Classification of Matrix-Product Unitaries with Symmetries

Zongping Gong, Christoph Sünderhauf, Norbert Schuch, J. Ignacio Cirac

arXiv:1812.09183

SPT phase classification of MPUs & beyond

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Phases of matter

- Same phase↔continuously connect

Water

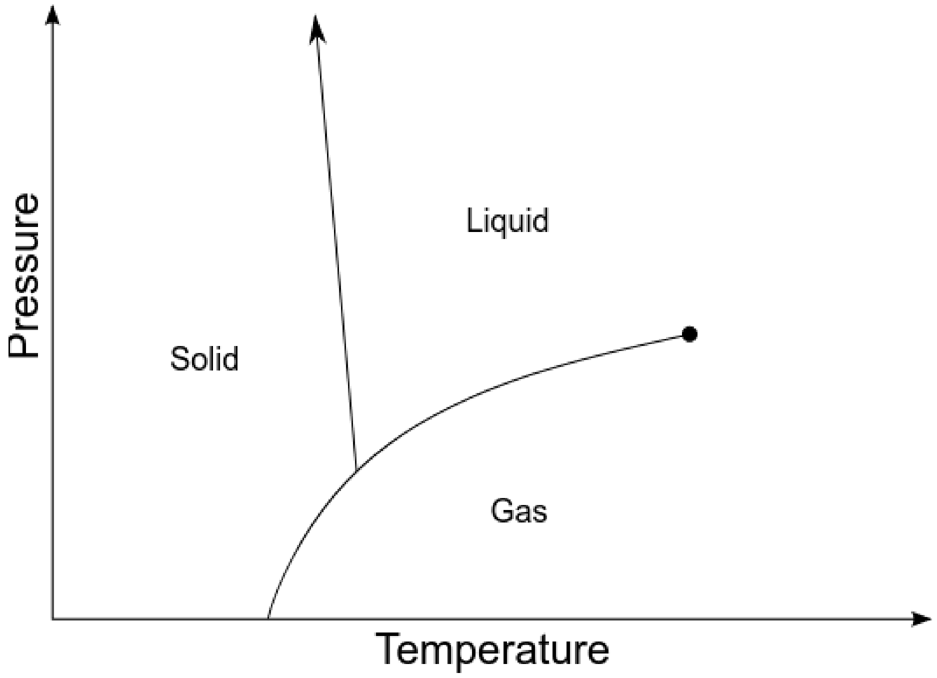
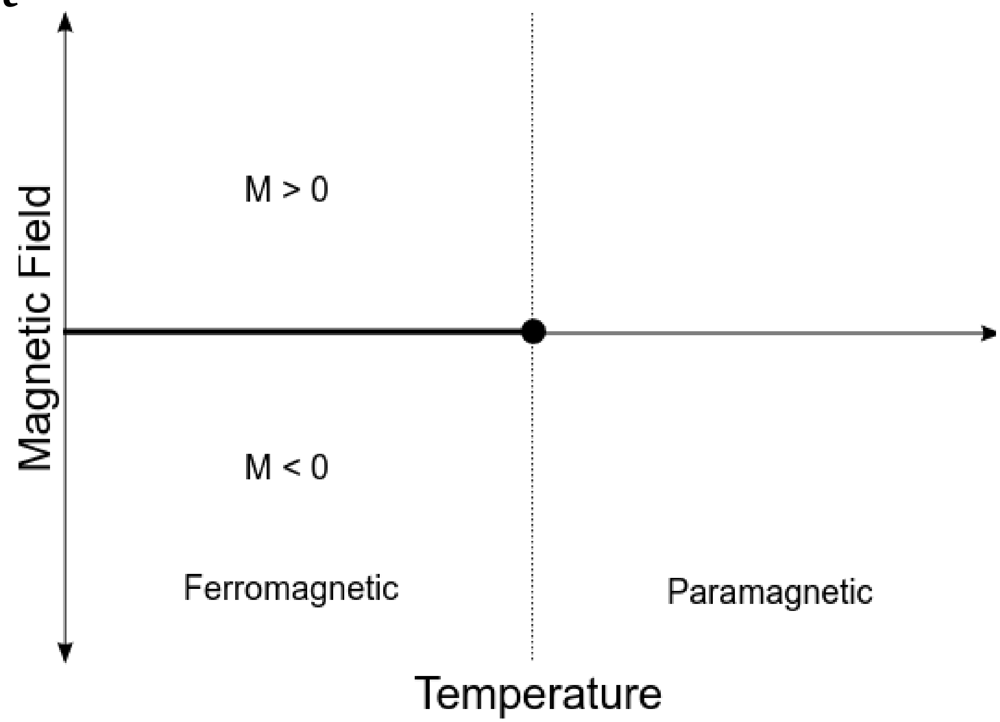


Image: https://web.stanford.edu/~peastman/statmech/phasetransitions.html

Phases of matter

- Same phase↔continuously connect
- 1937: Landau theory

Ising model



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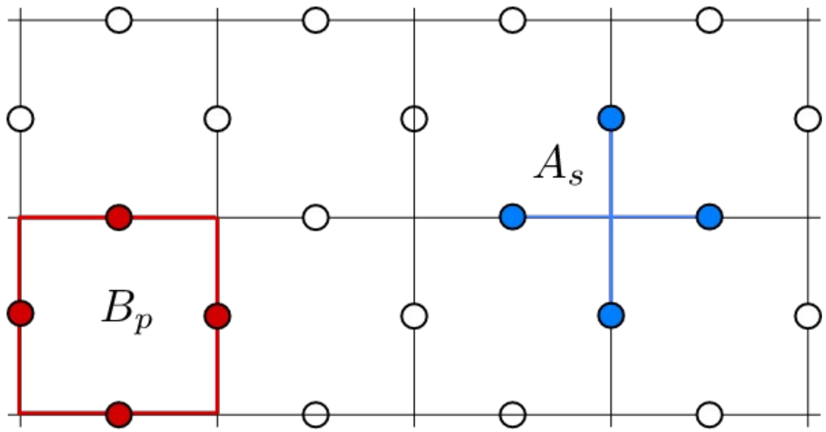
Phases of matter

- Same phase↔continuously connect
- 1937: Landau theory
- 1972: Topological order

Toric code



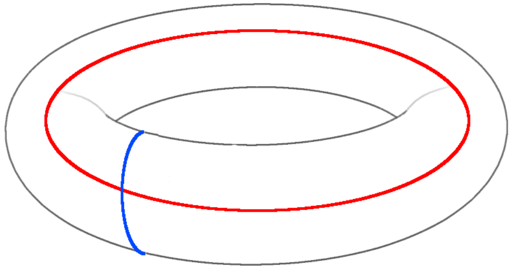


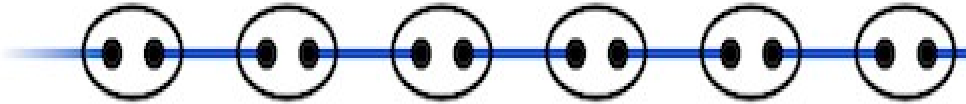
Image: https://en.wikipedia.org/wiki/Toric_code, https://web.stanford.edu/~peastman/statmech/phasetransitions.html

Phases of matter

- Same phase↔continuously connect
- 1937: Landau theory
- 1972: Topological order
- More recent: SPT (symmetry-protected topological) phases

  ⟶ continuously connect without breaking symmetry

AKLT model



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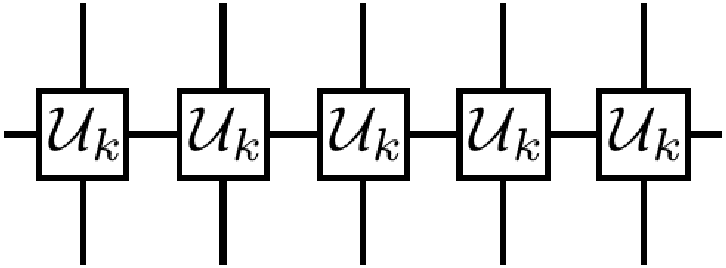
MPUs (Matrix-Product Unitaries)



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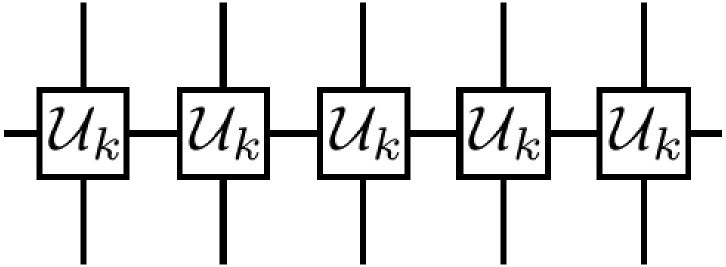
MPUs (Matrix-Product Unitaries)

Each MPU has a standard form with unitaries u,v



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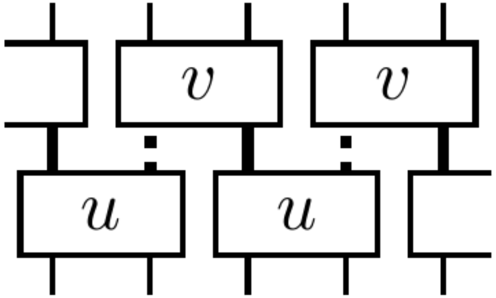

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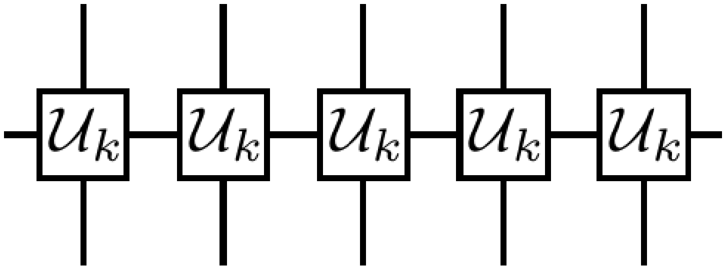

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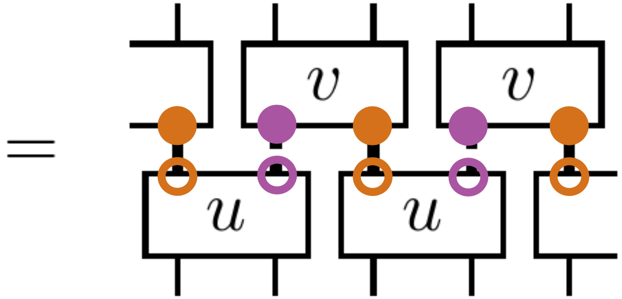
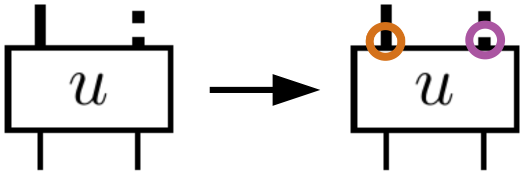
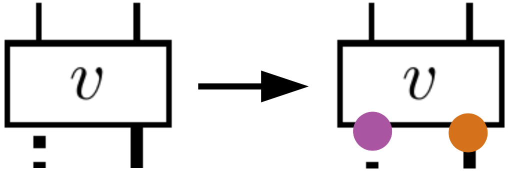


unique up to blocking & gauge transformations.





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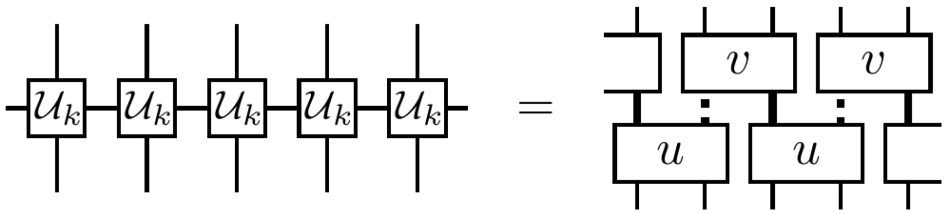
Connecting MPUs without symmetries



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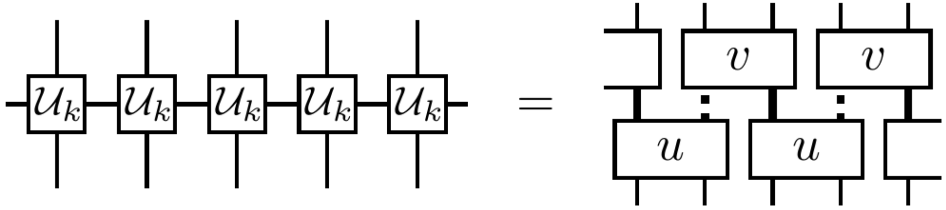
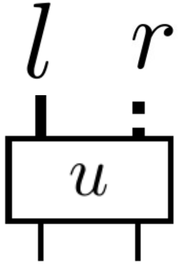


MPUs continuously connectable

$\iff$ same indices

Definition index



$$\text{ind} \equiv \frac{1}{2} \log \frac{r}{l}$$

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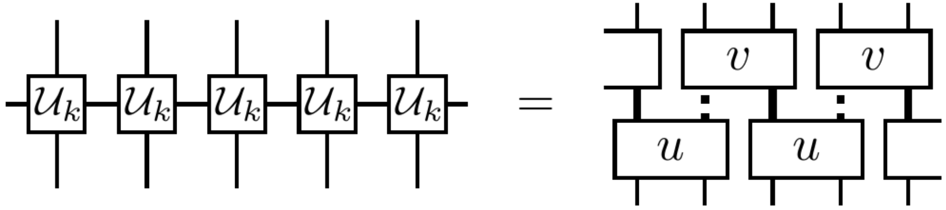
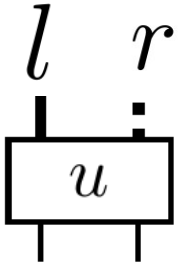


MPUs continuously connectable
$\Leftrightarrow$ same indices

r=2, l=1: 

Bonus: u, v continuous along path

Definition index



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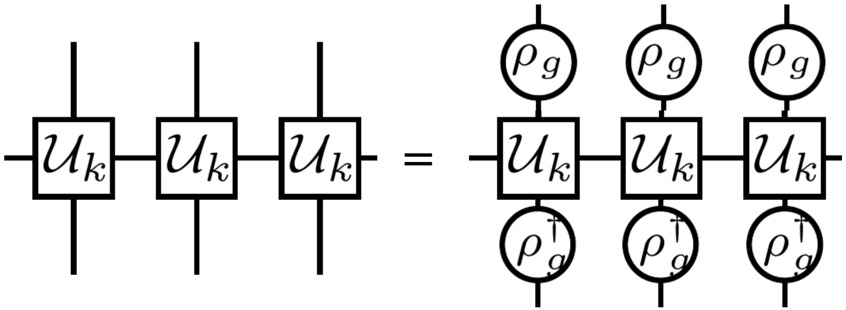
MPUs with symmetry G

Acts on MPU as a linear representation ρ_g

$$\rho_g \rho_h = \rho_{gh} \text{ for } g, h \in G$$



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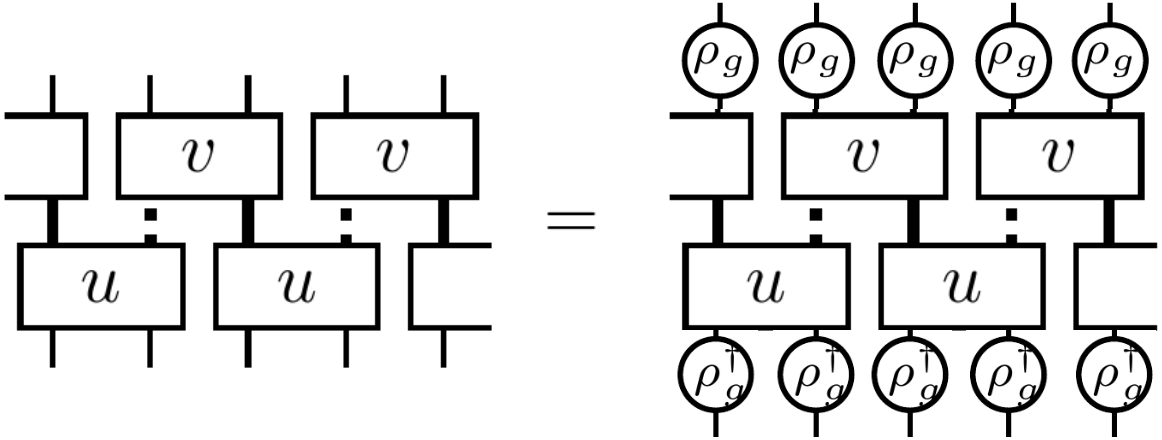

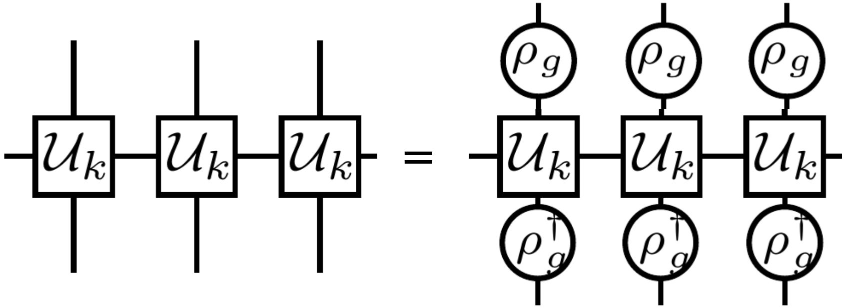

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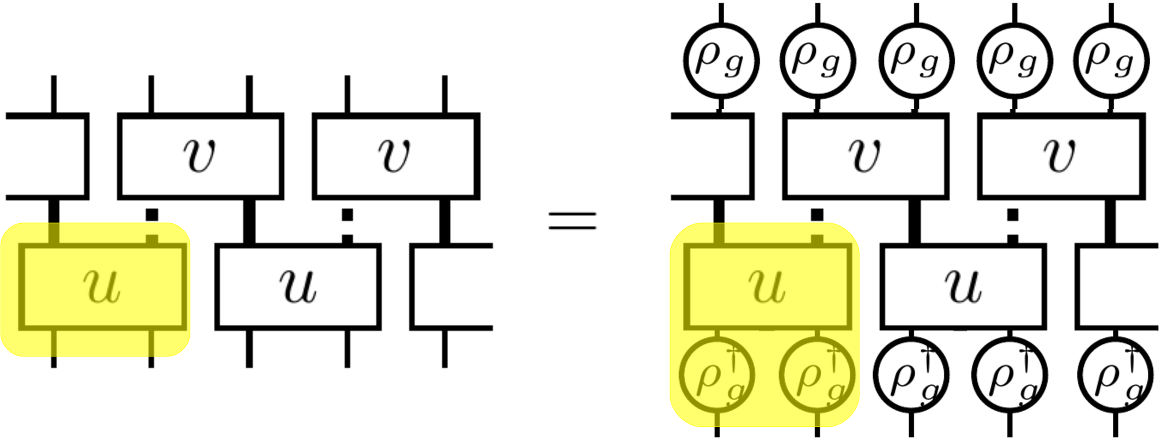

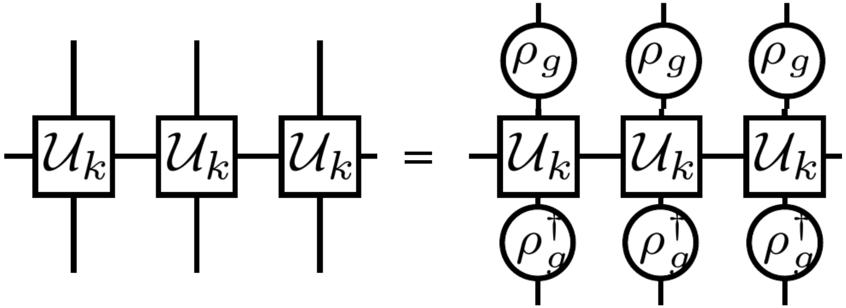

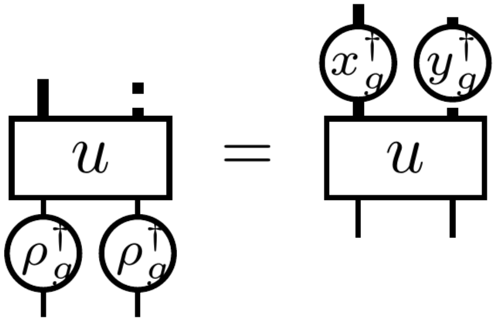

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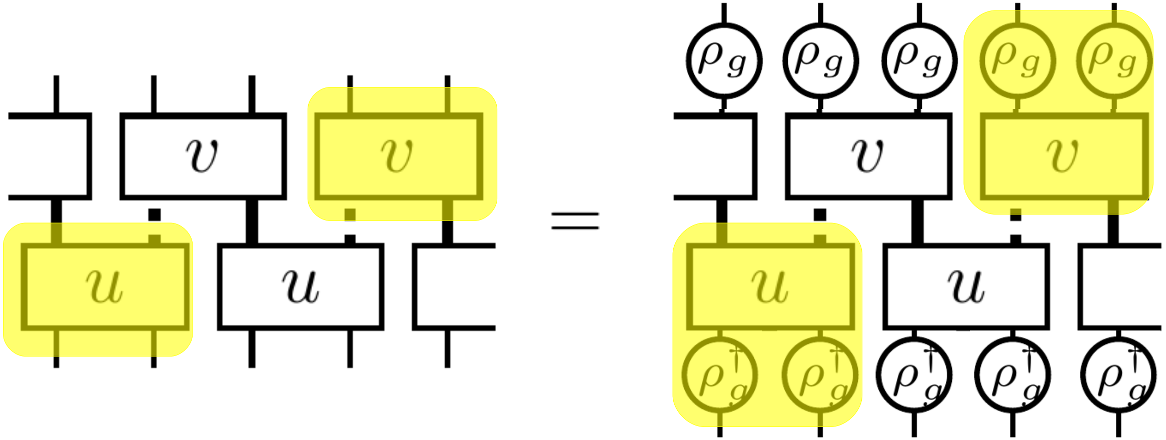
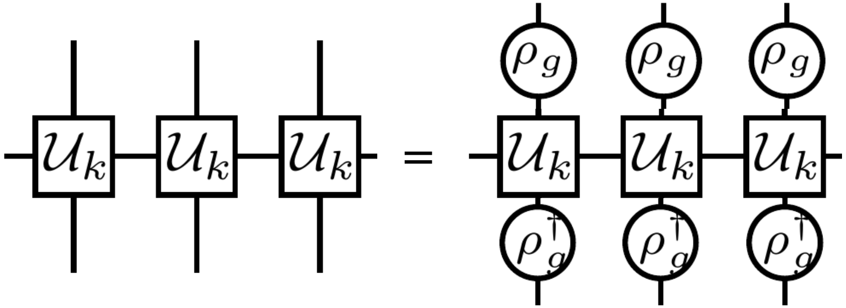
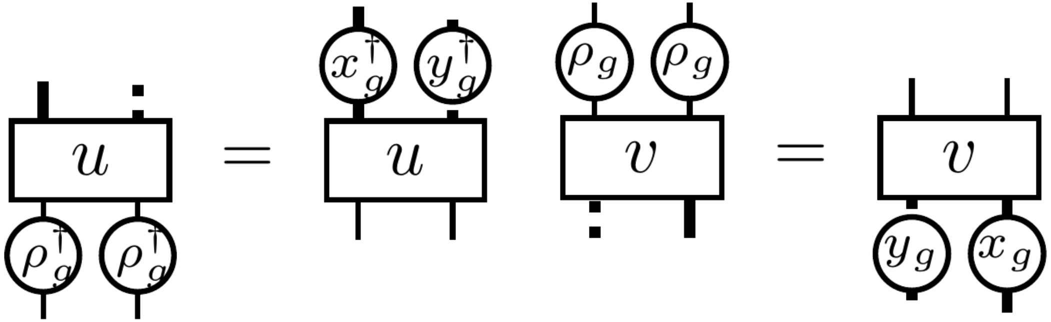






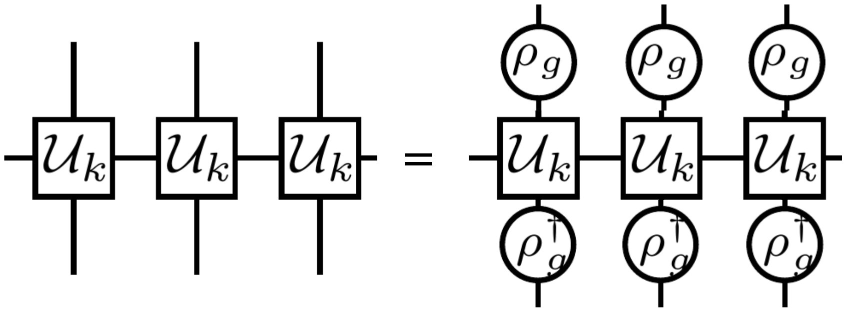
MPUs with symmetry G

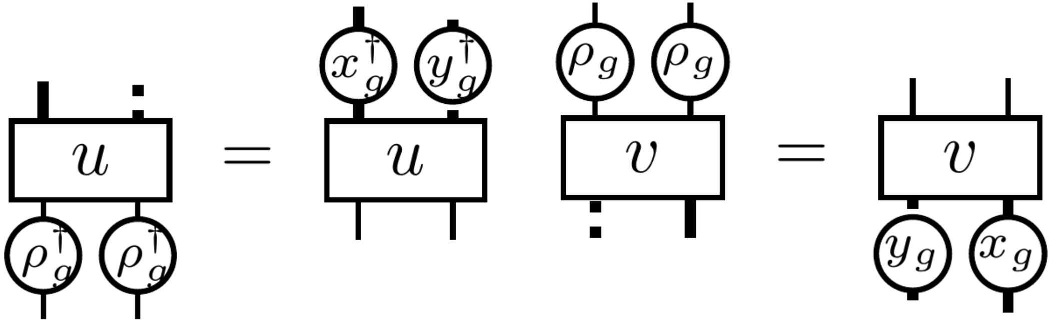
Acts on MPU as a linear representation ρ_g

$$\rho_g \rho_h = \rho_{gh} \text{ for } g, h \in G$$

Acts on standard form with a linear representation

$$x_g \otimes y_g \cong \rho_g \otimes \rho_g$$





Equivalence & cohomology class

MPUs are equivalent

$\iff$

can continuously connect them
without breaking symmetry, after

- blocking
- adding ancillas with arbitrary rep. of our choosing

Equivalence & cohomology class

MPUs equivalent
$\Leftrightarrow$ same indices & same cohomology classes

Definition cohomology class

$x_g \otimes y_g \cong \rho_g \otimes \rho_g$ linear rep.

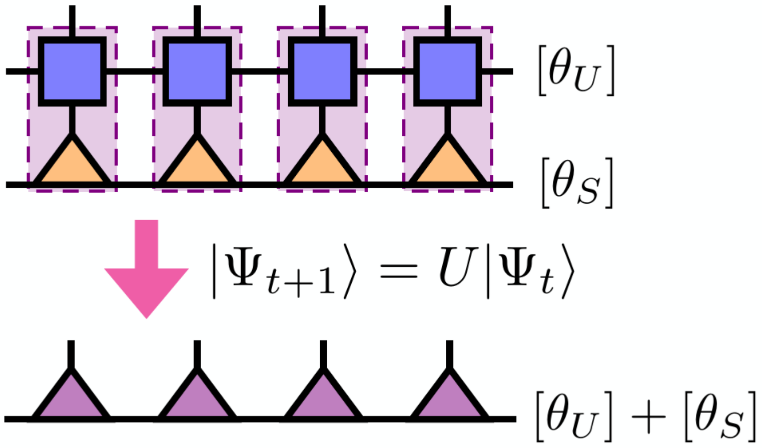
$\Rightarrow x_g$ is a projective rep.:

$$x_g x_h = e^{i\theta(g,h)} x_{gh}$$

Each set of [θ(g,h)]
(up to absorption into x)
is a cohomology class

Example: nontrivial cohomology class

"topological time crystal"



Example: nontrivial cohomology class

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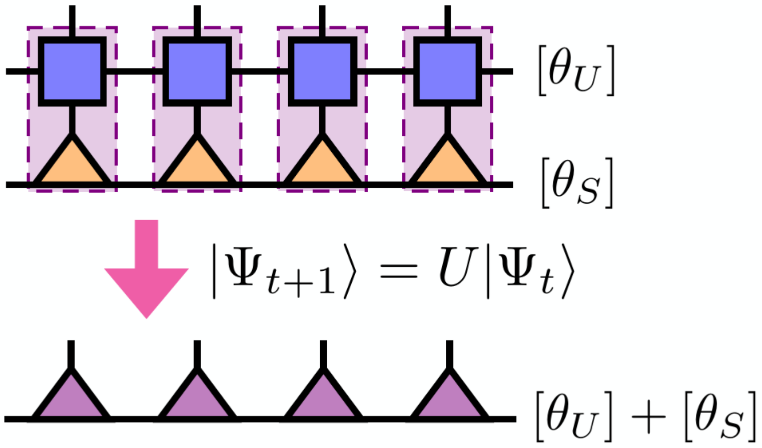
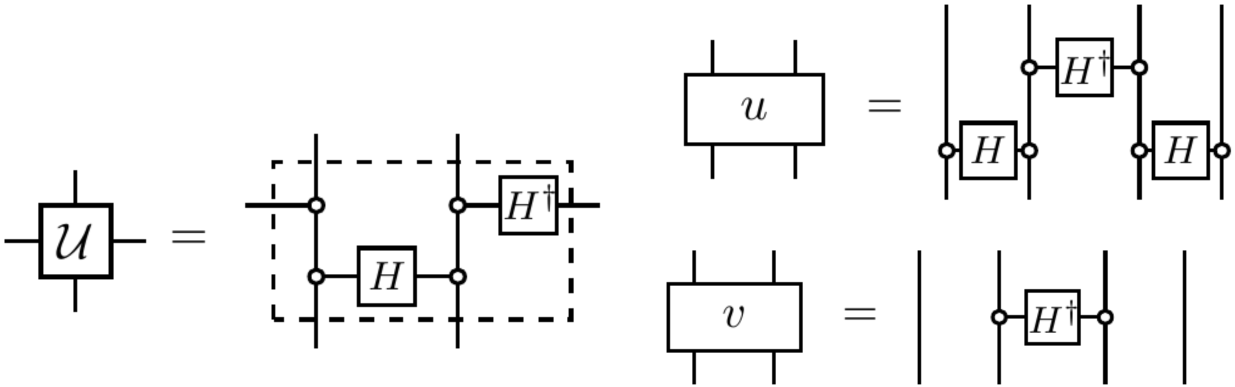


$$G = \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$\rho_{(m,n)} = Z^m \otimes Z^n$$
$$x_{(m,n)} = -X^n Z^m \otimes Z^n$$
$$y_{(m,n)} = Z^m \otimes X^m Z^n$$

ind 0, no SPIs

nontrivial cohomology



Strong equivalence & SPI

MPUs are strongly equivalent

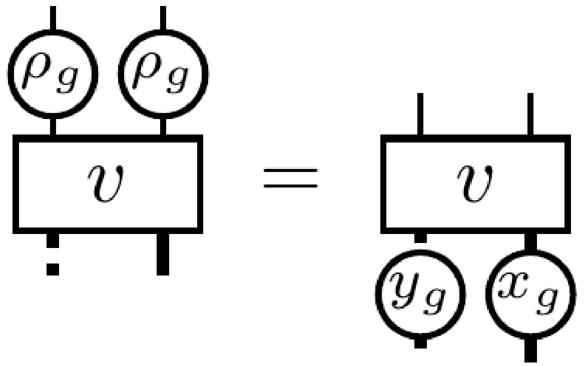
$\Longleftrightarrow$

can continuously connect them
without breaking the symmetry, after
• blocking
• adding ancillas with same rep. as original MPU

Strong equivalence & SPI

MPUs strongly equivalent
$\Longrightarrow$ same SPIs & same cohomology classes

Definition SPI (symm. protected index)



For $\mathrm{Tr}\,\rho_g \neq 0$:

$$\mathrm{ind}_g \equiv \frac{1}{2}\log\left|\frac{\mathrm{Tr}\,y_g}{\mathrm{Tr}\,x_g}\right|$$

Strong equivalence & SPI

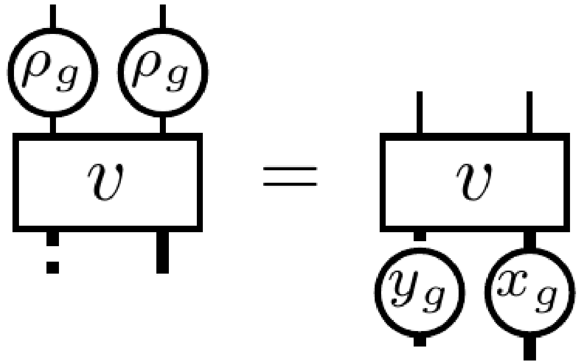
MPUs strongly equivalent
$\Rightarrow$ same SPIs & same cohomology classes

Can measure the relative SPI

$$\mathrm{ind}_g - \mathrm{ind}_e$$

with a Lohschmidt echo experiment

Definition SPI (symm. protected index)



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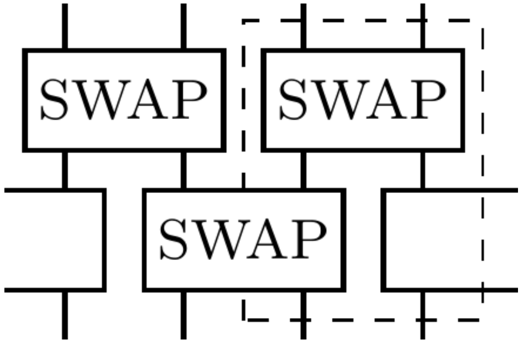
Example: trivial ind, nontrivial SPI

bilayer SWAP circuit



Example: trivial ind, nontrivial SPI

$G = \mathbb{Z}_3 = \{0, 1, 2\}$ has three irreducible reps: $\omega_g^t = e^{gt/3\, i\pi}$

for t=0,1,2 and g∈G

$x_g \otimes y_g \cong \rho_g \otimes \rho_g$

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$$\rho_g \cong 2\omega_g^0 \oplus 2\omega_g^1$$

$$x_g \cong 4\omega_g^0$$

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$$\rho_g \otimes \rho_g \cong 4\omega_g^0 \oplus 8\omega_g^1 \oplus 4\omega_g^2 \cong x_g \otimes y_g$$

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$$\mathrm{ind}_g \equiv \frac{1}{2} \log \left| \frac{\mathrm{Tr}\, y_g}{\mathrm{Tr}\, x_g} \right|$$

$\mathrm{ind}_0 = 0$, but $\mathrm{ind}_1 = \log |\cos \pi/3| \neq 0$

↓ equivalent to identity

↓ not strongly equivalent to identity

Applications beyond MPUs

- Quantum cellular automata

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- Inhomogeneous locality-preserving unitaries

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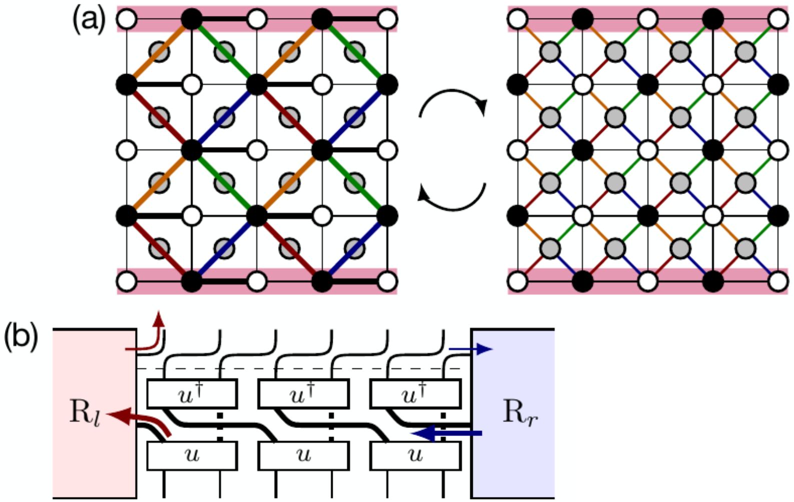
Applications beyond MPUs

- Quantum cellular automata
- Inhomogeneous locality-preserving unitaries
- Finite time dynamics of 1D local Hamiltonians
- Edge dynamics of many-body localised Floquet systems

$\longrightarrow$ find new class of 2D SPT phases characterised by SPI

fractional symmetry charge pump picture



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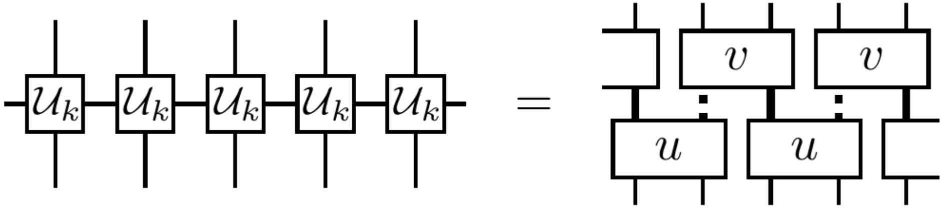
MPU standard form



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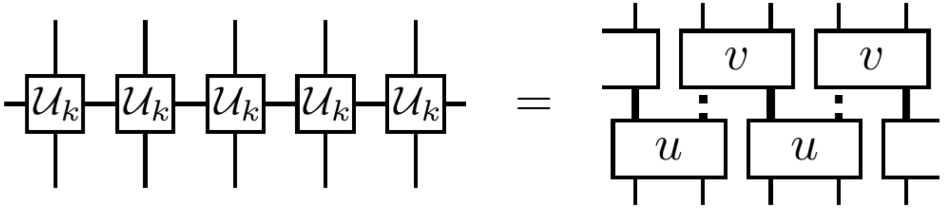
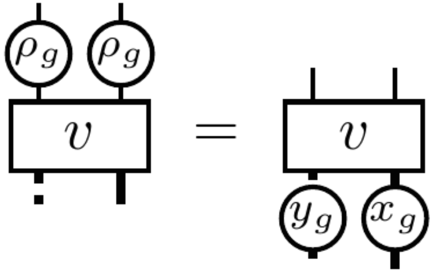


Symmetry action



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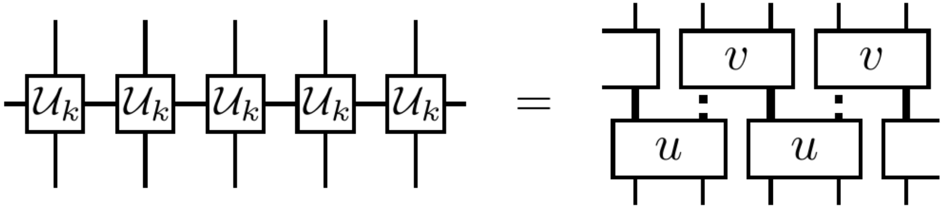

Symmetry protected index (SPI)

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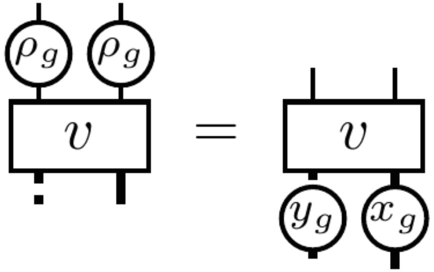

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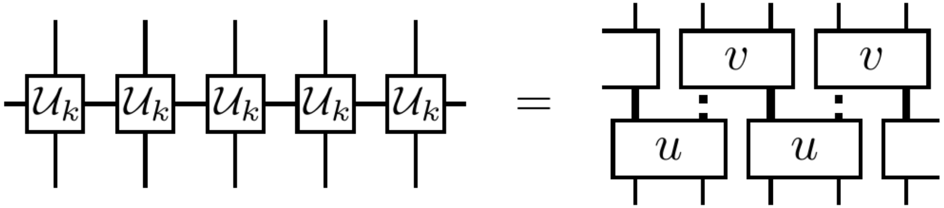

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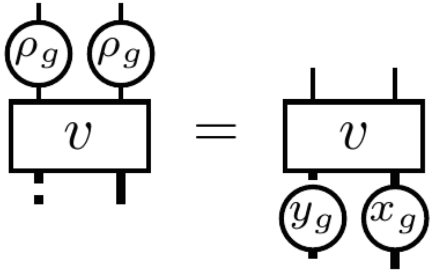

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Equivalence of MPUs
$\Longleftrightarrow$ same indices & same cohomology

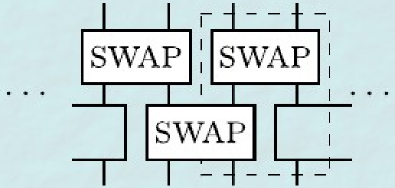
Strong equivalence of MPUs
$\Longrightarrow$ same SPIs & same cohomology

C Further non-trivial SPI

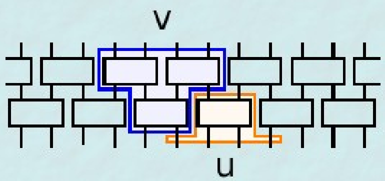
For $G=\mathbb{Z}_2$, we can construct a non-trivial factorisation of Equation 1:
$(6\rho_0 \oplus 2\rho_1)^{\otimes 2} \cong (5\rho_0 \oplus 3\rho_1) \otimes (8\rho_0)$.
The corresponding MPU with local Hilbert space dimension 8 has trivial ind and cohomology, but non-trivial SPI: $\mathrm{ind}_h = \log(8/(5-3))/2 = \log(2)$.

Details bilayer SWAP circuit

B Bilayer SWAP circuit



- $G = \mathbb{Z}_n = \{e, h, \ldots, h^n\}$, $n \geq 3$
- $\rho_h = \mathbb{1} \otimes W$ two qubits per site,
- $x_h = \mathbb{1} \otimes \mathbb{1}$ $W = |0\rangle\langle 0| + e^{2\pi i/n} |1\rangle\langle 1|$
- $y_h = W \otimes W$
- ind = 0, SPI: $\text{ind}_h = \log|\cos \pi/n| \neq 0$
- trivial cohomology



This MPU is equivalent to the identity (Theorem 1).
But it is not strongly equivalent to the identity (Theorem 2).

$\mathbb{Z}_n$ has n irreducible linear reps $\rho_0, \ldots, \rho_{n-1}$ with $\rho_i \otimes \rho_j = \rho_{i+j}$.
In terms of these:
$\rho_g \cong 2\rho_0 \oplus 2\rho_1$, implemented on two qubits as $\cong (2\rho_0) \otimes (\rho_0 \oplus \rho_1)$
$x_g \cong 4\rho_0$, implemented as $\cong (2\rho_0) \otimes (2\rho_0)$
$y_g \cong \rho_0 \oplus 2\rho_1 \oplus \rho_2$, implemented as $\cong (\rho_0 \oplus \rho_1) \otimes (\rho_0 \oplus \rho_1)$

This gives a non-trivial factorisation of the Equation 1:
$(2\rho_0 \oplus 2\rho_1)^{\otimes 2} \cong (4\rho_0) \otimes (\rho_0 \oplus 2\rho_1 \oplus \rho_2)$

For n=2, this non-trivial factorisation is not stable under blocking:
$(\rho_g \otimes \rho_g)^{\otimes 2} \cong (x_g \otimes \rho_g) \otimes (y_g \otimes \rho_g)$ trivially as $x_g \otimes \rho_g \cong y_g \otimes \rho_g \cong \rho_g \otimes \rho_g$.
In fact, the SPI is not defined as $\text{Tr}(\rho_h) = 0$. Further, we can show the n=2 MPU is strongly equivalent to the identity.