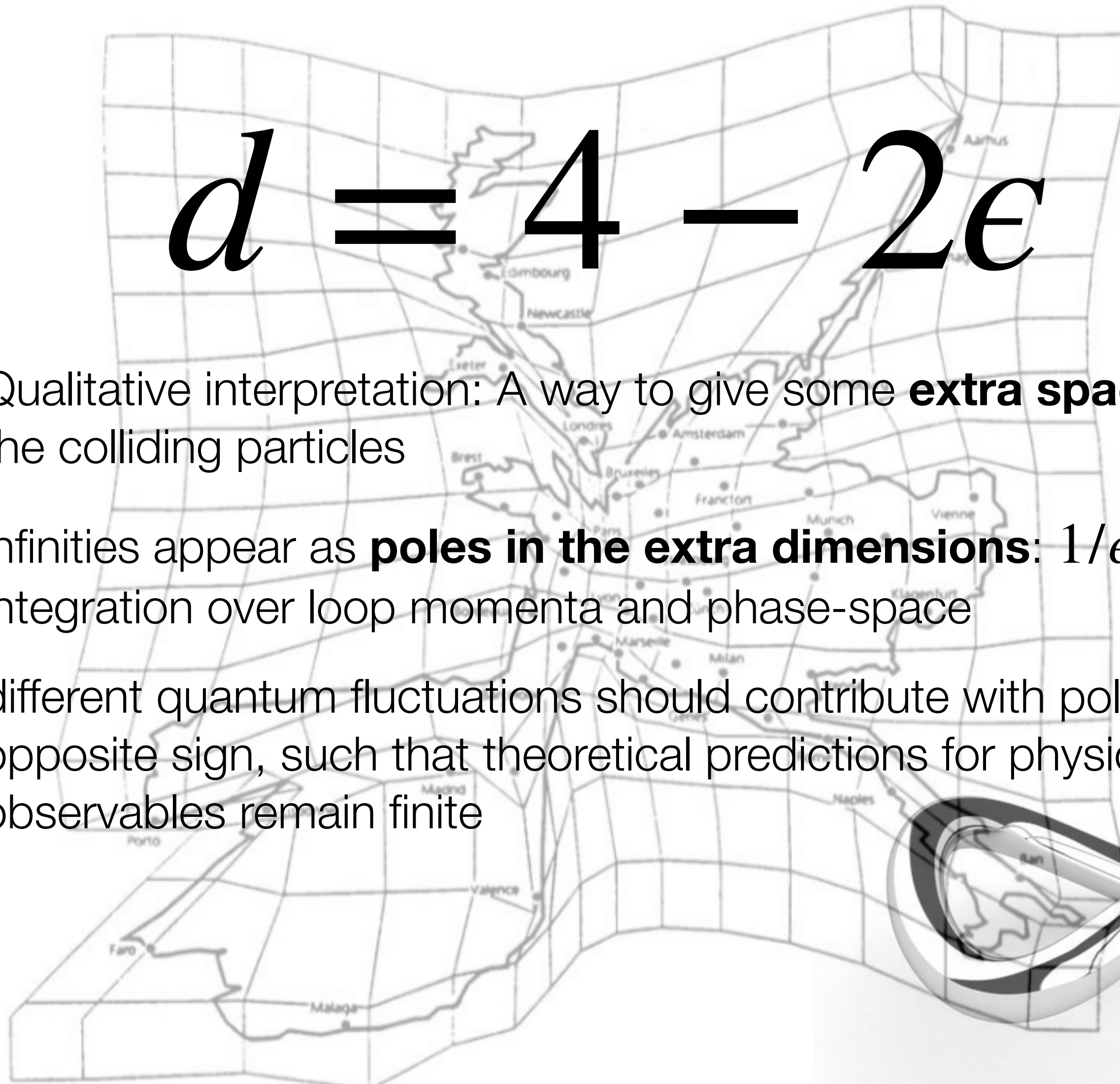
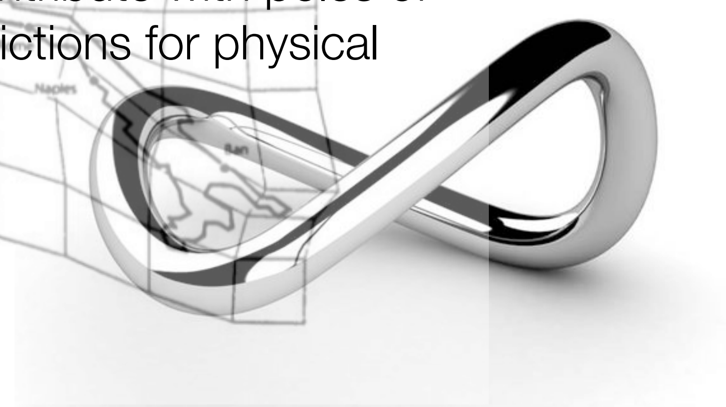


$$d = 4 - 2\epsilon$$

- Qualitative interpretation: A way to give some **extra space** to the colliding particles
- infinities appear as **poles in the extra dimensions**: $1/\epsilon$ after integration over loop momenta and phase-space
- different quantum fluctuations should contribute with poles of opposite sign, such that theoretical predictions for physical observables remain finite



Isolating the poles in ϵ

- **Slicing** method: split phase-space in two regions

$$\int_0^1 \frac{f(x)}{x} \to \int_0^1 x^{-1+\epsilon} f(x) \simeq f(0) \int_0^\omega x^{-1+\epsilon} + \int_\omega^1 \frac{f(x)}{x}$$

$$= f(0) \left(\frac{1}{\epsilon} + \log(\omega) \right) + \int_\omega^1 \frac{f(x)}{x}$$

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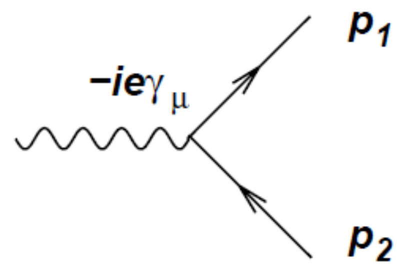
- **Subtraction** method: add and subtract back an approximation having the same singular behaviour

$$\int_0^1 x^{-1+\epsilon} f(x) = f(0) \int_0^1 x^{-1+\epsilon} + \int_0^1 \frac{f(x) - f(0)}{x}$$

e+e-: soft-collinear gluon amplitude

- At leading-order (LO):

$$\mathcal{M}^{(0)}_{q\bar{q}} = (-\imath e_q)\,\bar{u}(p_1)\,\gamma^\mu\,v(p_2)$$



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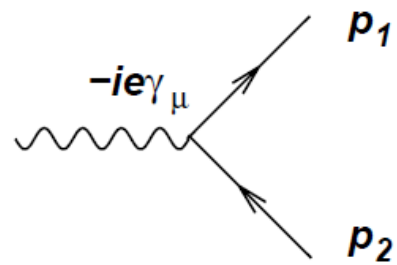
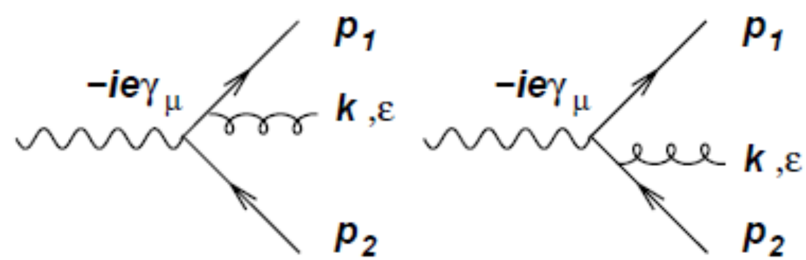


- Then emit one gluon

$$\mathcal{M}^{(0)}_{q\bar{q}g} = (-\imath e_q)\, (\imath g_S)\, \mathbf{T}^a\, \bar{u}(p_1) \left(\not{\varepsilon}(k) \frac{\imath}{\not{p}_1 + \not{k}} \gamma^\mu - \gamma^\mu \frac{\imath}{\not{p}_2 + \not{k}} \not{\varepsilon}(k) \right) v(p_2)$$



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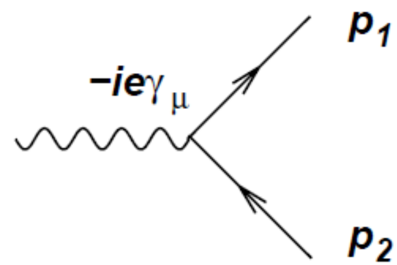

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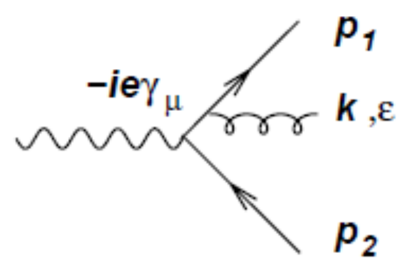
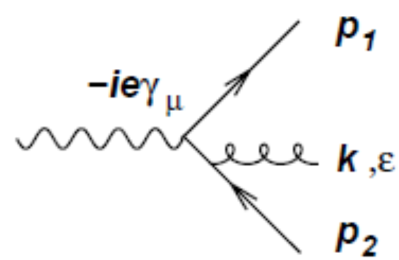
Using $\not p_2 \not\varepsilon = 2\varepsilon \cdot p_2 - \not\varepsilon \not p_2 \mid \not k \not\varepsilon = -\not\varepsilon \not k$ and the equation of motion $\not p_2\, v(p_2) = 0$ in the soft ($\not k \to 0$) and collinear with p_2 ($\not k\, v(p_2) \to 0$) limits





$$(\not p_2 + \not k)\,\not\varepsilon(k)\,v(p_2) \simeq 2\varepsilon \cdot p_2\, v(p_2)$$

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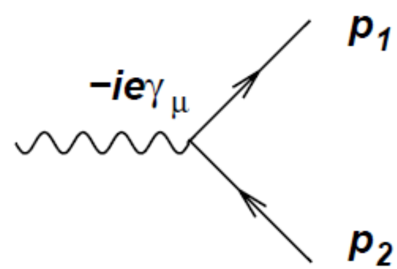

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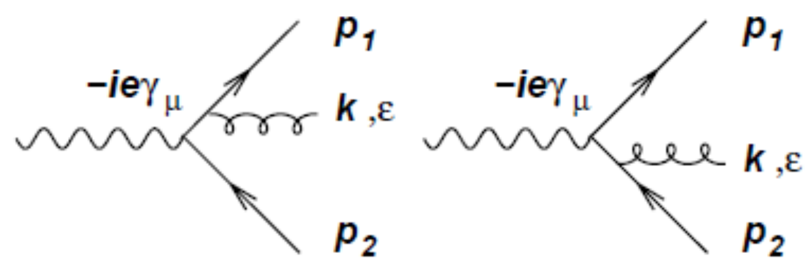

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Then

$$\mathcal{M}^{(0)}_{q\bar{q}g} \simeq (-\imath e_q)\,(\imath g_S)\,\mathbf{T}^a\,\bar{u}(p_1)\,\gamma^\mu\,v(p_2)\left(\frac{p_1\cdot\varepsilon}{p_1\cdot k} - \frac{p_2\cdot\varepsilon}{p_2\cdot k}\right)$$

e+e-: square amplitude

$$|\mathscr{M}^{(0)}_{q\bar{q}g}|^2 \simeq \sum_{a,\mathrm{pol}} \left| \mathscr{M}^{(0)}_{q\bar{q}} \, (\imath g_{\mathrm{S}}) \, \mathbf{T}^a \left(\frac{p_1 \cdot \varepsilon}{p_1 \cdot k} - \frac{p_2 \cdot \varepsilon}{p_2 \cdot k} \right) \right|^2$$

$$= - |\mathscr{M}^{(0)}_{q\bar{q}}| \, g_{\mathrm{S}}^2 C_F \left(\frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} \right)^2 = |\mathscr{M}^{(0)}_{q\bar{q}}| \, g_{\mathrm{S}}^2 C_F \, \frac{2 p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$$

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Include phase space

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Note **factorisation** into hard and soft-collinear-gluon emission

e^+e^-: square amplitude

The squared matrix element in terms of energy* and angle

$$\frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} = \frac{4}{E^2\,(1 - \cos^2\theta)}$$

- It diverges for $E \to 0$: **infrared (soft) emission**
- It diverges for $\theta \to 0$ and $\theta \to \pi$: **collinear singularities**

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Use **dimensional regularisation (DREG)** to integrate analytically over the soft and collinear regions of the phase-space

$$\frac{d^3k}{2E(2\pi)^3} \to \frac{d^{d-1}k}{2E(2\pi)^{d-1}} \qquad d = 4 - 2\epsilon$$

leads to poles in $1/\epsilon^2, 1/\epsilon$, and a finite remainder

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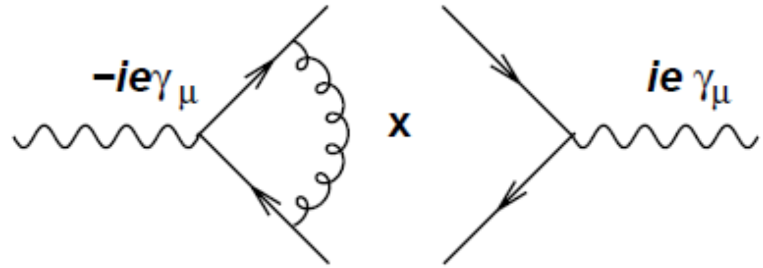
* Are there soft singularities if $1/E$?

e+e-: virtual amplitude

- The one-loop amplitude (Feynman gauge):



$$\int_\ell = -\imath \mu^{4-d} \int \frac{d^d\ell}{(2\pi)^d}$$

$$\mathcal{M}^{(1)}_{q\bar{q}} = (-\imath e_q)\, g_S^2\, C_F\, \bar{u}(p_1) \left[\int_\ell \frac{\gamma^\nu (\not\ell - \not p_1) \gamma^\mu (\not\ell + \not p_2) \gamma_\nu}{[(\ell - p_1)^2 + \imath 0][(\ell + p_2)^2 + \imath 0][\ell^2 + \imath 0]} \right] v(p_2)$$

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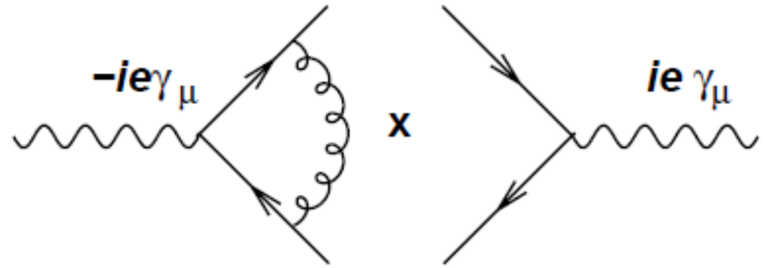

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- set the virtual gluon on-shell: $\dfrac{1}{\ell^2+\imath 0} \to -\imath 2\pi\,\theta(\ell_0)\,\delta(\ell^2) \equiv \tilde{\delta}(\ell)$

$$\bar{u}(p_1)\,\gamma^\nu\,(\not\ell-\not p_1) = \bar{u}(p_1)\,[2(\ell-p_1)^\nu - \not\ell\gamma^\nu]$$
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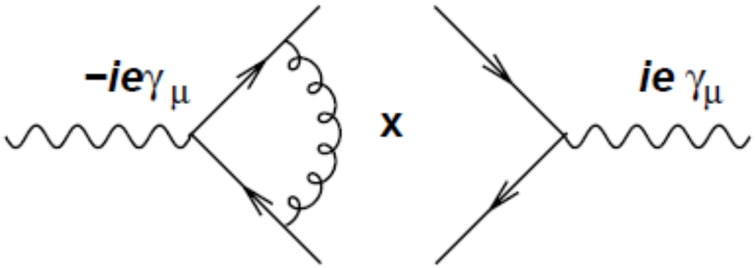

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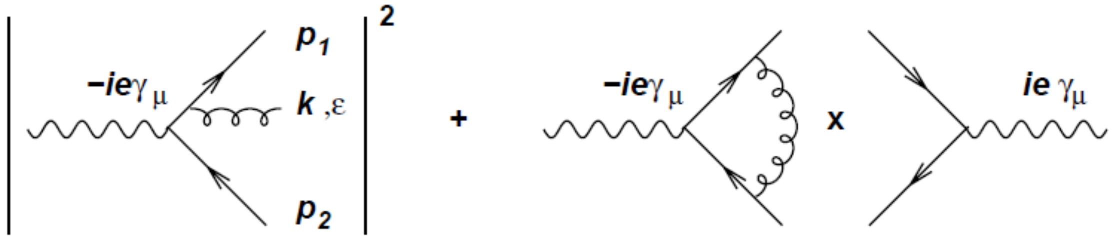
- In the limit where ℓ becomes soft, or collinear with either of the external particles

$$\mathcal{M}^{(1)}_{q\bar{q}} \simeq -\mathcal{M}^{(0)}_{q\bar{q}}\, g_S^2 C_F \int_\ell \frac{p_1 \cdot p_2}{(p_1 \cdot \ell)(p_2 \cdot \ell)} \tilde{\delta}(\ell)$$

e+e-: total cross-section

- the **total cross-section** is the sum of all quantum fluctuations



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$$\int d\Phi_{q\bar{q}} \, 2\mathrm{Re}\langle \mathcal{M}^{(1)}_{q\bar{q}} | \mathcal{M}^{(0)}_{q\bar{q}} \rangle \simeq - \int d\Phi_{q\bar{q}} \, |\mathcal{M}^{(0)}_{q\bar{q}}|^2 \, g_S^2 C_F \int_{\ell} \frac{2p_1 \cdot p_2}{(p_1 \cdot \ell)(p_2 \cdot \ell)} \tilde{\delta}(\ell)$$

- The total cross-section must be finite: if real part has poles in $1/\epsilon$, then the loop integration should exhibit the same poles of opposite sign (**unitarity, conservation of probability**)

e^+e^-: total cross-section

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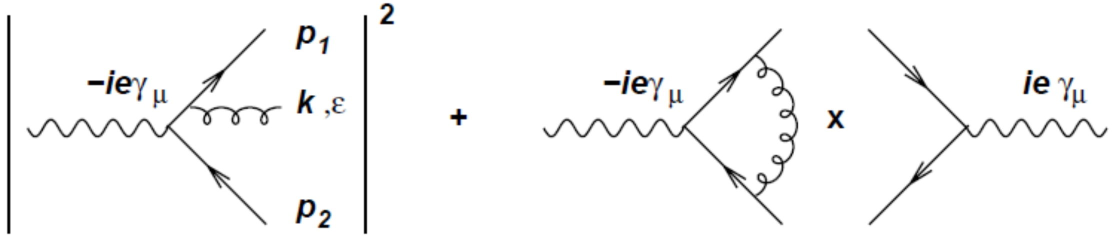

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- Alternatively: **local cancellation** before integration (loop-tree duality LTD / four-dimensional unsubtraction FDU) [Catani et al. 2008]

NNLO ingredients

$$\sigma^{\mathrm{NLO}} = \int_{N} d\sigma^{\mathrm{V}} + \int_{N+1} d\sigma^{\mathrm{R}}$$

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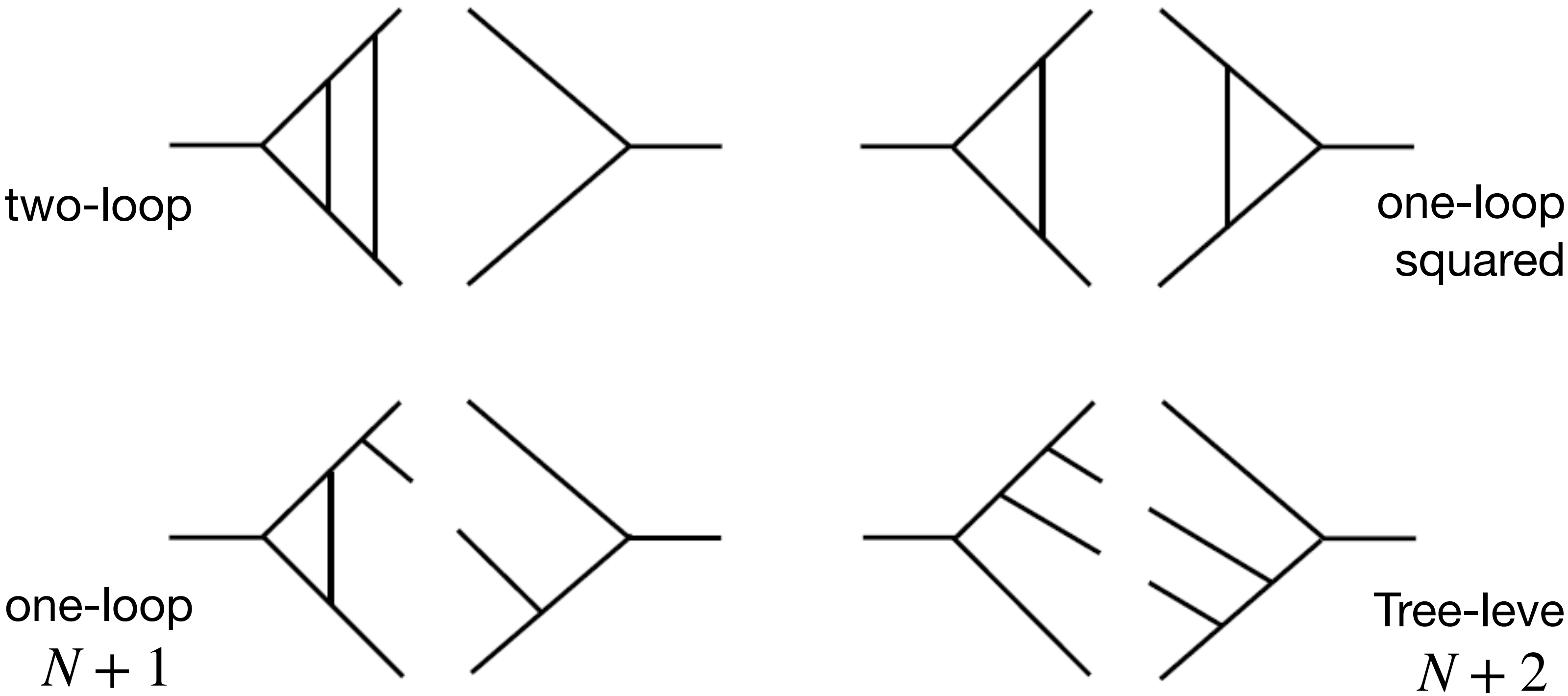

Theorems about cancellation of singularities

- **BN (Bloch-Nordsiek 1937):** QED (with finite fermion mass) IR divergences cancel if sum over soft (unobserved) photons in the final state

- **KLN (Kinoshita 1962 | Lee, Nauenberg 1964):** soft and collinear divergences cancel if sum over degenerate final and initial states ($\gamma^* \to$ hadrons need only sum in final state)

Definition of infrared and collinear safety

For an observable's distribution to be calculable in [fixed order] perturbation theory, the observable should be infrared safe, i.e. insensitive to the emission of soft or collinear partons. In particular, if $\vec{p}_i$ is any momentum occurring in its definition, it must be invariant under the branching

$$\vec{p}_i \to \vec{p}_j + \vec{p}_k$$

Whenever $\vec{p}_j$ and $\vec{p}_k$ are parallel (collinear) of one of them is small (soft)
[Ellis, Stirling, Webber, QCD and Collider Physics]

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Examples	
Multiplicity of gluons	
Energy of the hardest particle	?
Jet cross-sections	

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For an observable's distribution to be calculable in [fixed order] perturbation theory, the observable should be infrared safe, i.e. insensitive to the emission of soft or collinear partons. In particular, if $\vec{p}_i$ is any momentum occurring in its definition, it must be invariant under the branching

$$\vec{p}_i \to \vec{p}_j + \vec{p}_k$$

Whenever $\vec{p}_j$ and $\vec{p}_k$ are parallel (collinear) of one of them is small (soft)
[Ellis, Stirling, Webber, QCD and Collider Physics]

Examples	
Multiplicity of gluons	not IRC safe, modified by soft/collinear splitting
Energy of the hardest particle	not IRC safe, modified by collinear splitting
Jet cross-sections	is IRC safe, collinear particles get merged into a single parent particle in the first steeps