Origin of the slow growth of entanglement entropy in long-range interacting systems

Silvia Pappalardi (Sissa & ICTP, Trieste, Italy)

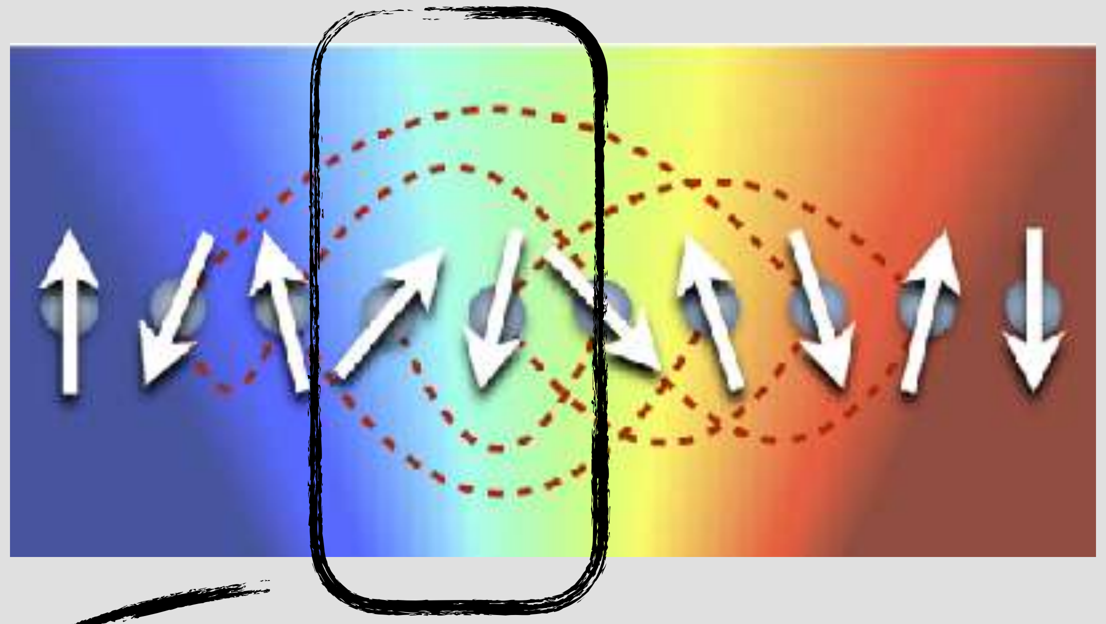
February 10th, 2020 - Benasque

with **Alessio Lerose** (University of Geneva)

[to appear on arXiv]
[to appear in Phys.Rev.Research (RC) (arXiv:1811.05505)]

Non-equilibrium behaviour of an isolated quantum system

$$|\psi(t)\rangle = e^{iHt}$$

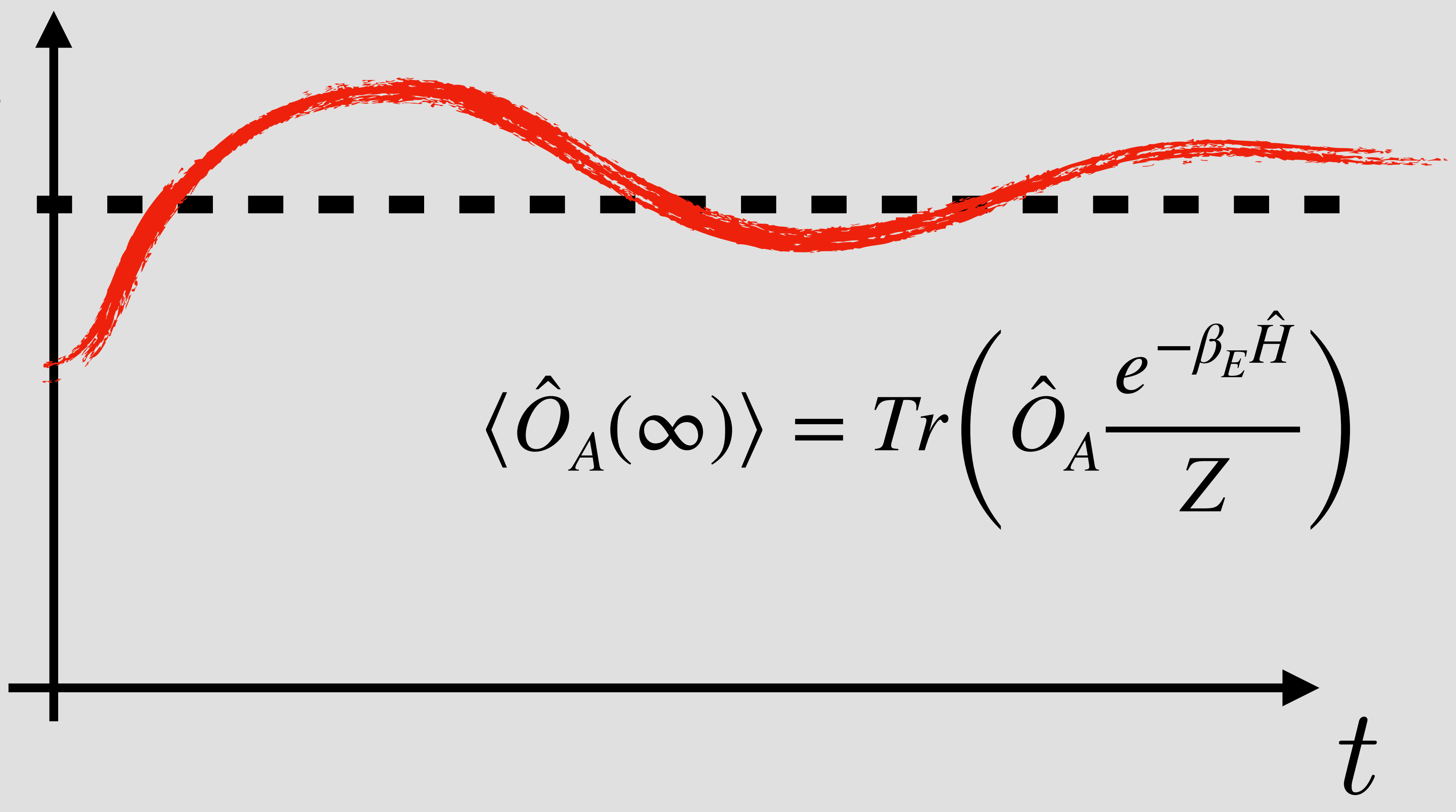


A

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Growth of quantum correlations between A and the rest

Erasure of LOCAL information despite global evolution is unitary



$$\langle \hat{O}_A(\infty) \rangle = Tr\left(\hat{O}_A \frac{e^{-\beta_E \hat{H}}}{Z} \right) \qquad \langle \hat{H} \rangle_\beta = E$$

Bipartite entanglement entropy

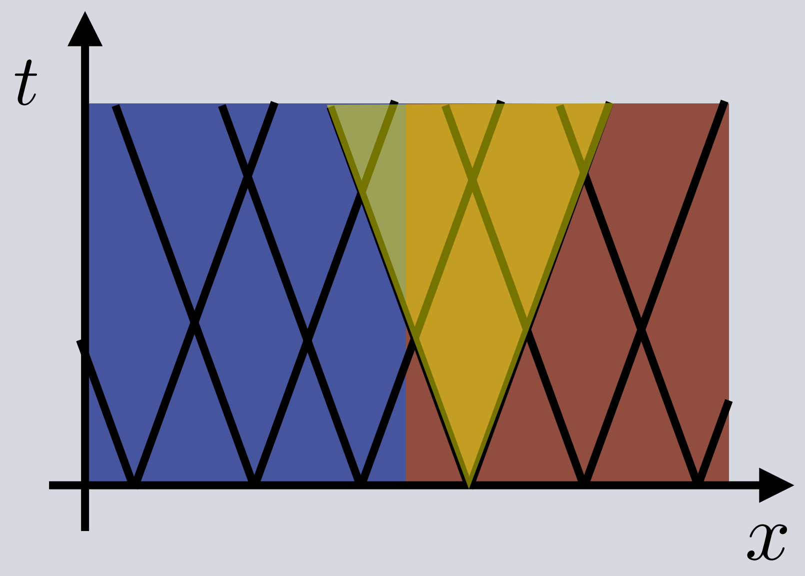
$$S_A(t) = -\mathrm{Tr}\, \hat{\rho}_A(t) \log \hat{\rho}_A(t)$$

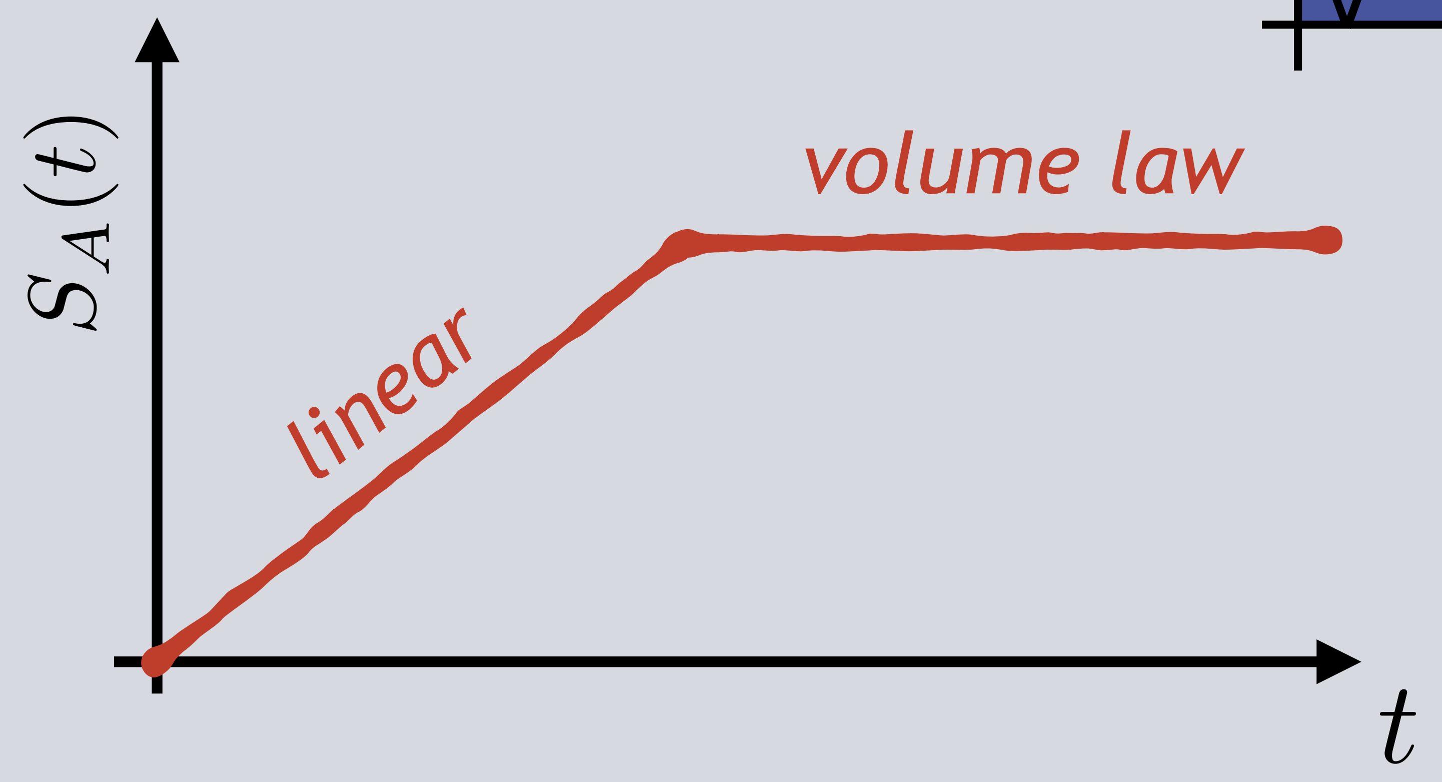
$$\hat{\rho}_A(t) = \mathrm{Tr}_B |\psi(t)\rangle\langle\psi(t)|$$

Entanglement entropy evolution

Short-range paradigm

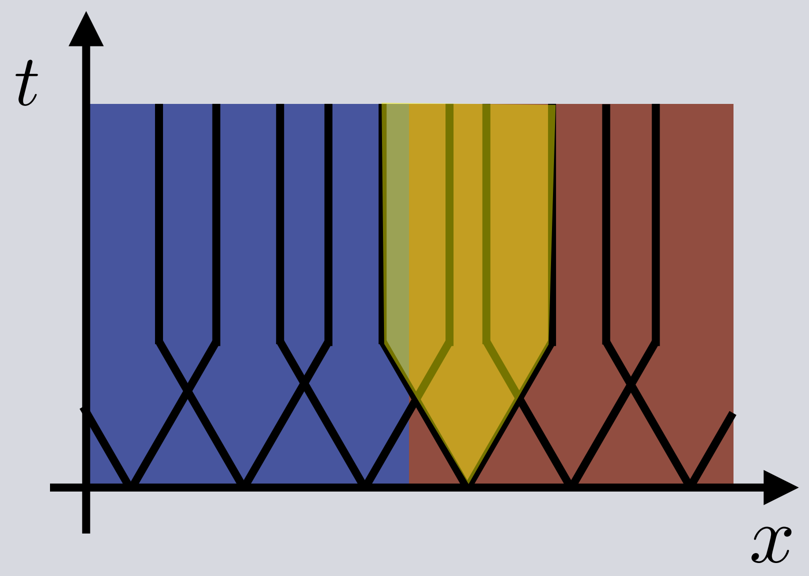
General behaviour

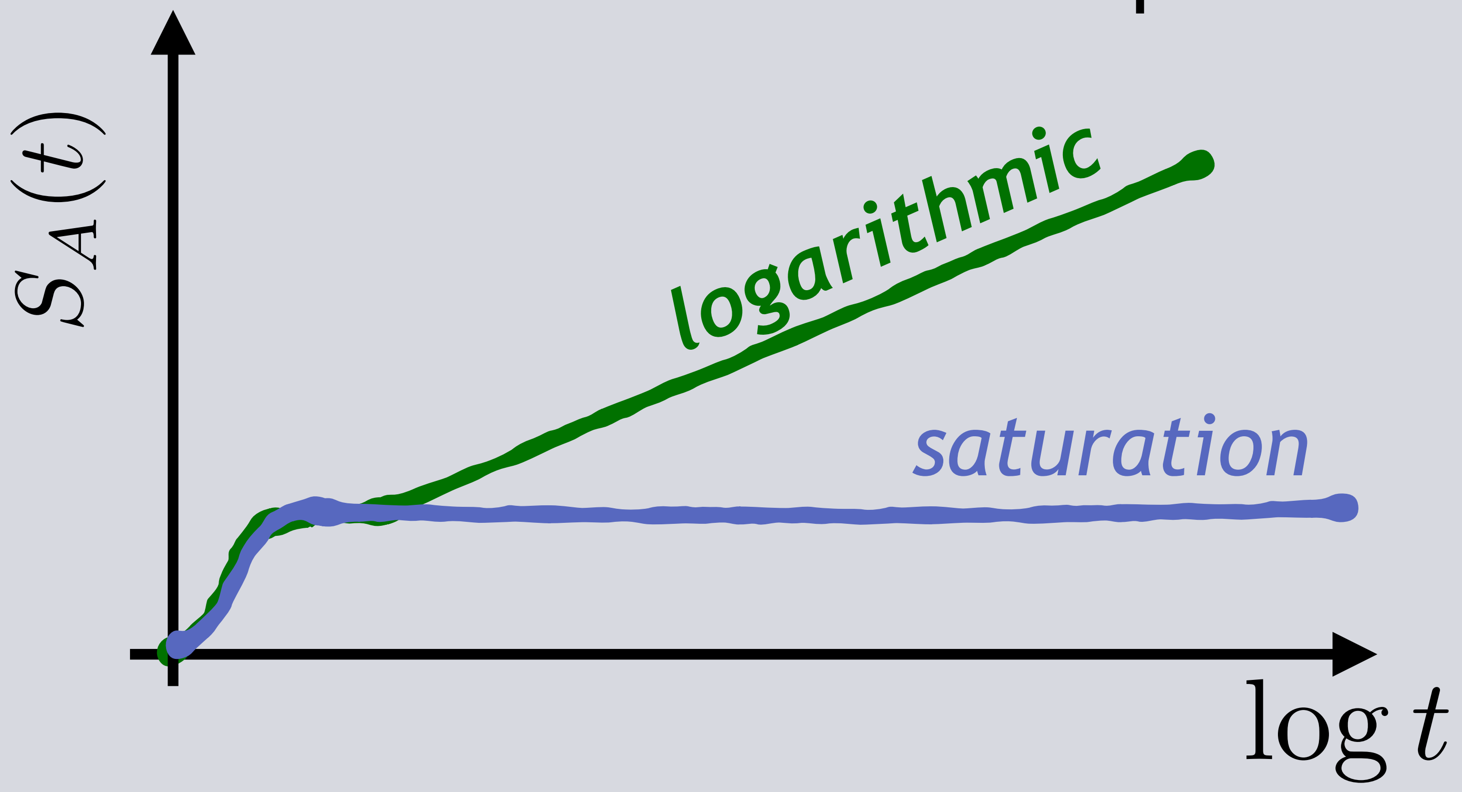




[Calabrese, Cardy - JSTAT, 2005]
[...]
[Nahum, Ruhman, Vijay, Haah - Phys. Rev. X, 2017]

Ergodicity breaking





[Žnidarič, Prosen, Prelovšek - Phys. Rev. B, 2008]
[Bardarson, Pollmann, Moore - Phys. Rev. Lett., 2012]
[Serbyn, Papić, Abanin - Phys. Rev. Lett., 2013]
[...]

Long-range systems

Classical physics:

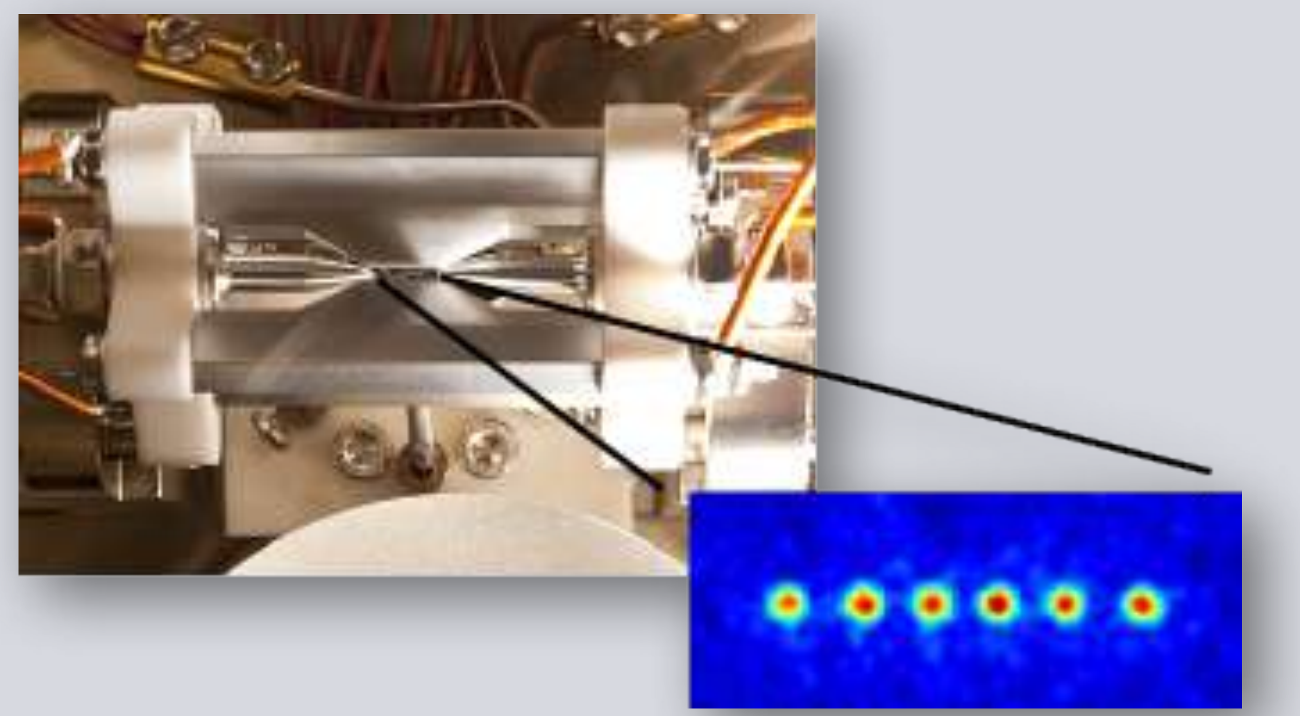
$$J_{ij} \sim \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|^\alpha}$$

$\alpha \leq d$

d - *dimensional*

[Campa, Dauxois, Fanelli, Ruffo - UOP Oxford, 2014]

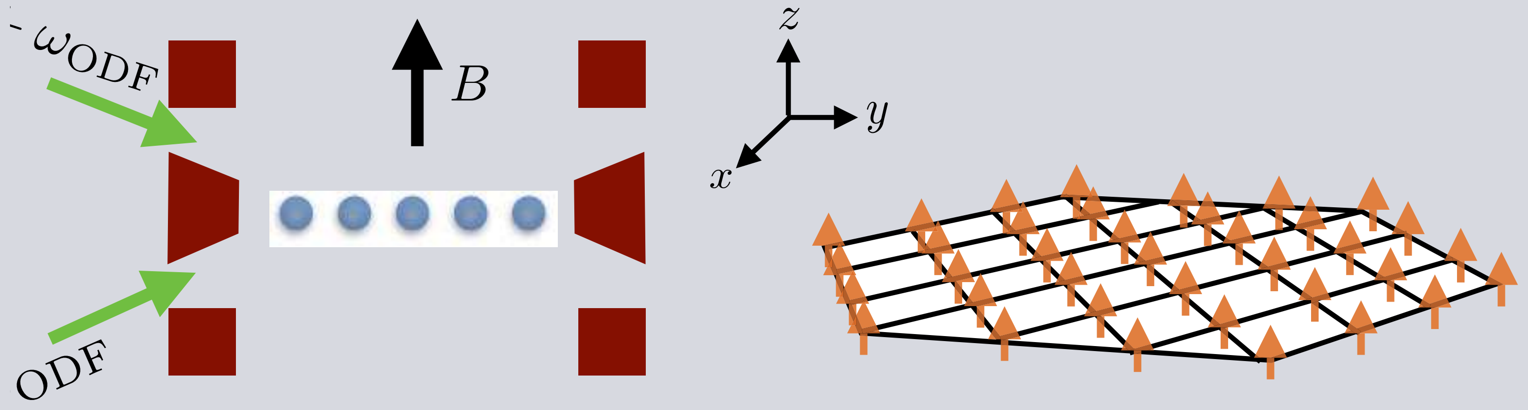
Quantum experiments in AMO physics: trapped ions, Rydberg atoms, Spinor Condensates, Cavity QED...



Paul trap

$0.5 < \alpha < 1.8$

Maryland (C. Monroe)
Innsbruck (R. Blatt)



Penning trap

$0.02 < \alpha < 0.2$

Colorado (J. Bollinger)

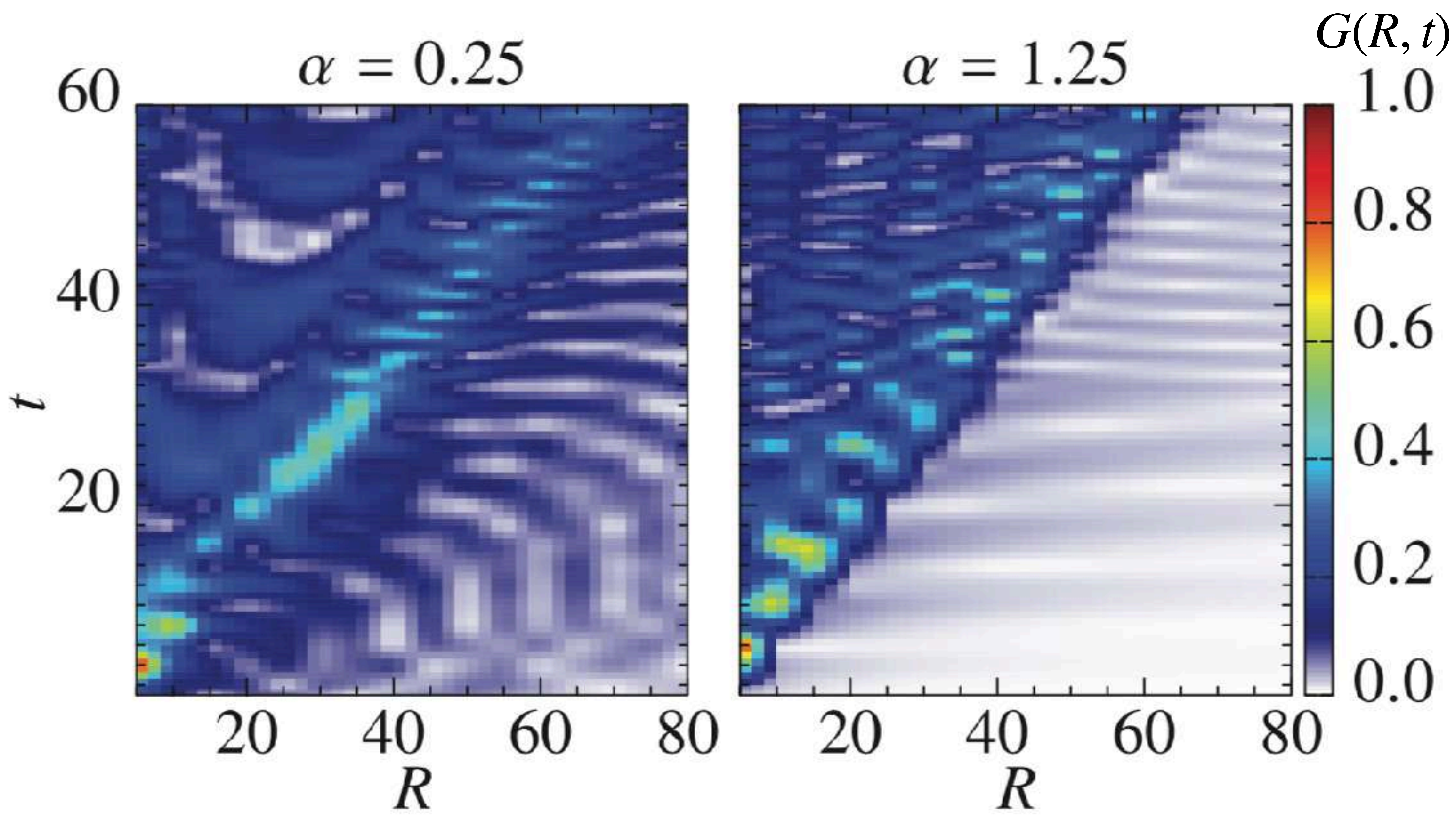
Quantum non-equilibrium physics:

- new dynamical phases (DPT, Time Crystals, etc.)
- pre-thermalization and hints of ergodicity breaking
- existence of MBL phase

[Sciolla, Biroli - JSTAT, 2011]
[Gong, Duan - NJP, 2013]
[Russomanno, Iemini, Dalmonte, Fazio - Phy. Rev. B, 2017]
[Nandkishore, Sondhi, PRX, 2017]
...
[Mori - Journ. Phys. A, 2018]
[Roy, Logan - arXiv 1903.04851, 2019]

Fast correlation spreading with long-range interactions

correlation between
site 0 and site R



Fast buildup of long
distance correlations

Violations of linear
light-cone spreading

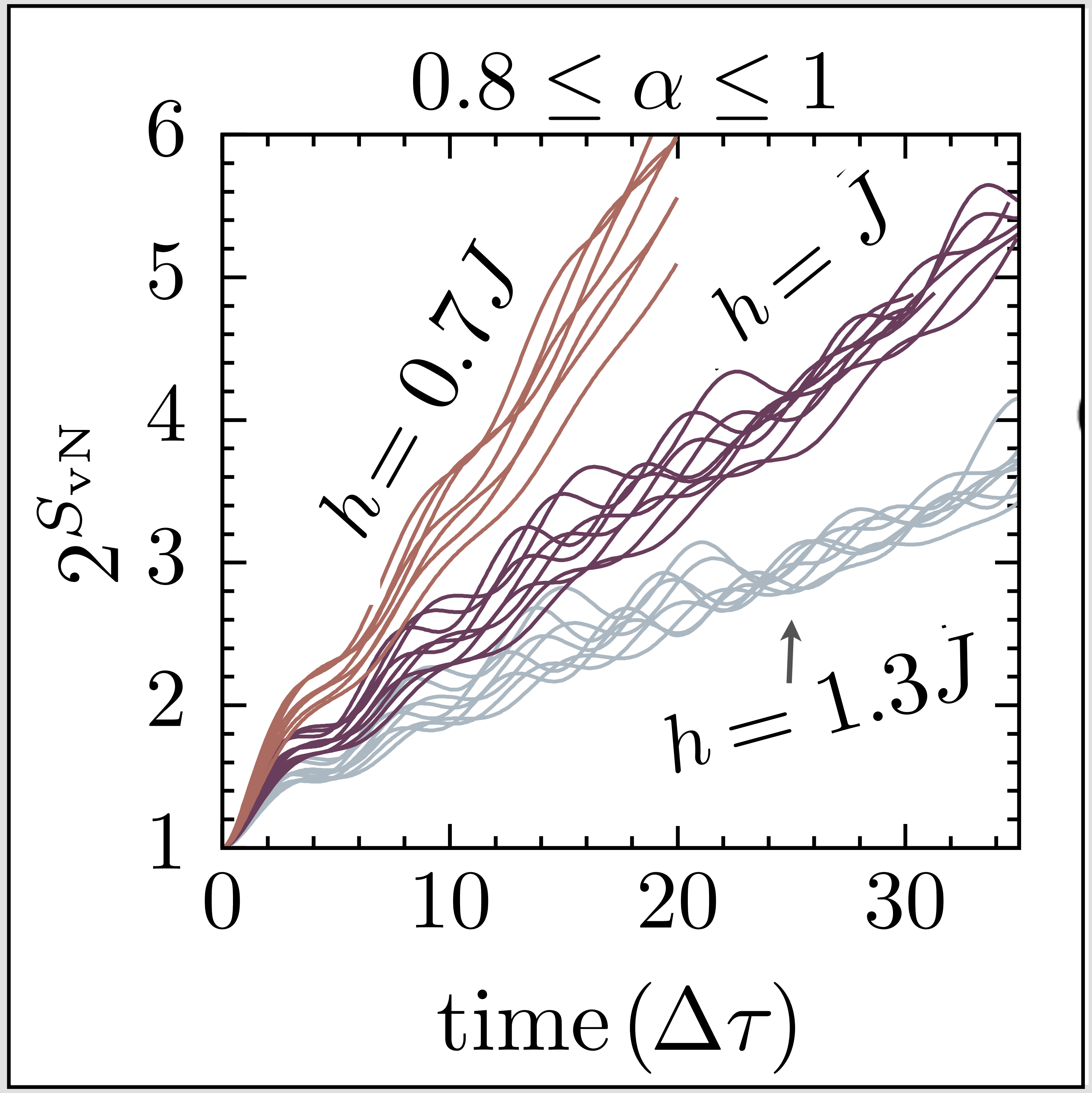
Typical behavior of spatiotemporal correlations

(Picture from Lepori, Trombettoni, Vodola, JStat '17)

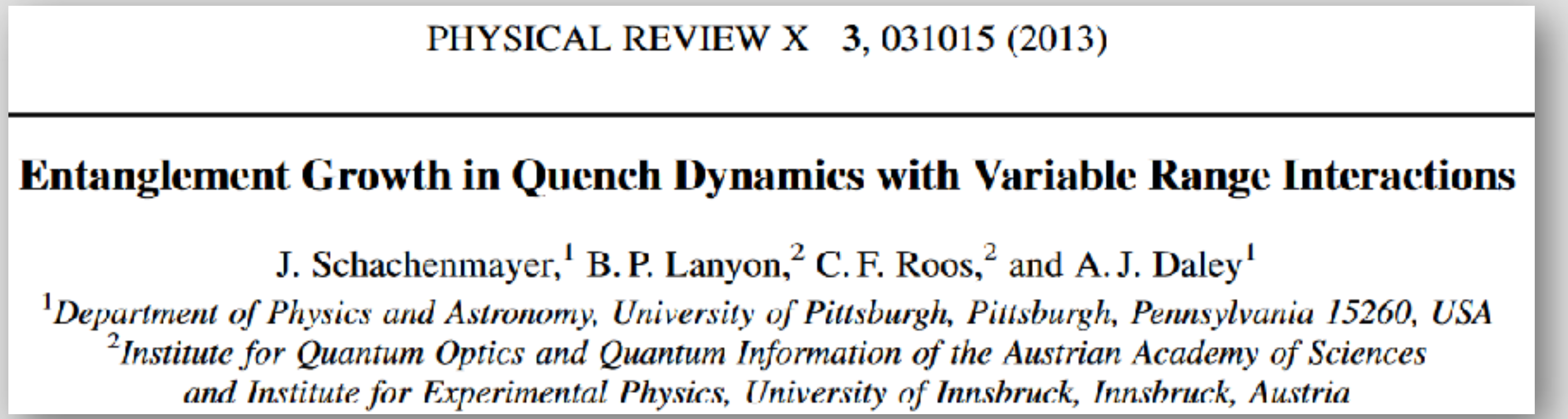
[Hauke, Tagliacozzo, Eisert, Lewenstein, Kastner, Gorshkov, Carleo, Cevolani, Sanchez-Palencia, Wouters, Essler, Daley, Rey, Roscilde, Pupillo, Frerot, Naldesi, Trombettoni, Nakamura, Nayak, Yao,…]

Long-range chains: hints from numerics

Quench from $|\psi_0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$ with $\hat{H} = -J\sum_{i\neq j}^{N} \frac{\hat{\sigma}_i^x \hat{\sigma}_j^x}{|i-j|^\alpha} - h\sum_i^N \hat{\sigma}_i^z$



$N = 30,\, 40,\, 50 \quad D_{\mathrm{MPS}} = 120$



PHYSICAL REVIEW X **3**, 031015 (2013)

Entanglement Growth in Quench Dynamics with Variable Range Interactions

J. Schachenmayer,[1] B. P. Lanyon,[2] C. F. Roos,[2] and A. J. Daley[1]

[1]*Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA*
[2]*Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences and Institute for Experimental Physics, University of Innsbruck, Innsbruck, Austria*

different behavior. Counterintuitively, quenches above the critical point for these long-range interactions lead only to a logarithmic increase of bipartite entanglement in time, so that in this regime, long-range interactions produce a slower growth of entanglement than short-range interactions. This can be understood by the fact that the dynamics

[Schachenmayer, Lanyon, Roos, Daley - Phys. Rev. X, 2013]
[Buyskikh, Fagotti, Schachenmayer, Essler, Daley - Phys. Rev. A, 2016]

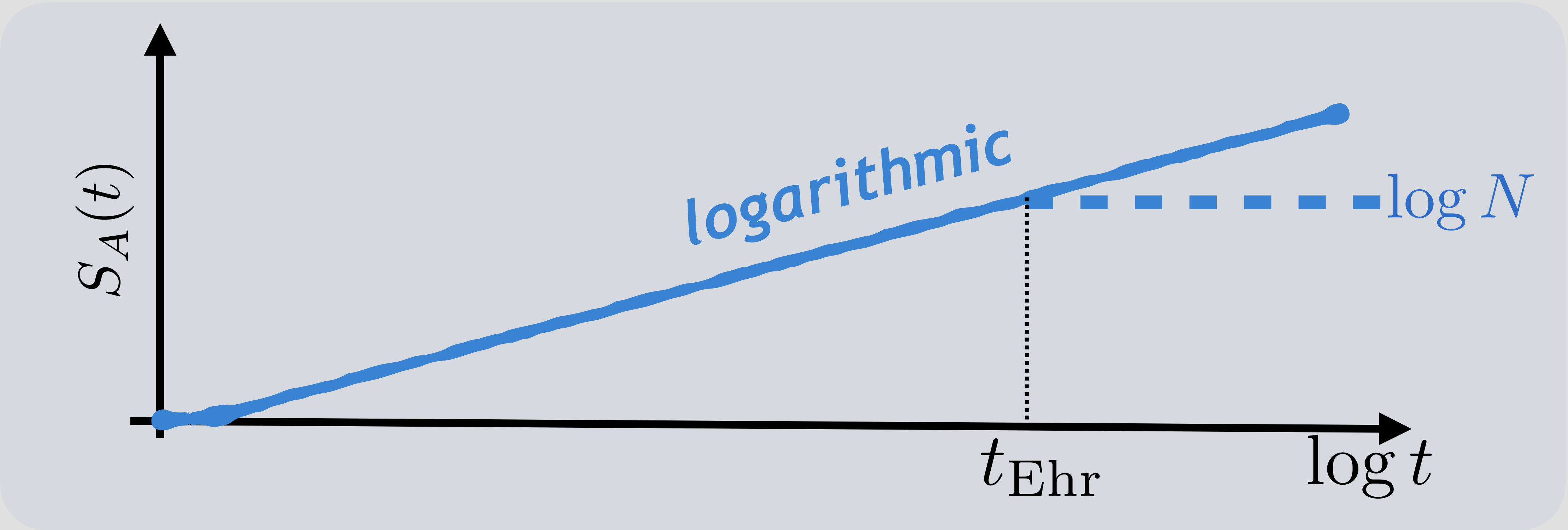
Origin of the slow growth of entanglement entropy in long-range interacting systems

Goal: understand why this happens.

$0 < \alpha < d$

✗ Breakdown of the quasi-particle picture

✓ Dominated by semi-classical **collective squeezing induced entanglement** growth → different mechanism!



Semi-classical entanglement dynamics $(\alpha = 0)$

Dynamics with all-to-all interactions ($\alpha = 0$)

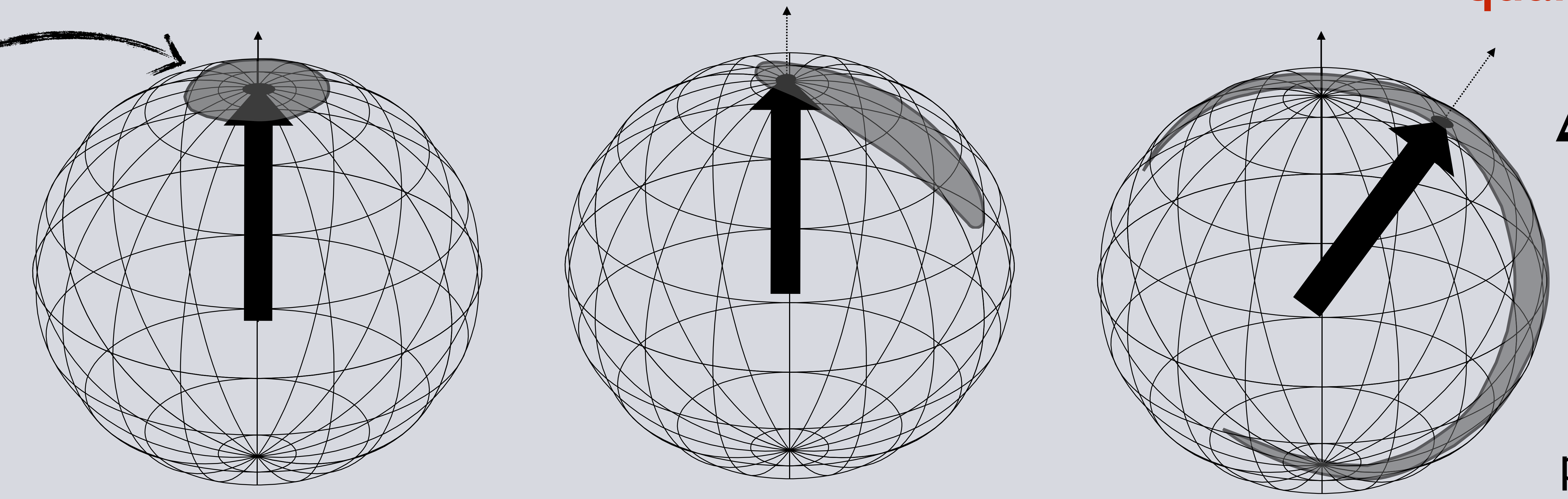
$$\hat{H} = -\frac{J}{N}\sum_{ij}^{N} \hat{\sigma}_i^x \hat{\sigma}_j^x - h\sum_{i}^{N} \hat{\sigma}_i^z$$

$\propto \left(\widetilde{\sigma}^x_{k=0}\right)^2$ $\qquad \propto \widetilde{\sigma}^z_{k=0}$

- Collective spin $\vec{S} = \sum_{i=1}^{N} \vec{\sigma}_i \propto N$
- extensive
- conserved $\left[\,\left|\vec{S}\right|^2, H\right] = 0$
- small Hilbert space $dim\mathscr{H} = N + 1$

Nonequilibrium Dynamics = Classical trajectory on the sphere

$\Delta \sim \hbar_{eff} \sim \hbar / N$



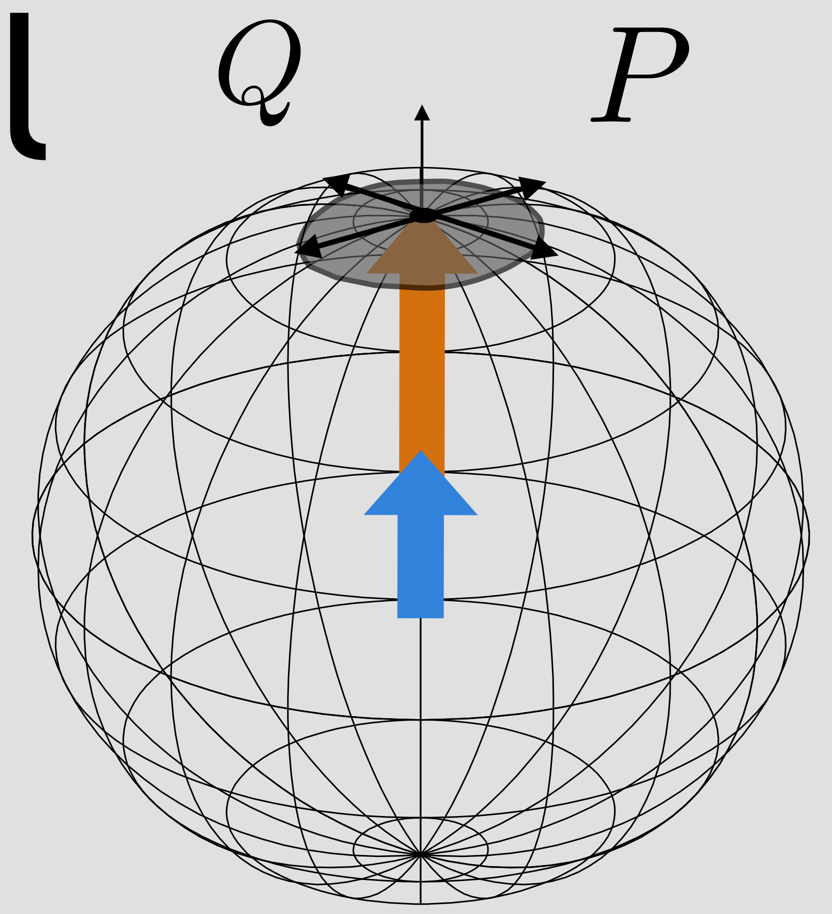
quantum fluctuations!

$\Delta \sim 1$

Ehrenfest time t_{Ehr}
purely quantum dynamics

[Polkovnikov - Annals of Physics, 2010]

Entanglement dynamics of a collective model



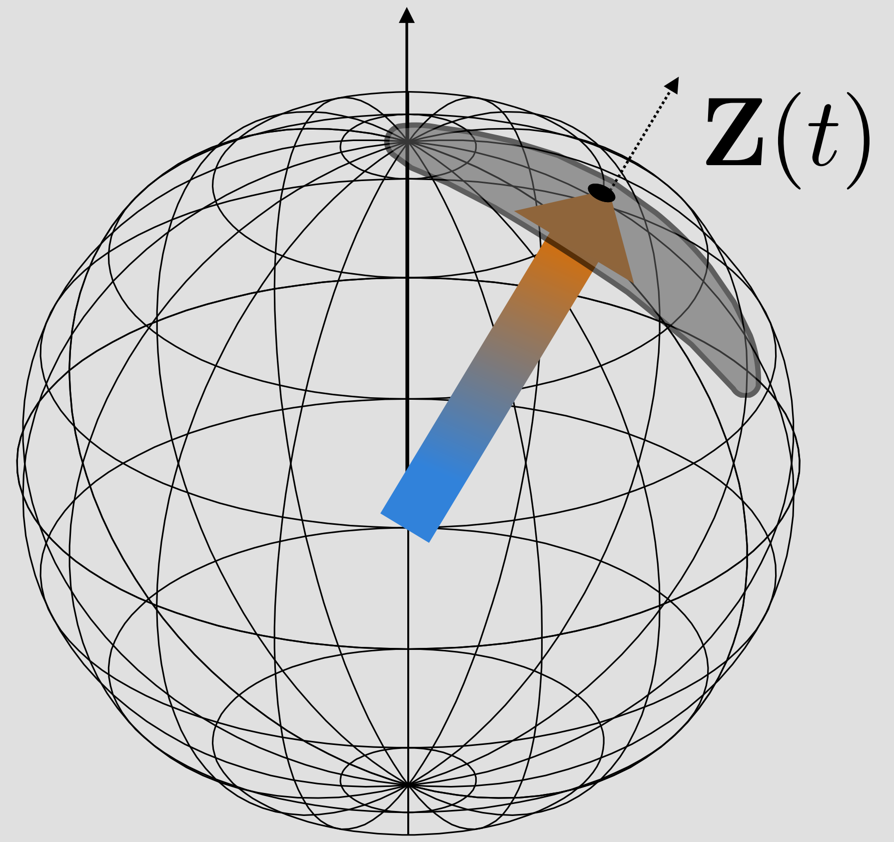
- decompose the collective spin $\hat{\mathbf{S}} = \hat{\mathbf{S}}_A + \hat{\mathbf{S}}_B$

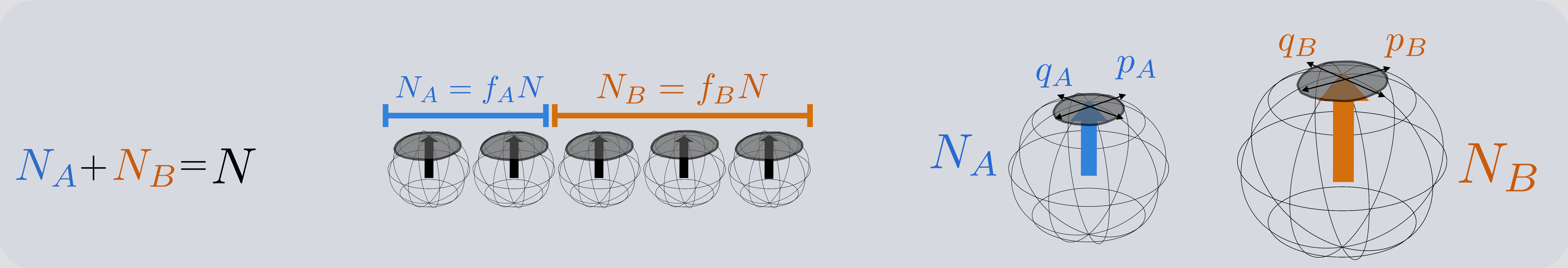
[Vidal, Dusuel, Barthel - JSTAT, 2007]

- Holstein-Primakoff: treat spin fluctuations as bosons

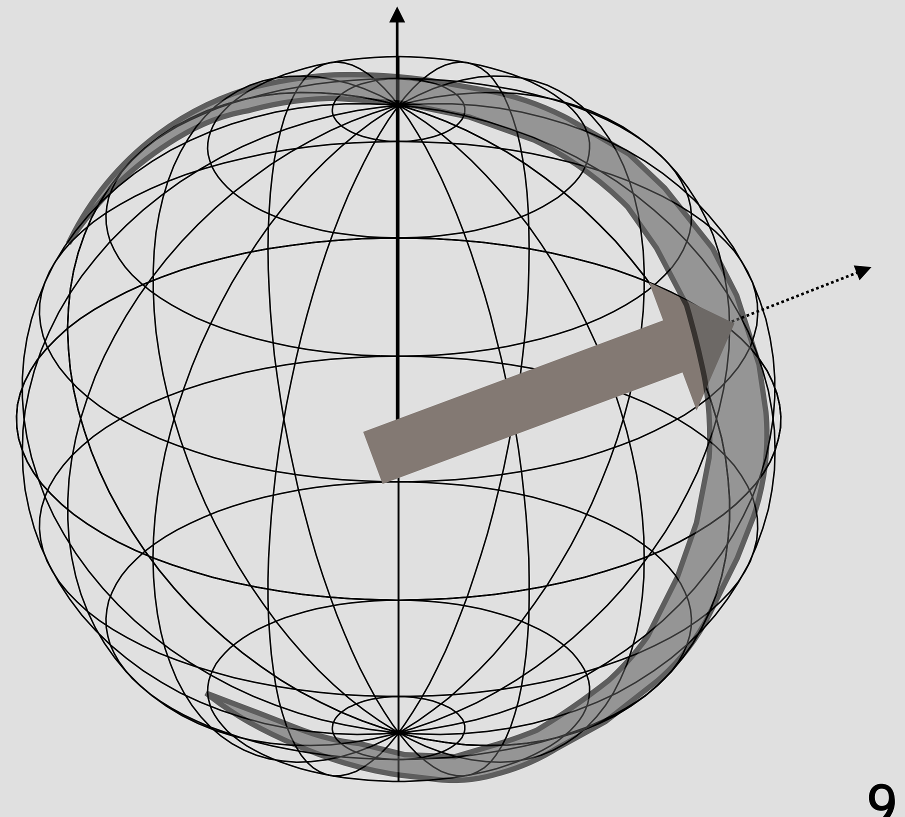
$(\hat{q}_A, \hat{p}_A)\ (\hat{q}_B, \hat{p}_B) \longleftrightarrow (\hat{Q}, \hat{P})$

collective fluctuations





- Work in the reference frame of the classical spin $\tilde{H}(t) = \hat{H} - \omega(t) \cdot \mathbf{S}$

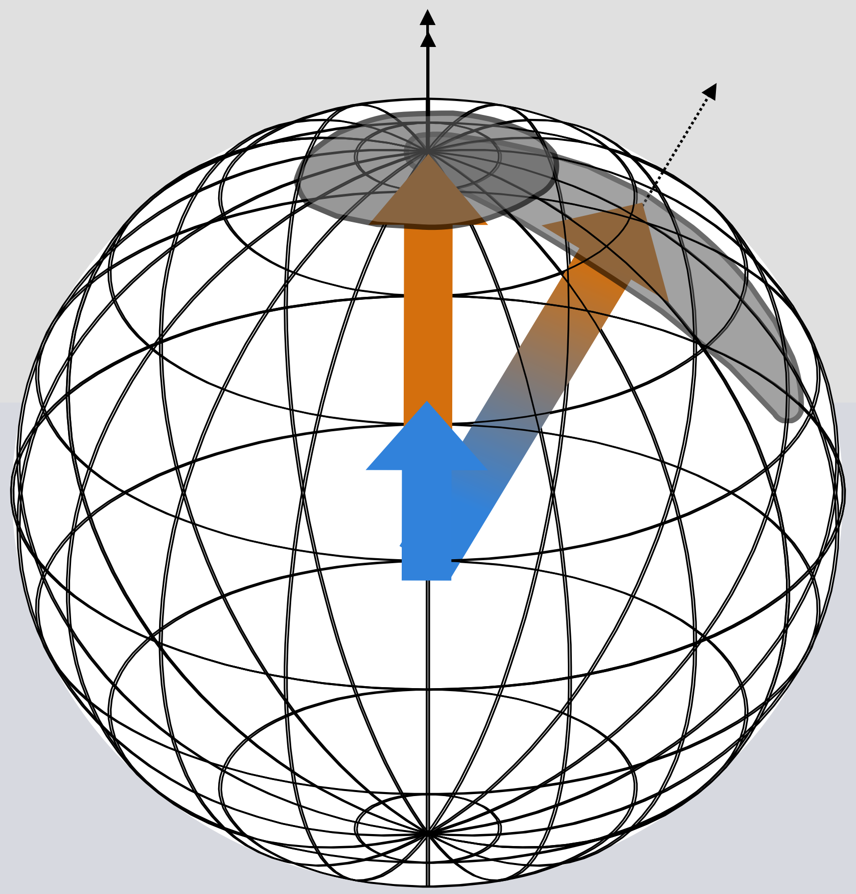


- quadratic Hamiltonian for the fluctuations

$$\tilde{\hat{H}}(t) = h^{(2)}_{QQ}(t)\,\frac{\hat{Q}^2}{2} + h^{(2)}_{PP}(t)\,\frac{\hat{P}^2}{2} + h^{(2)}_{QP}(t)\,\frac{\hat{Q}\hat{P} + \hat{P}\hat{Q}}{2} + \mathcal{O}\left(1/\sqrt{N}\right)$$

[Lerose, Marino, Žunkovič, Gambassi and Silva, PRL , 2018]
[SP, Russomanno, Žunkovič, Iemini, Silva, Fazio, PRB, 2018]

$S_A(t)$ and collective excitations



Entanglement between bosons (q_A, p_A) and (q_B, p_B)

the system is quadratic: $\hat{\rho}_A$ is gaussian

$$S_A = \sqrt{1+4f_A f_B\,\langle\hat{n}_{exc}\rangle}\,\text{arccoth}\left(\sqrt{1+4f_A f_B\,\langle\hat{n}_{exc}\rangle}\right) + \frac{1}{2}\log\left(f_A f_B\,\langle\hat{n}_{exc}\rangle\right)$$

$$\langle\hat{n}_{exc}\rangle = \frac{\langle Q^2\rangle + \langle P^2\rangle - 1}{2}$$

entangled $\langle\hat{n}_{\text{exc}}\rangle \gg 1$

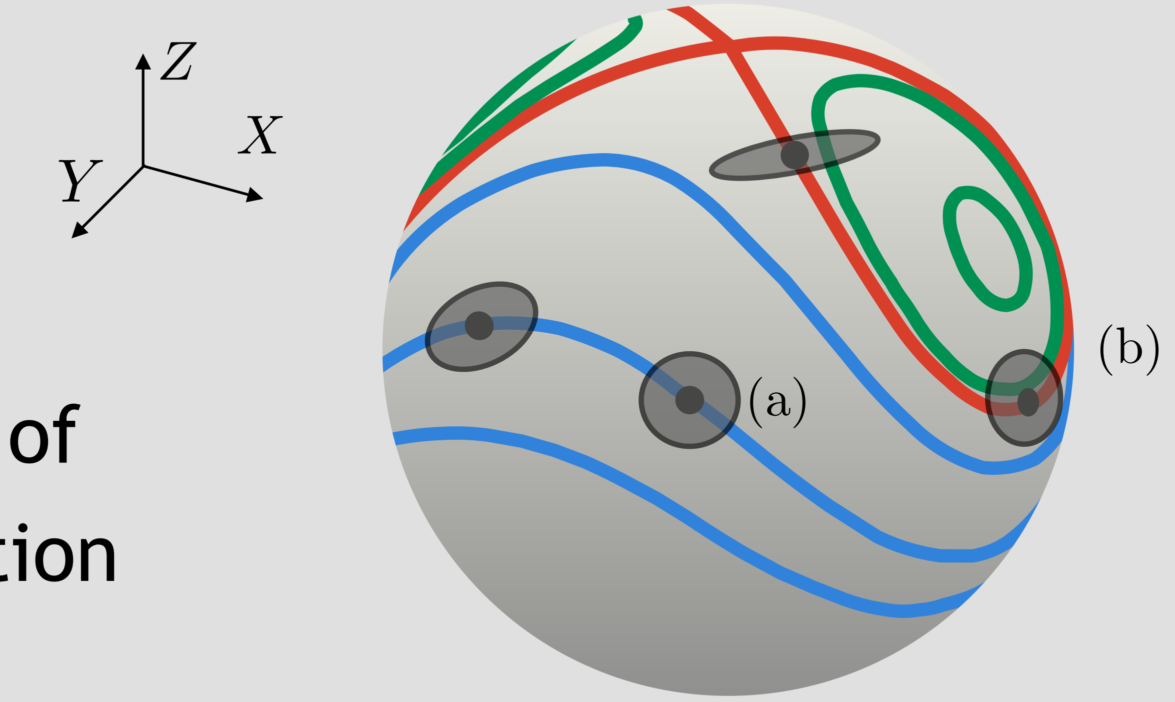
$$S_A \sim \frac{1}{2}\log\langle\hat{n}_{\text{exc}}\rangle + 1 + \frac{1}{2}\log f_A f_B$$

separable states $\langle\hat{n}_{\text{exc}}\rangle = 0$

$$S_A = 0$$

[Barthel, Chung, Schollwock - Phys. Rev. A, 2006]
[Hackl, Bianchi, Modak, Rigol - Phys.Rev.A, 2018]

Relation to semiclassical trajectories

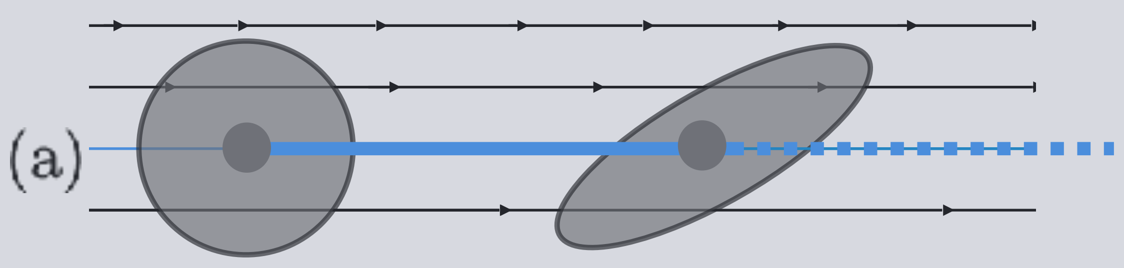


[Sciolla, Biroli - JSTAT, 2011]

$$S_A(t) \sim 1 + \frac{1}{2}\log f_A f_B + \frac{1}{2}\log\langle \hat{n}_{\text{exc}}(t)\rangle$$

The rate of $\langle \hat{n}_{exc}(t)\rangle$ is determined by the classical flow of the small displacements (Q, P) around the classical solution

Generic quenches

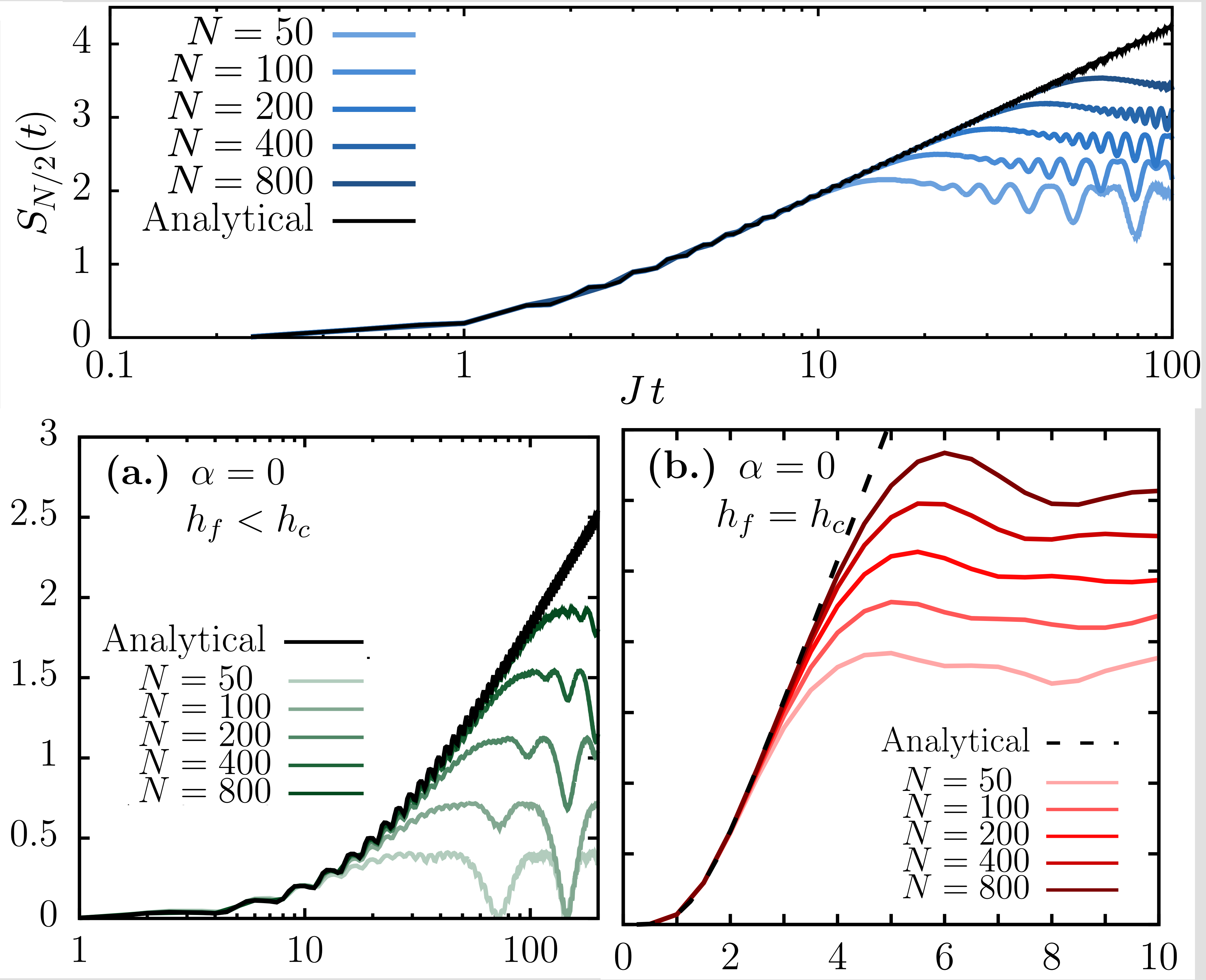


$$\langle \hat{n}_{\text{exc}}(t)\rangle \sim t^2$$

$$S_A(t) \sim \log t$$

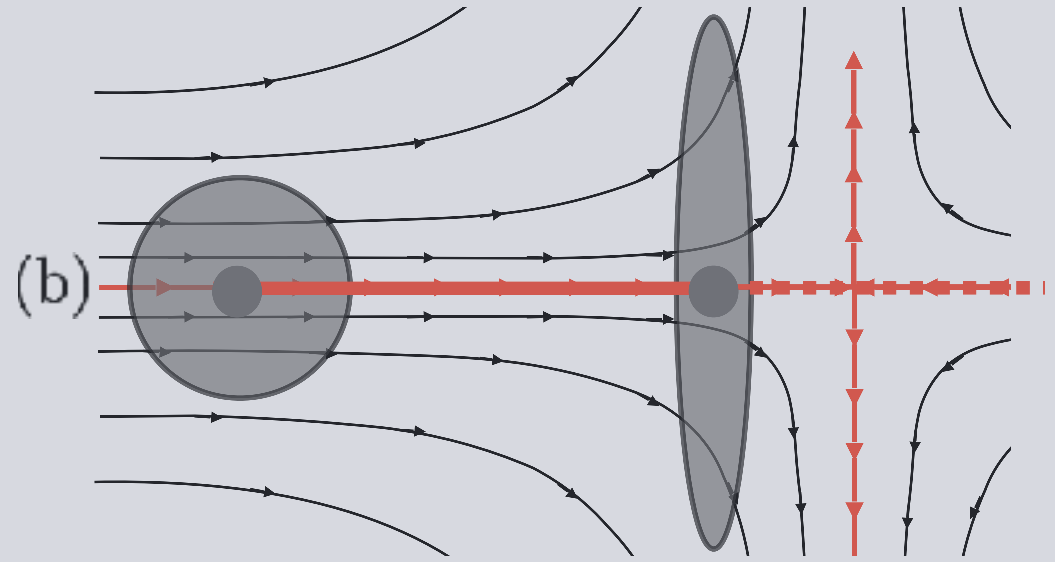
$$t_{\text{Ehr}} \sim \sqrt{N}$$

Numerical simulations for fully-connected Ising model



validity before the **Ehrenfest time** $\langle \hat{n}_{\text{exc}}\rangle \sim N$

Unstable trajectory



$$\langle \hat{n}_{\text{exc}}(t)\rangle \sim e^{2\lambda t}$$

$$S_A(t) \sim \lambda_{h_c}\, t$$

$$t_{\text{Ehr}} \sim \log N$$

Entanglement dynamics in semi-classical models

[Lerose, Pappalardi, to appear]

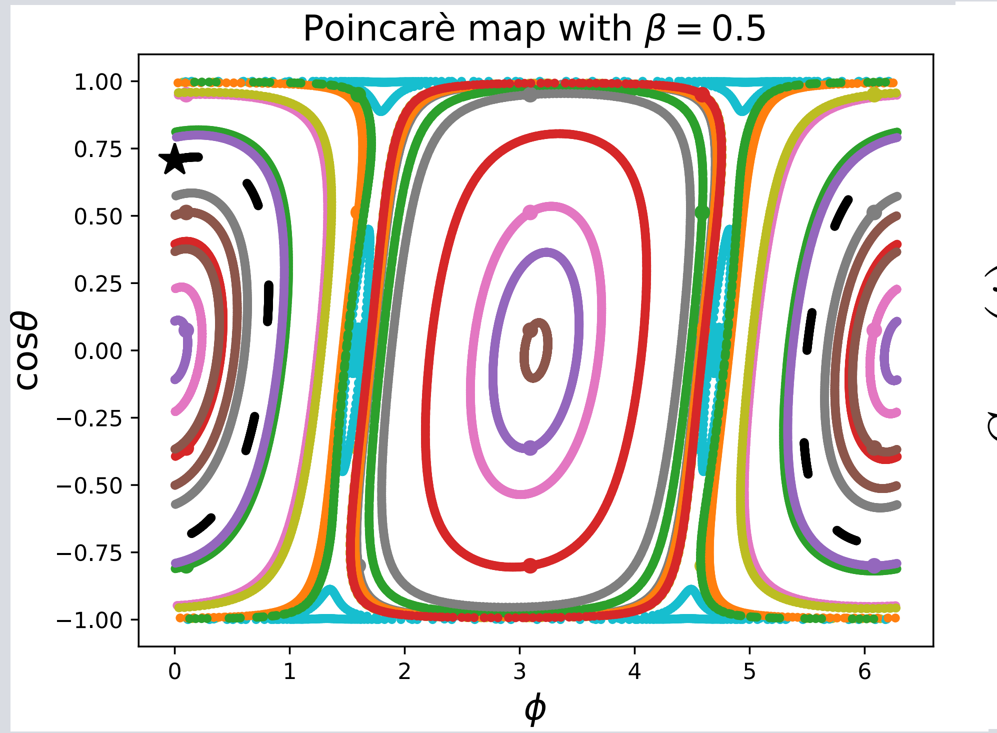
classical dynamics ⟷ bipartite entanglement (**entanglement entropy**), multipartite entanglement (**quantum Fisher information**) , otoc, etc.

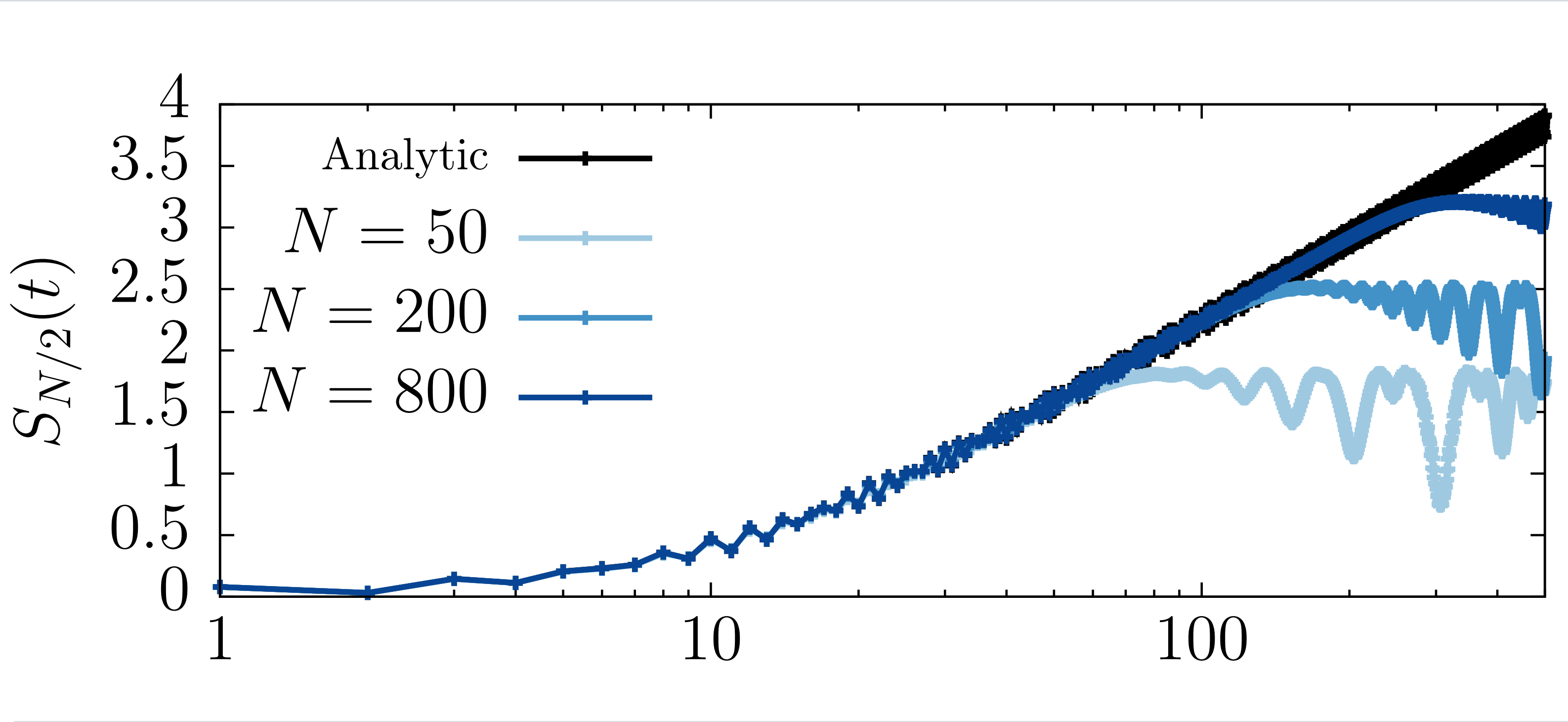
Example: Kicked top

$$\hat{H}(t) = \alpha \hat{S}_x + \frac{\beta}{2Ns}\hat{S}_z^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

Regular Phase (KAM)

$$S_{\text{ent}} \sim \log t$$

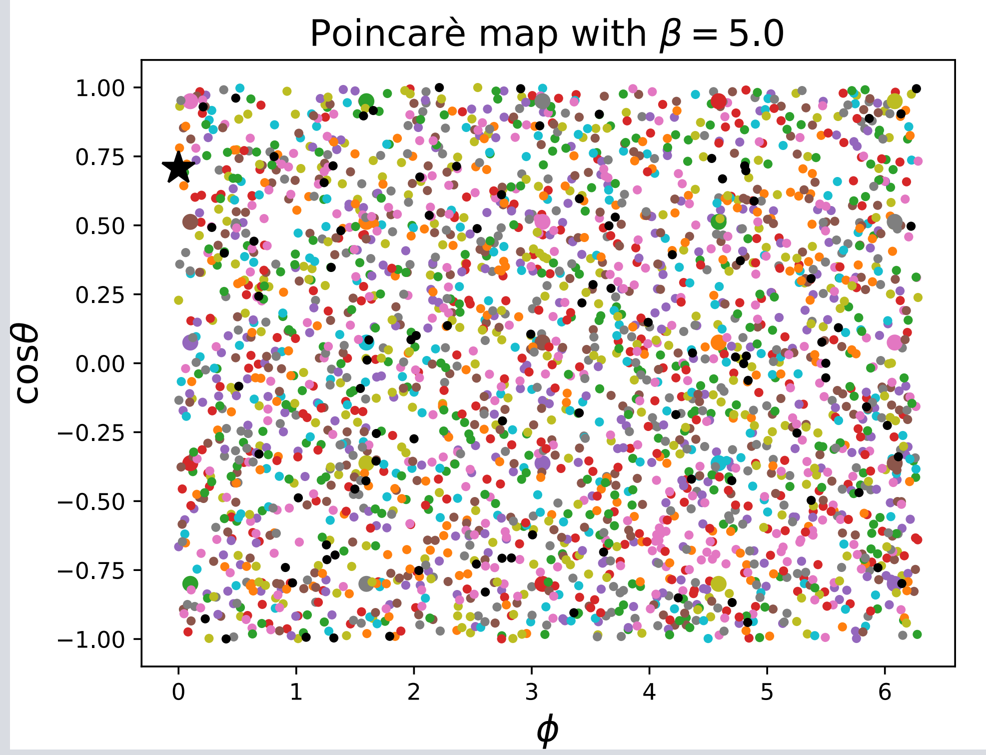


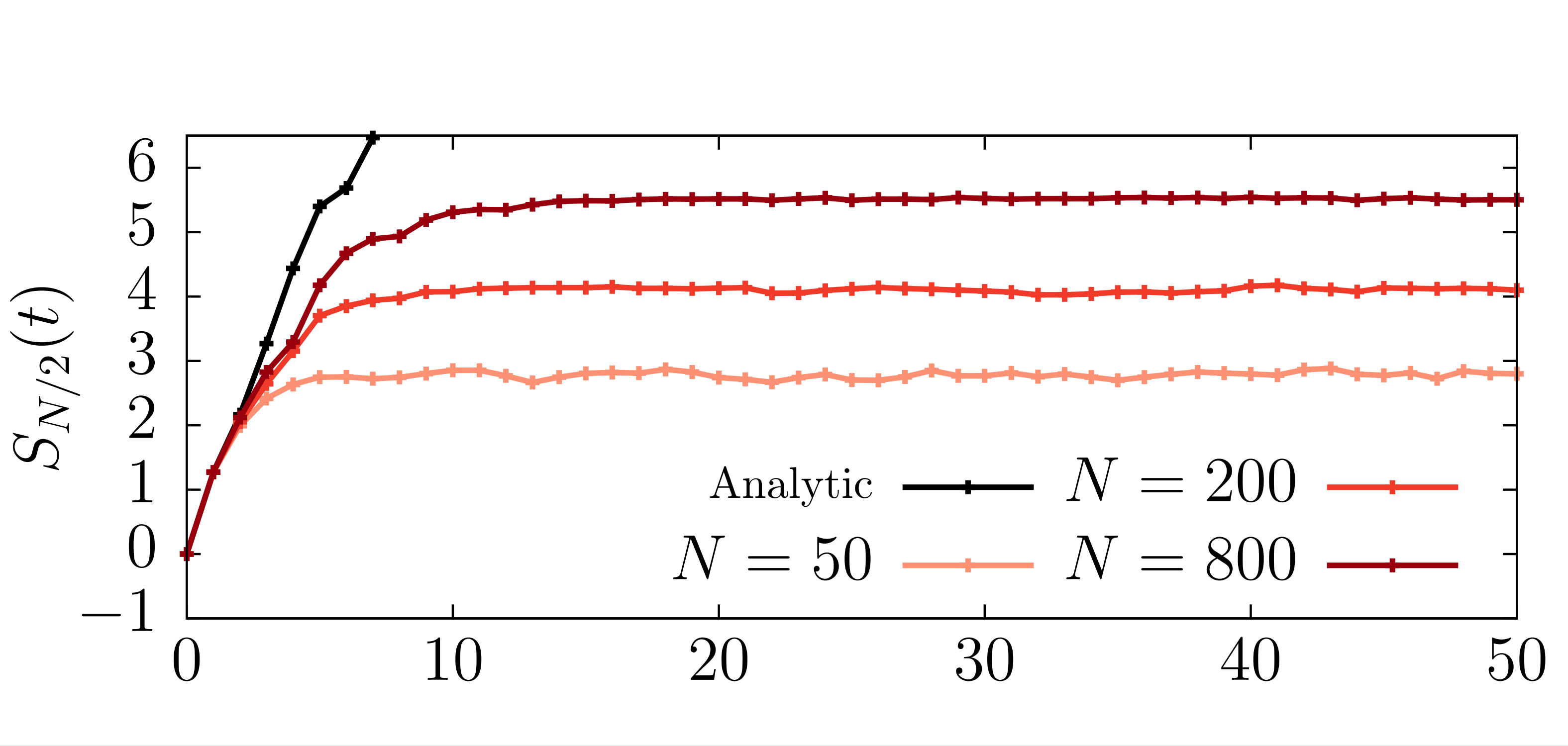


Chaotic Phase

Kolmogorov-Sinai entropy

$$S_{\text{ent}} \sim \left(\sum_{\lambda_i > 0} \lambda_i \right) t$$





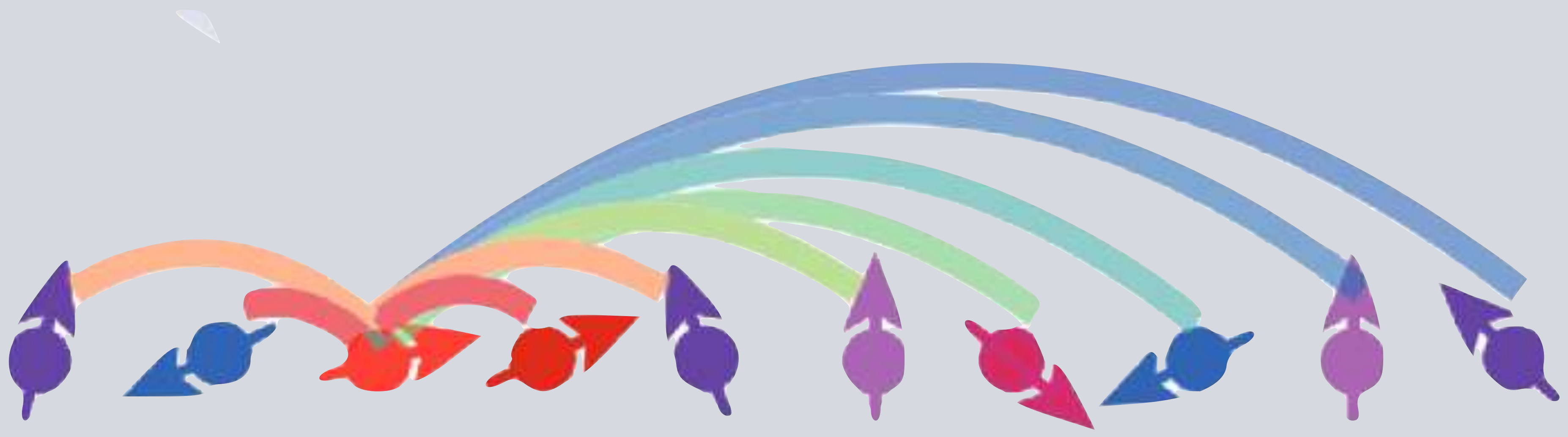
[Zurek, Paz - Physica D: Nonlinear Phenomena, 1995]
[Hackl, Bianchi, Modak, Rigol - Phys.Rev.A, 2018]

[Haake, Quantum signatures of chaos 1991]

Spatially decaying interactions $\alpha \neq 0$

Spatially decaying interactions

$$\hat{H} = -\frac{J}{\mathcal{N}_{\alpha,N}} \sum_{i\neq j} \frac{\hat{\sigma}_i^x \hat{\sigma}_j^x}{|\mathbf{r}_i - \mathbf{r}_j|^\alpha} - h \sum_i \hat{\sigma}_i^z$$



Kač normalization $\mathcal{N}_{\alpha,N} = \frac{1}{N} \sum_{i\neq j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|^\alpha}$ $\quad \mathcal{N}_{0,N} = N$

$$\tilde{\hat{H}}_\alpha(t) = \tilde{\hat{H}}_0(t) + \hat{H}_{\mathrm{sw}}(t)$$

Approach

$k = 0$ *collective mode*

$$\hat{H} = -\frac{1}{N} \sum_k \tilde{J}_k(\alpha)\, \tilde{\sigma}_k^x\, \tilde{\sigma}_{-k}^x - h\, \tilde{\sigma}_{k=0}^z$$

- Fourier Transform $\mathbf{k} = \frac{2\pi}{L}(n_1, \ldots, n_d),\ \ n_\mu = 0, 1, \ldots, L-1$
- Time-dependent rotation $\tilde{\hat{H}}(t) = \hat{H} - \boldsymbol{\omega}(t) \cdot \hat{\mathbf{S}}$
- Holstein-Primakoff on the individual spins $\tilde{q}_{\mathbf{k}} = L^{-d/2} \sum_j e^{-i\mathbf{k}\cdot\mathbf{r}_j} \hat{q}_j \quad \tilde{p}_{\mathbf{k}} = L^{-d/2} \sum_j e^{-i\mathbf{k}\cdot\mathbf{r}_j} \hat{p}_j$

Spatially decaying interactions

$$\tilde{\hat{H}}_\alpha(t) = \tilde{\hat{H}}_0(t) + \hat{H}_{\mathrm{sw}}(t)$$

zero mode Hamiltonian

integrable

$$\tilde{\hat{H}}_{\alpha=0}(t)$$

$$\langle \hat{n}_{\mathrm{sw}} \rangle \equiv 0 \qquad [\hat{n}_{\mathbf{k}}, \tilde{\hat{H}}_0] = 0 \quad \text{for all} \quad \mathbf{k} \neq 0$$

$$\hat{n}_{\mathrm{sw}}(t) = \sum_{\mathbf{k}\neq\mathbf{0}} \hat{n}_{\mathbf{k}} \equiv \sum_{\mathbf{k}\neq\mathbf{0}} \frac{\tilde{q}_{\mathbf{k}}\tilde{q}_{-\mathbf{k}} + \tilde{p}_{\mathbf{k}}\tilde{p}_{-\mathbf{k}} - 1}{2}$$

spin-wave Hamiltonian

$$\hat{H}_{\mathrm{sw}}(t) = \sum_{\mathbf{k}\neq 0} \widetilde{f}_{\alpha,\mathbf{k}} \left[J_{qq}(t)\, \frac{\tilde{q}_{\mathbf{k}}\tilde{q}_{-\mathbf{k}}}{2} + J_{pp}(t)\, \frac{\tilde{p}_{\mathbf{k}}\tilde{p}_{-\mathbf{k}}}{2} + J_{qp}(t)\, \frac{\tilde{q}_{\mathbf{k}}\tilde{p}_{-\mathbf{k}} + \tilde{p}_{\mathbf{k}}\tilde{q}_{-\mathbf{k}}}{2} \right]$$

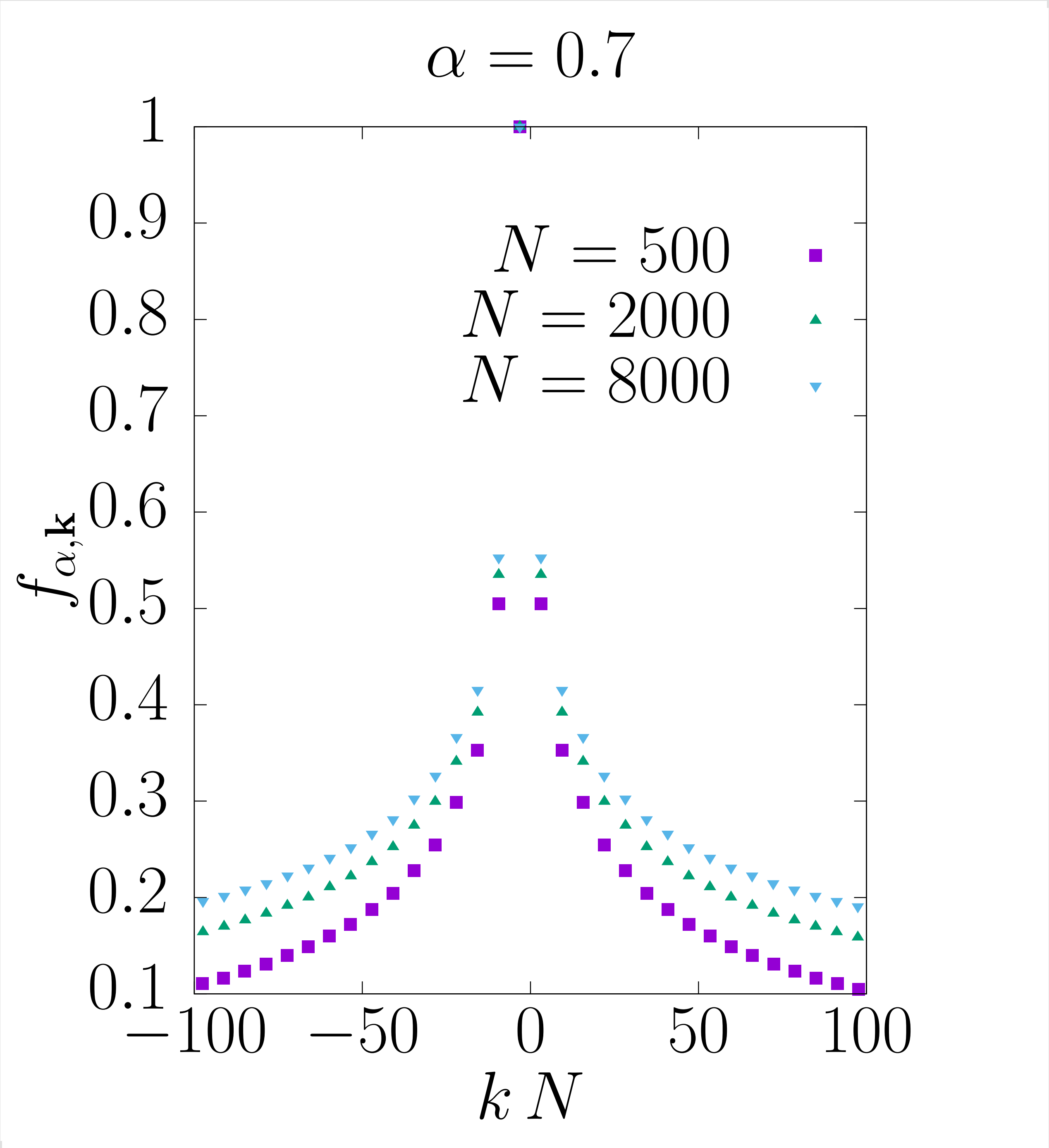
spin waves generated by the dynamics $\langle \hat{n}_{\mathrm{sw}}(t) \rangle \neq 0$

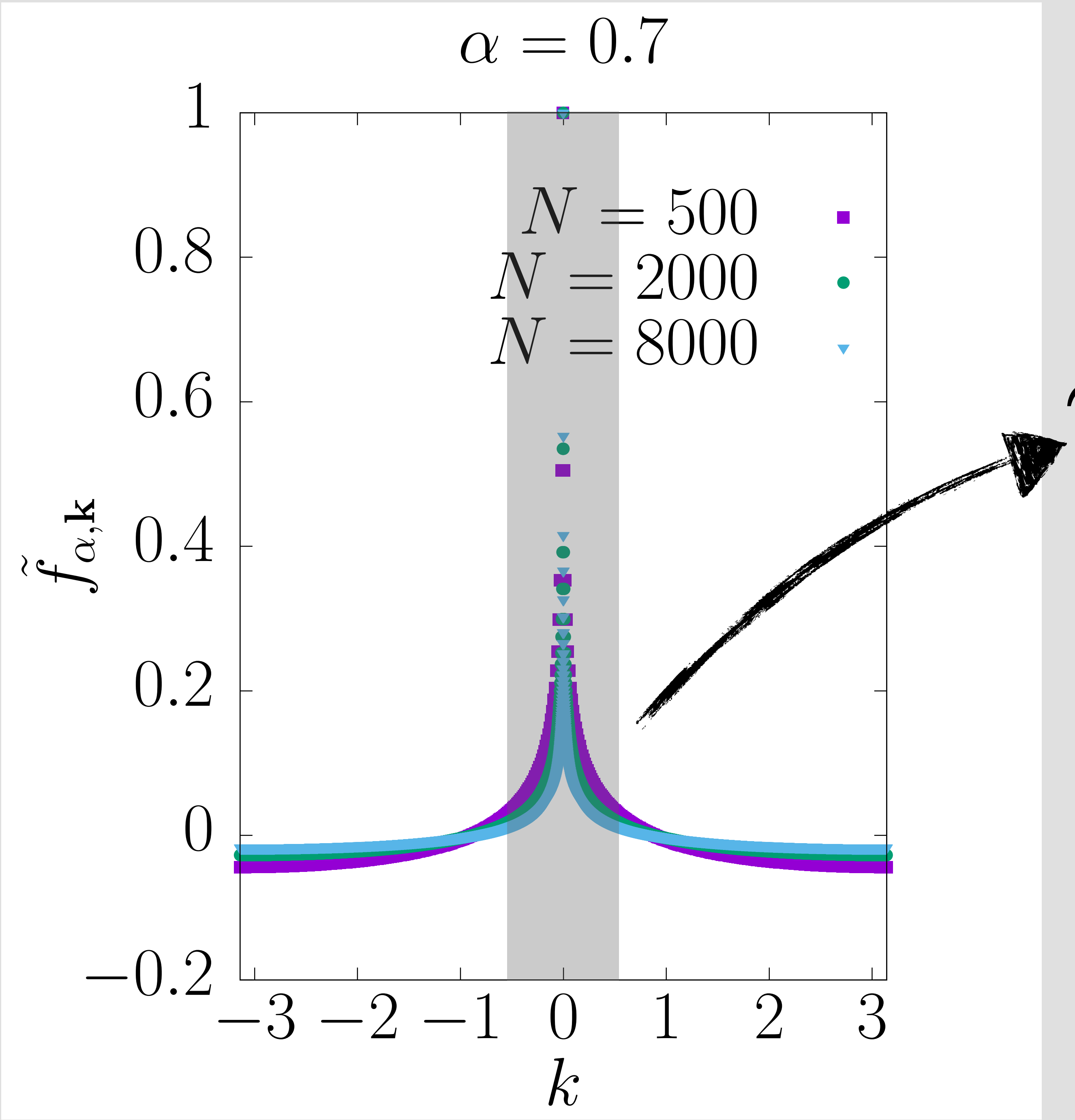
breaking of integrability

Quasi-conservation of spin waves for $\alpha \leq d$

$$\tilde{f}_{\alpha,k} = \frac{1}{\mathcal{N}_{\alpha,N}} \sum_{ij} \frac{e^{-i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)}}{|\mathbf{r}_i - \mathbf{r}_j|^{\alpha}}$$

small $k \sim \frac{1}{N}$





$$\sim \frac{J}{(|k|N)^{1-\alpha}}$$

Example:
1D Ising long-range

$$|\dot{n}_{\mathbf{k}\neq 0}(t)| = \left| \left\langle \left[n_{\mathbf{k}\neq\mathbf{0}}, \widetilde{H}(t) \right] \right\rangle \right| \sim \frac{J}{(|\mathbf{k}|L)^{\beta}} \qquad \beta \equiv \mathrm{Min}(d-\alpha, 1)$$

long **pre-thermalization regime** $T_{\text{pre-th}} \sim N^{\beta/d}$

- the system stays trapped near a small submanifold of the full Hilbert space $\sim N$
- Long-wavelength modes $k \sim 1/L$ might break permutation invariance.

[Mori - Journ. Phys. A, 2018]

Spin-waves contribution

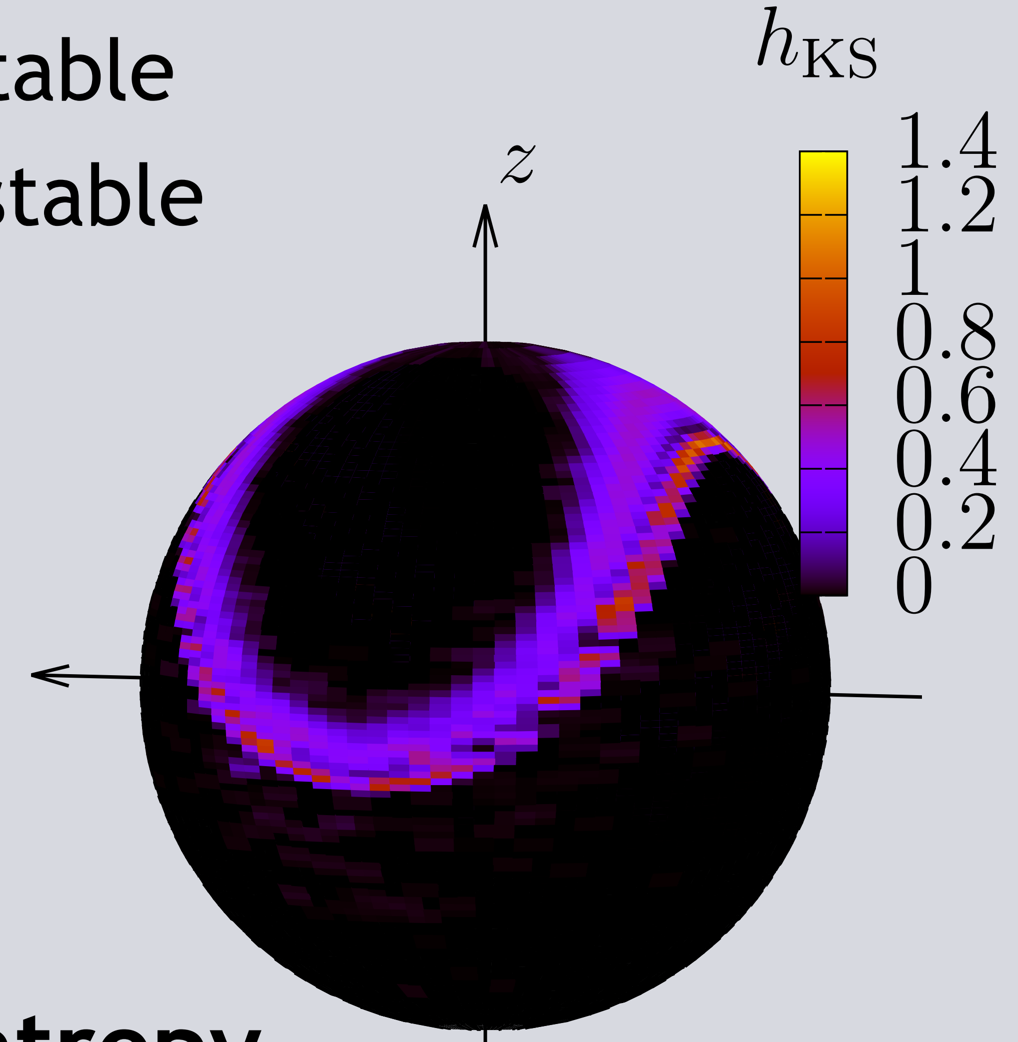
$$\tilde{\hat{H}}(t) = \tilde{\hat{H}}_0(t) + \sum_{\mathbf{k}\neq 0} \tilde{f}_{\alpha,\mathbf{k}} \left[J_{qq}(t)\,\frac{\tilde{q}_{\mathbf{k}}\tilde{q}_{-\mathbf{k}}}{2} + J_{pp}(t)\,\frac{\tilde{p}_{\mathbf{k}}\tilde{p}_{-\mathbf{k}}}{2} + J_{qp}(t)\,\frac{\tilde{q}_{\mathbf{k}}\tilde{p}_{-\mathbf{k}} + \tilde{p}_{\mathbf{k}}\tilde{q}_{-\mathbf{k}}}{2} \right]$$

The dynamics is described by a *discrete* set of **periodically driven harmonic oscillators** (drive = classical motion)

Stability analysis at the classical period T_{cl}

$e^{\pm\lambda_{\mathbf{k}} T_{cl}}$

- $\lambda_{\mathbf{k}} = i\omega_{\mathbf{k}}$ stable
- $\lambda_{\mathbf{k}}$ real, unstable

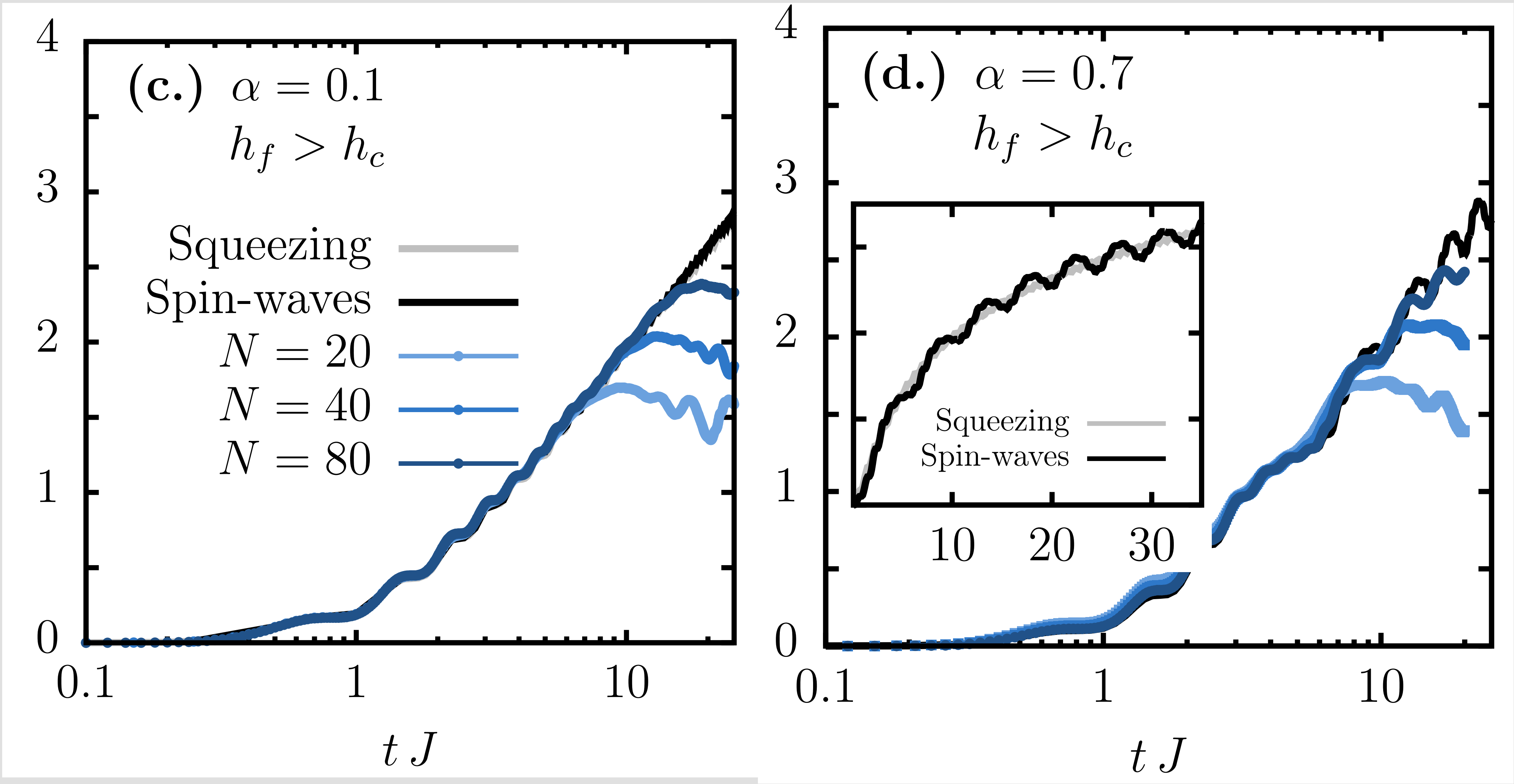


The Kolmogorov-Sinai entropy

$$h_{KS}(\theta_0, \phi_0) = \sum_k \Re[\lambda_k(\theta_0, \phi_0)]$$

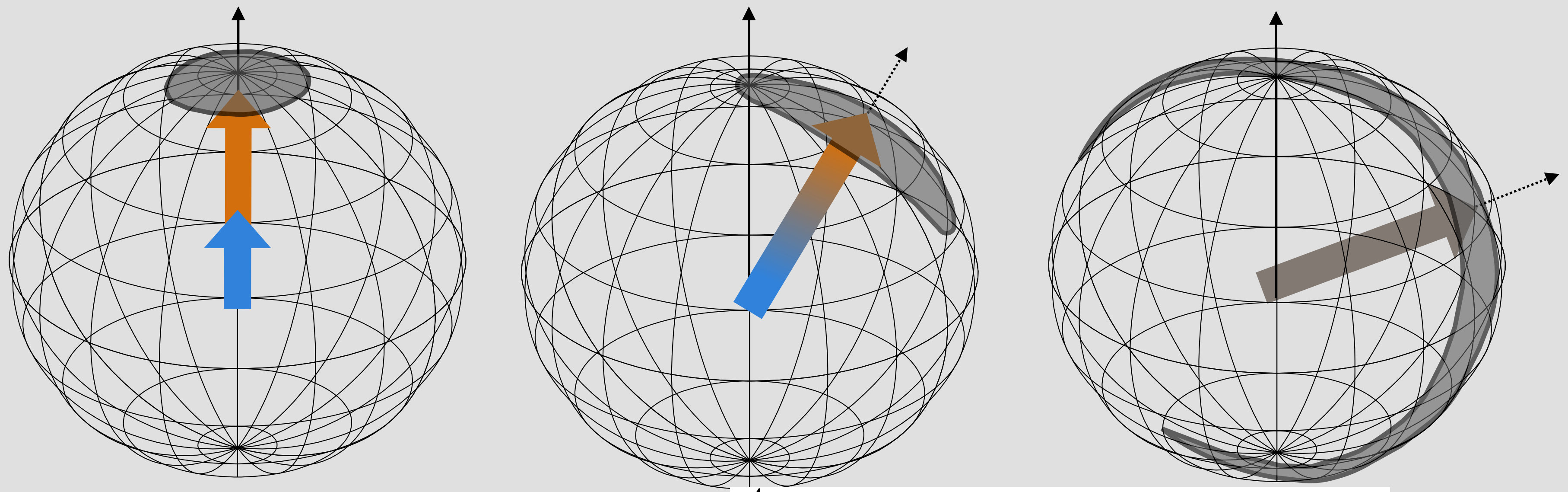
for different initial conditions

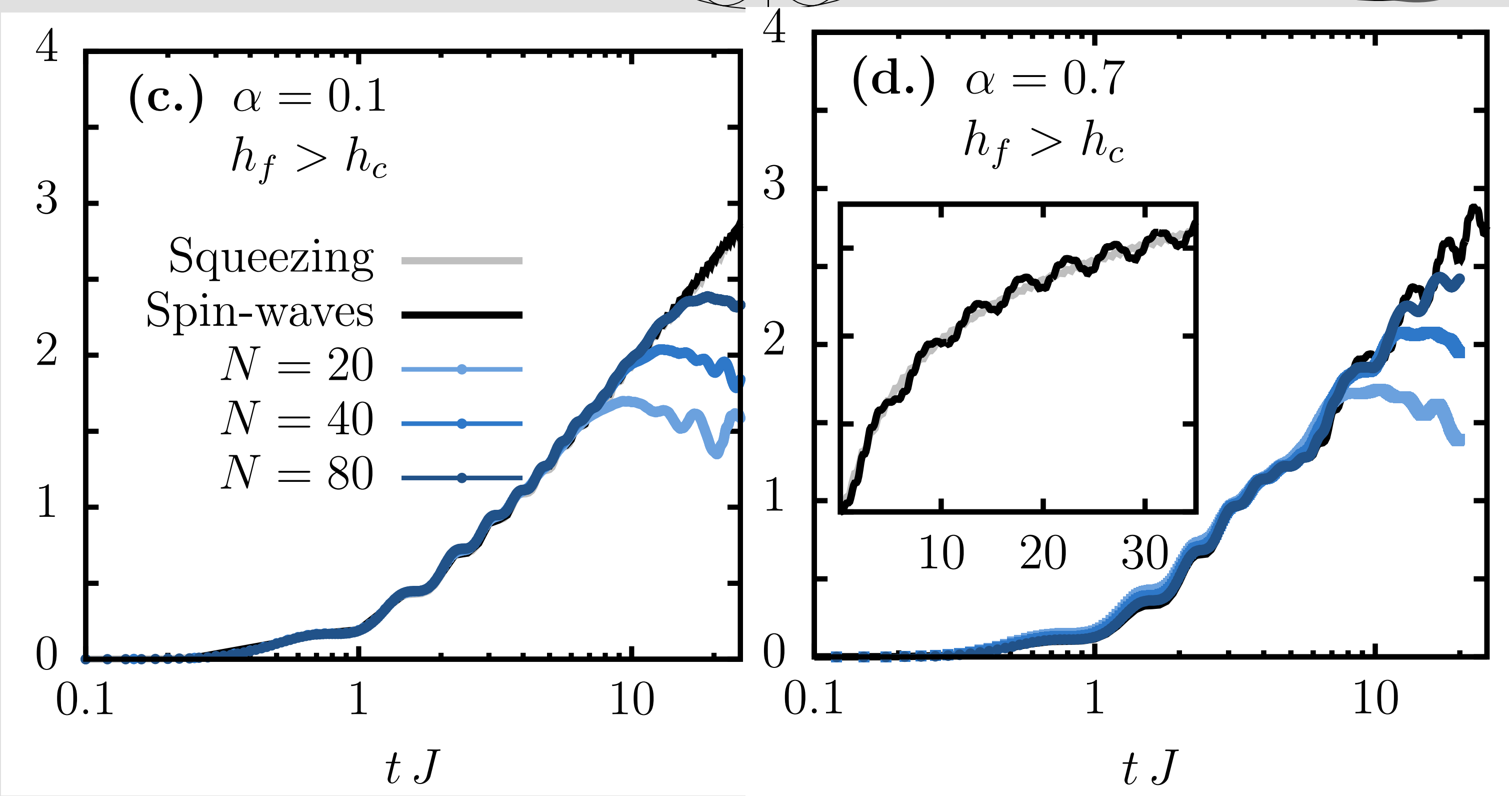
Numerical simulations by MPS-TDVP
(converged with bond dimension D=128)



new contributions with standard boson techniques
[Hackl, Bianchi, Modak, Rigol - Phys.Rev.A, 2018]

Conclusions





1. **semi-classical $S_A(t)$: collective squeezing induce**
   - Entanglement entropy directly **experimentally** measurable
   - connection with quantum Fisher information, otoc, etc.

2. **analytical $S_A(t)$ beyond the short-range paradigm with quasiparticle picture**
   - picture in the presence of a dominant zero-mode;
   - typical stability of spin-waves excitations (ergodicity breaking of long-range systems)
   - ‘efficiency’ of classical simulations: TDVP, CTWA etc