The 16-fold way in the Kitaev honeycomb model

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60’

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Outline & main message

1) Kitaev honeycomb model as Majorana fermions in Z_2 gauge field (0 or π flux) characterized by integer Chern number, modulo 16

2) Triangular vortex lattices :

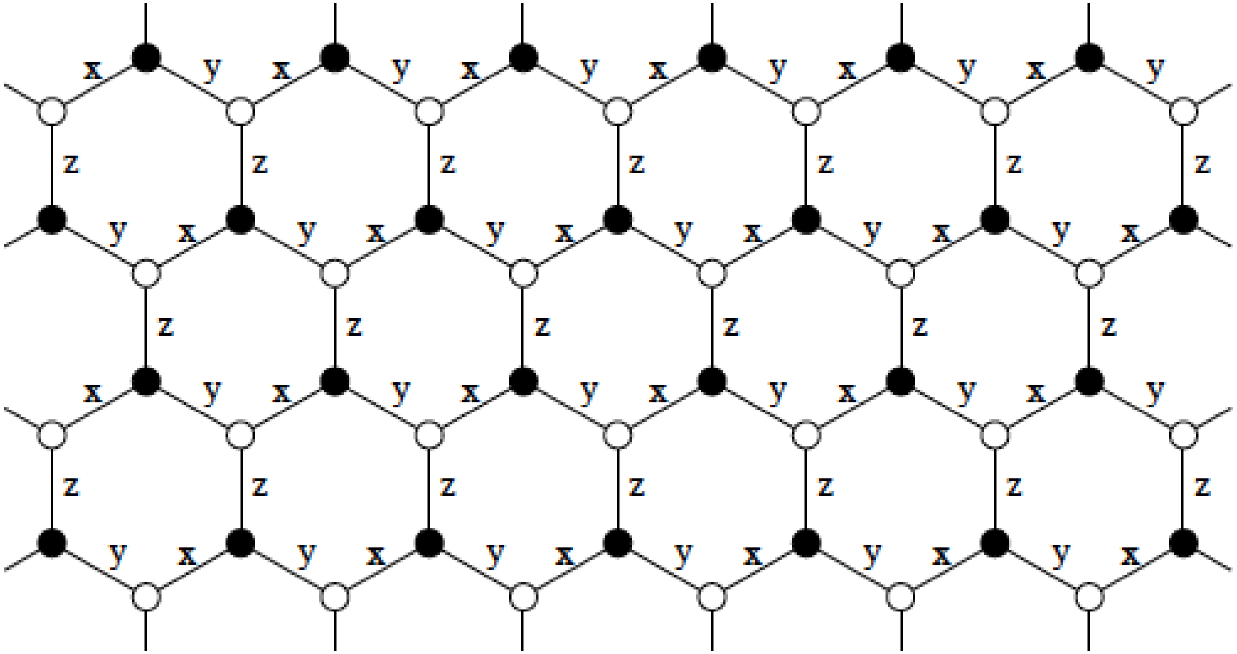
Chern = $0, \pm 1, \ldots, \pm 6, 8$ but not ± 7

3) Effective models in the dilute vortex limit

Kitaev's honeycomb model

Spin 1/2 on the honeycomb lattice with compass nearest neighbor exchange interaction

$$H = -\sum_{<i,j>} J_\alpha \sigma_i^\alpha \sigma_j^\alpha \qquad \alpha = x, y, z$$

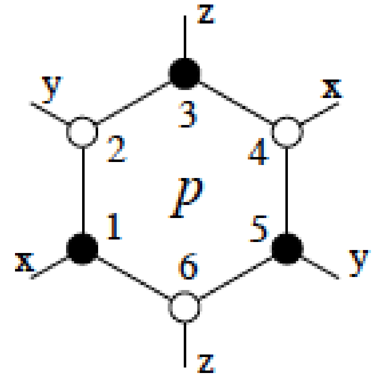


Kitaev 2006

Conserved quantities

For a plaquette p

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$



$$[H, W_p] = 0 \, \forall p \qquad \{w_p = \pm 1\}$$

A vortex is (by definition) $w_p = -1$

Hilbert space splits in uncoupled « vortex sectors »

$$\mathrm{dim}\mathcal{H} = 2^N = 2^{N/2} \times 2^{N/2}$$

Kitaev 2006

Majorana fermions & Z_2 gauge field

Problem maps to honeycomb tight-binding model of MF with anisotropic hopping (J_x,J_y,J_z) in static Z_2 gauge field : $u_{jk} = -u_{kj} = \pm 1$

$$H = \sum_{j,k} i \frac{J_\alpha}{2} u_{jk} c_j c_k \qquad \{c_i, c_j\} = 2\delta_{i,j}$$

Z_2 magnetic field : $w_p = \prod_{(i,j)\in p} u_{ij} = \pm 1$ $\quad 0 \text{ or } \pi - \text{flux}$

vortex

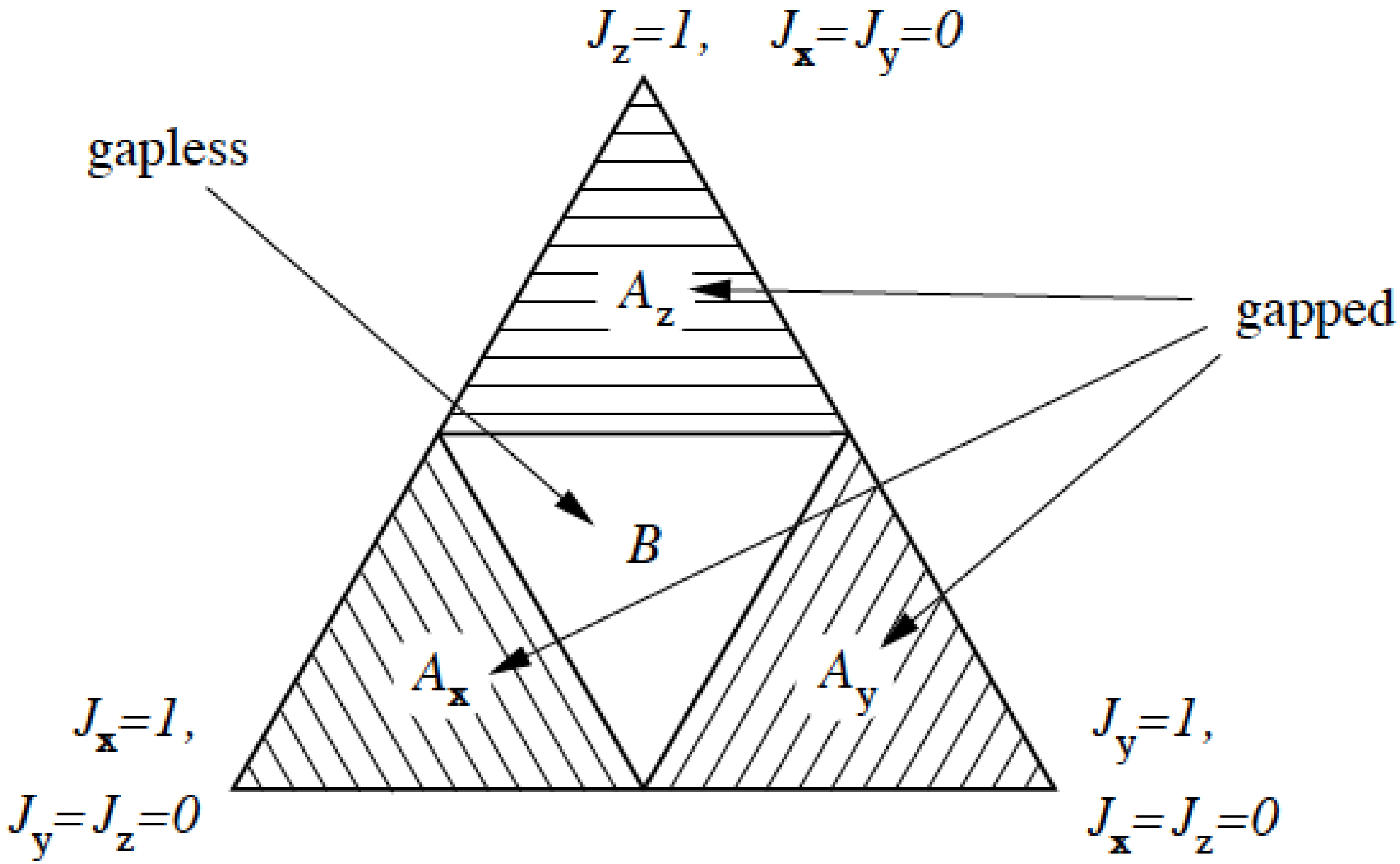
Quadratic hamiltonian → free MF

Doubled spectrum $\pm\epsilon_k \text{ with } k = 1, ..., N/2$

Groundstate in vortex-free sector

Kitaev 2006

Phase diagram of vortex-free sector



B : graphene-like, two Dirac cones : gapless nodal fermions + gapped vortices

A : beyond merging of Dirac cones : gapped fermions + gapped vortices (toric code)

Kitaev 2006

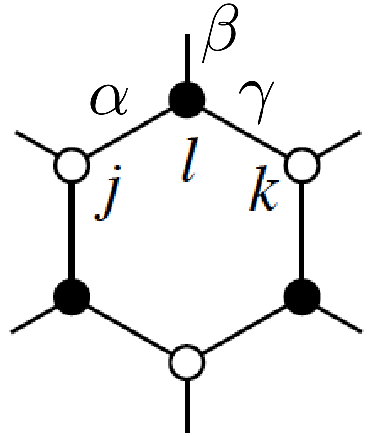
3-spin term breaking time-reversal

In order to gap the B phase in a chiral way

Restrict to isotropic exchange interaction

$$J_x = J_y = J_z = 1 \text{ and } \kappa$$

$$H_3 = \kappa \sum_{(j,l,k)\circlearrowright} \epsilon_{\alpha\beta\gamma} \sigma_j^\alpha \sigma_l^\beta \sigma_k^\gamma$$



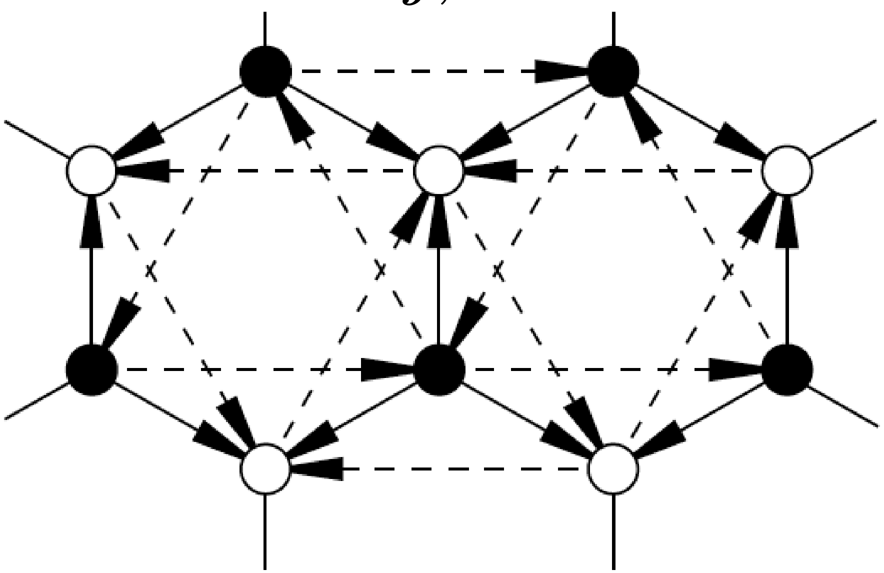
External magnetic field
Zeeman term

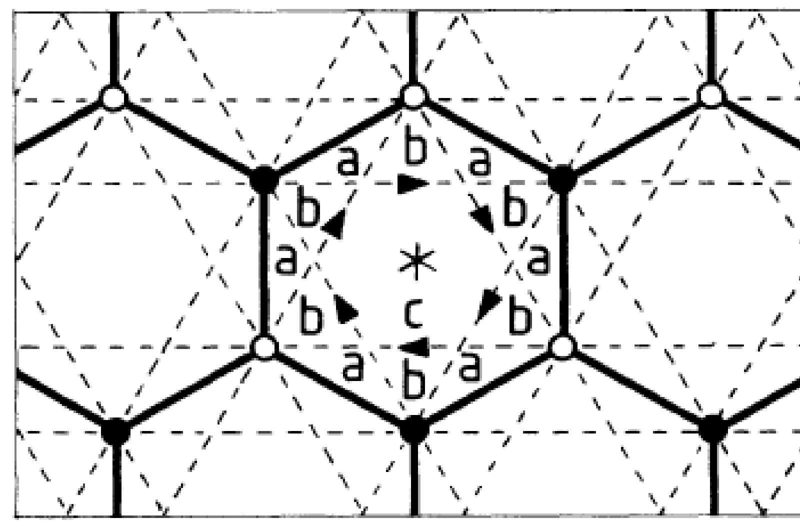
$$\kappa \sim \frac{h_x h_y h_z}{J^2}$$

Kitaev 2006

NNN hopping with inhomogeneous magnetic field within the unit cell

$$H = i\frac{J}{2}\sum_{j,k} u_{jk}c_jc_k + i\kappa \sum_{(j,k,l)\circlearrowright} u_{jk}u_{kl}c_jc_l$$





Equivalent to Haldane’s model of Chern insulator

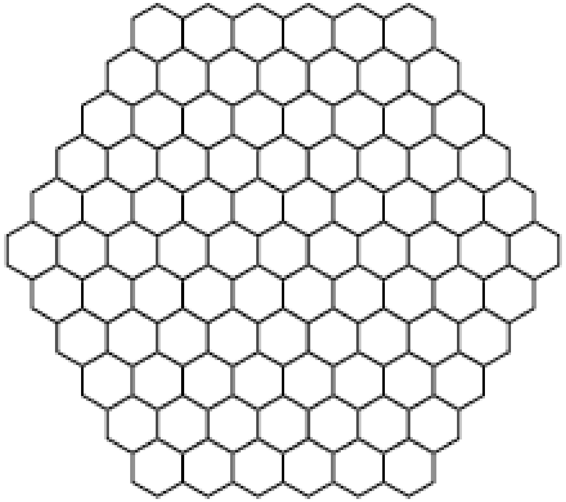
but for MF, with $M = 0$ and $\phi = \pi/2$

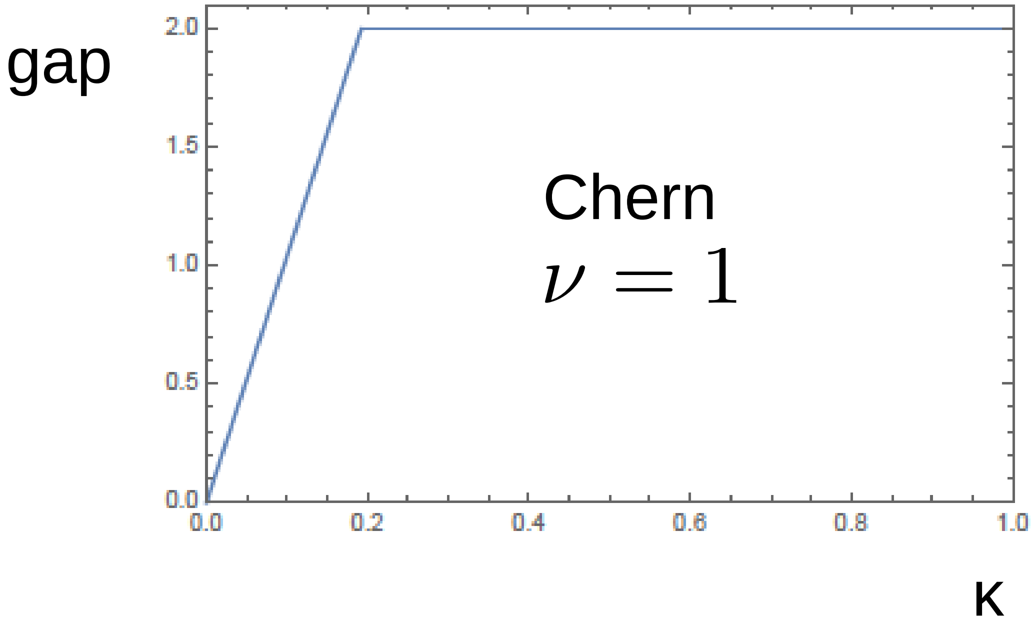
& with a « vortex pattern »

Haldane 1988

Vortex-free sector (white background)

$\rho = 0$





Kitaev 2006

Periodic table of gapped non-interacting fermions (10-fold way)

System	Cartan nomenclature	TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
standard (Wigner-Dyson)	A (unitary)	0	0	0	-	Z	-
	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	Z_2	Z_2
chiral (sublattice)	AIII (chiral unit.)	0	0	1	Z	-	Z
	BDI (chiral orthog.)	+1	+1	1	Z	-	-
	CII (chiral sympl.)	−1	−1	1	Z	-	Z_2
BdG (Majorana fermions)	D	0	+1	0	Z_2	Z	-
	C	0	−1	0	-	Z	-
	DIII	−1	+1	1	Z_2	Z_2	Z
	CI	+1	−1	1	-	-	Z

Haldane model: 2D topological insulator (quantum Hall effect)

Kitaev chain: 1D topological superconductor

Kitaev honeycomb model: 2D topological superconductor

Schnyder, Ryu, Furusaki & Ludwig 2008 ; Kitaev 2009

D class: 2D topological superconductor that breaks TRS

Bulk excitations :

- gapped Bogolubov quasiparticles, BdG bands carry a non-zero Chern number (Z topological invariant)

- alternatively: gapped (massive) MF in 2+1, Majorana bands carry a non-zero Chern number

ν integer

Volovik 1988

Chiral & real edge modes

Edge excitations :

ν = number of chiral-Majorana edge modes

(bulk-edge correspondance)

half an integer quantum edge channel : chiral central charge $c_- = \nu/2$

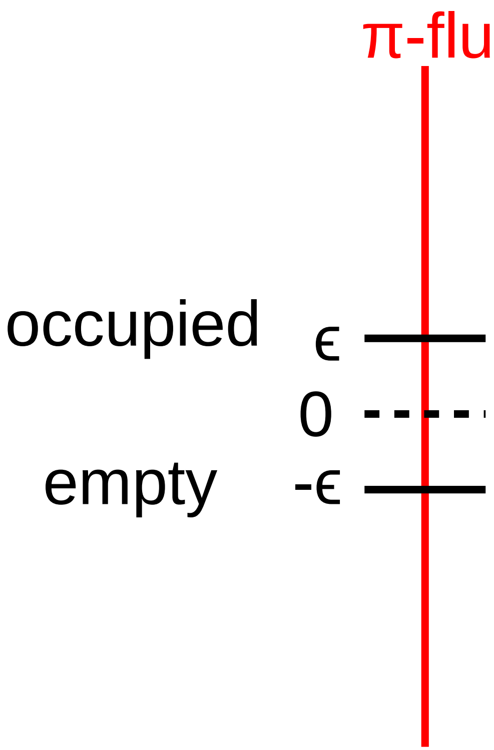
thermal transport $I = \frac{\pi}{12} c_- T^2$

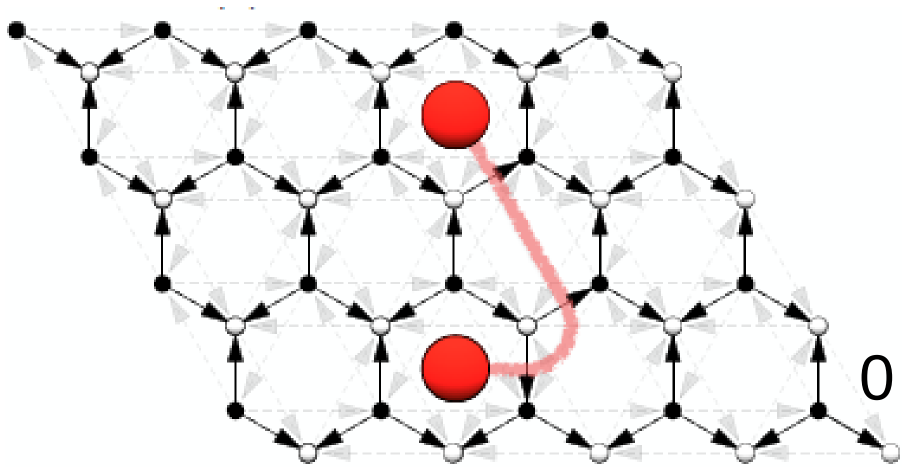
ν integer

Volovik 1988, Kitaev 2006

Excitations bound to a vortex

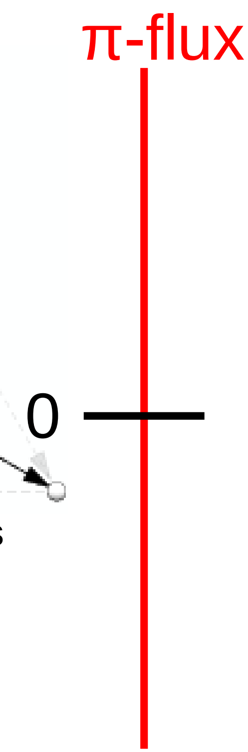
ν even





Picture from C. Self's PhD thesis

ν odd



ν integer mod 2

single complex fermion mode =
pair of Majorana zero modes (MZM)

$$H = 2\epsilon(a^\dagger a - 1/2) = 2\epsilon\, \mathrm{i} c\tilde{c}$$

$$\{a_i, a_j^\dagger\} = \delta_{i,j}$$

Caroli, de Gennes, Matricon 64

unpaired MZM
= separated MZM pair

$$H = t\, \mathrm{i} c_1 c_2 \qquad t \to 0$$

$$\{c_i, c_j\} = 2\delta_{i,j}$$

Kopnin-Salonaa 91; Volovik 99
see also Jackiw-Rossi 81

Fractionalized excitations : MZM are non-abelian anyons

MZM or Majorana bound state or Majorana anyon or Majorino (Wilczek)

= MF, at zero energy, localized in a vortex core (or another topological defect)

Ising anyon $\sigma \times \sigma = 1 + \epsilon$

Rk1: MF = fermion that is its own antiparticle ≠ MZM

Rk2: Jackiw's mechanism of fractionalization

Read & Green 00; Ivanov 01; see also Moore & Read 91

Exchange statistics of vortices : 16-fold way

1) 2 layers of topo SC with Chern ν = QHE with ν

2) π-flux binds a $\nu/2$ charge because $q=\sigma_{xy}\,\phi$

3) braiding of two charge-flux composites $(\nu/2,\pi)$

4) AB phase = $\pi\,\nu/2$

5) exchange phase = $\pi\,\nu/4$

6) exchange phase per layer = $\pi\,\nu/8 = 2\pi\,\nu/16$

<u>ν integer mod 16</u>

Kitaev 2006

Summary of anyon theories

- Odd ν=±1;±3;±5;±7 : non-abelian Ising anyons

  gs torus deg. = 3 ; planar deg. = 2^n (n vortex pairs)

  single non-abelian vortex $\sigma \times \sigma = 1 + \epsilon$

- Even ν=0;±2;±4;±6;±8 : abelian anyons

  gs torus deg. = 4 ; planar deg. = 1

  two abelian vortices (e & m or a & $\bar{a}$)

  a) ν=0;±4;8 : $Z_2 x Z_2$ anyons (toric code) $e \times m = \epsilon$

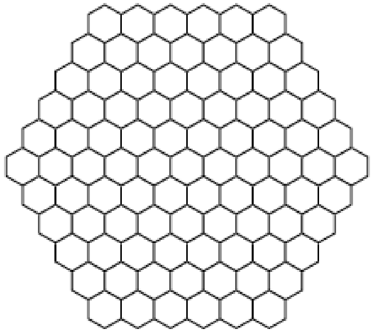
  b) ν=±2;±6 : Z_4 anyons $a \times a = \bar{a} \times \bar{a} = \epsilon$

Kitaev 2006 ; reviewed in lectures by Bernevig (video at PiTP 2015 and notes with Neupert, arXiv:1506.05805)

Kitaev’s model : known phases when $J_x = J_y = J_z = 1 \;\&\; 0 < \kappa \ll 1$

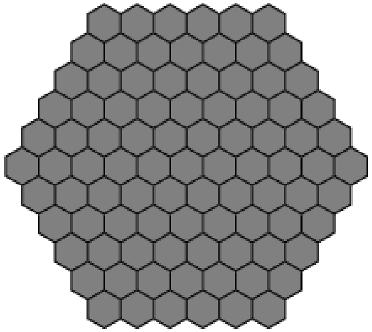
vortex free : ν=1

Kitaev 2006



vortex full : ν=2

Lahtinen, Pachos 2010

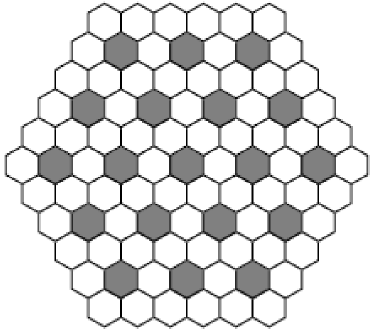


triangular vortex lattices

Lahtinen, Ludwig, Pachos,Trebst 2012: ν=4

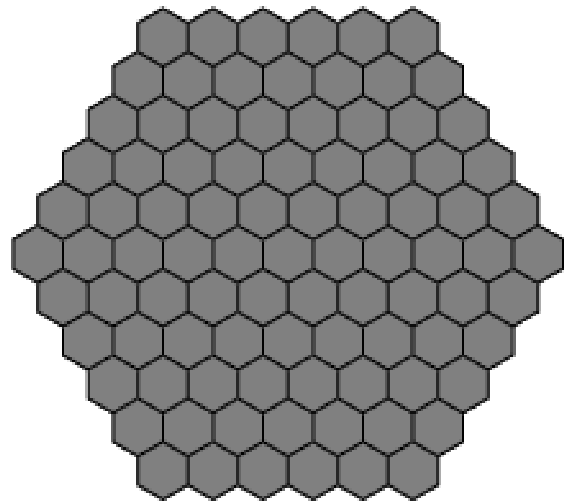
Zhang, Batista, Halasz 2019: ν=3,8

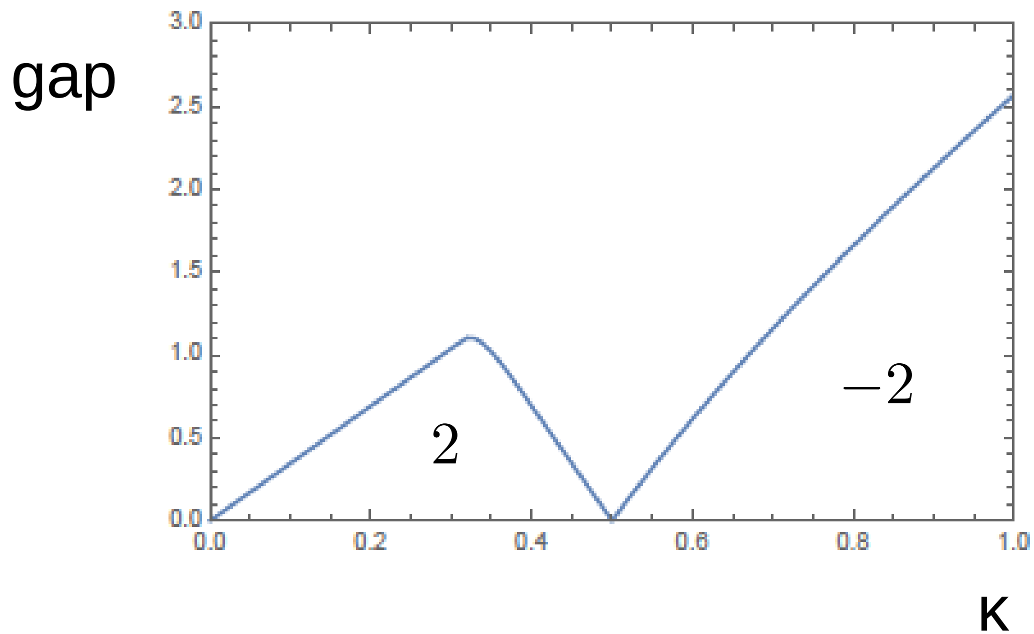
Us: ν=5,6 not 7



Vortex-full sector (black background)

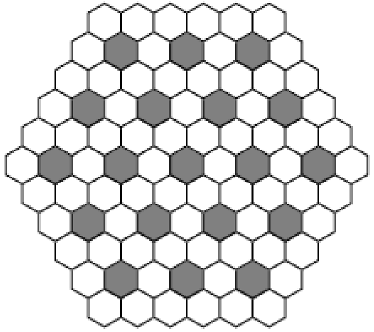
$$\rho = 1$$



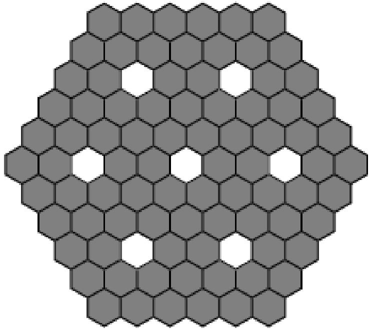


Triangular vortex lattices at any κ

Direct: black vortices in a vortex-free (white) sea



Dual: white vortices in a vortex-full (black) sea

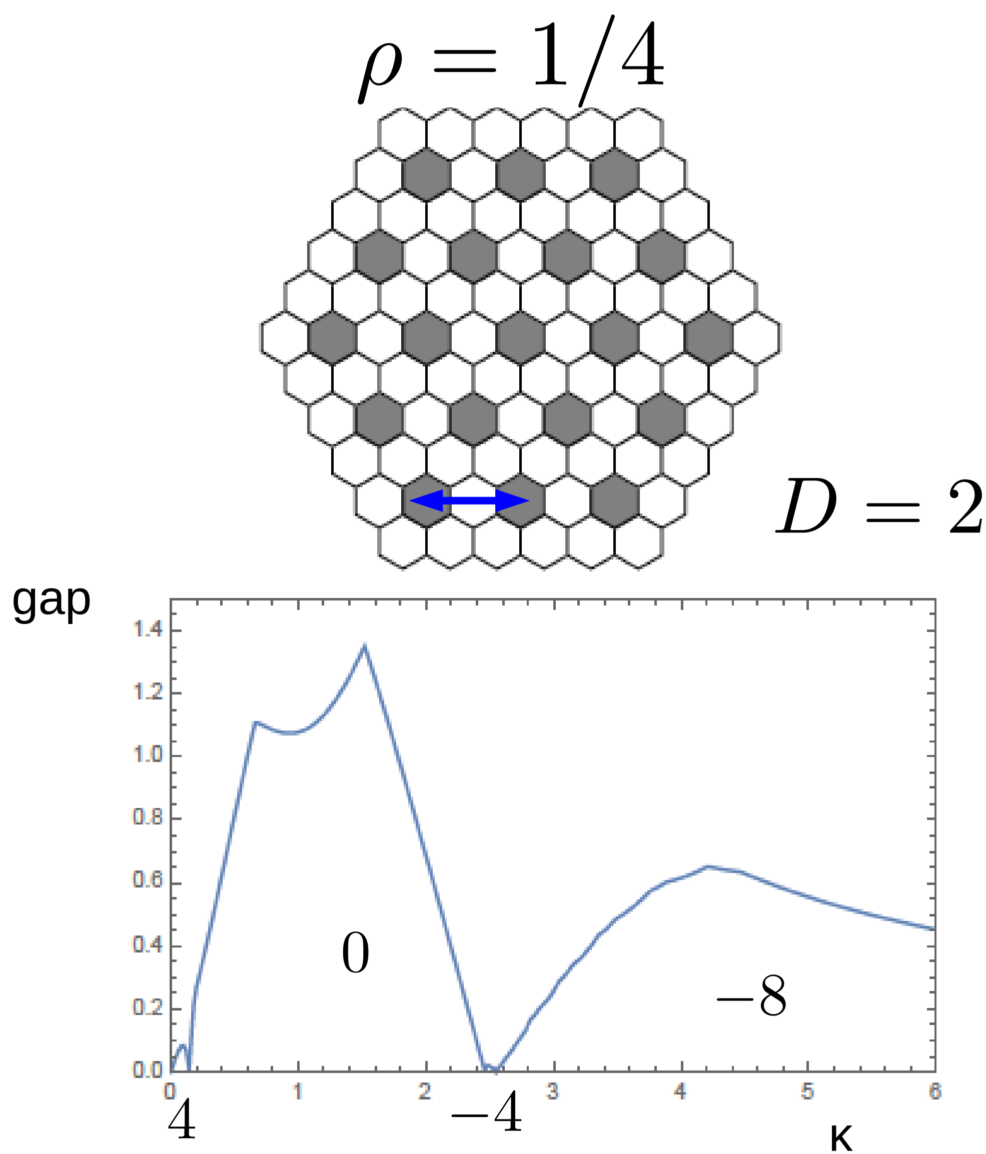


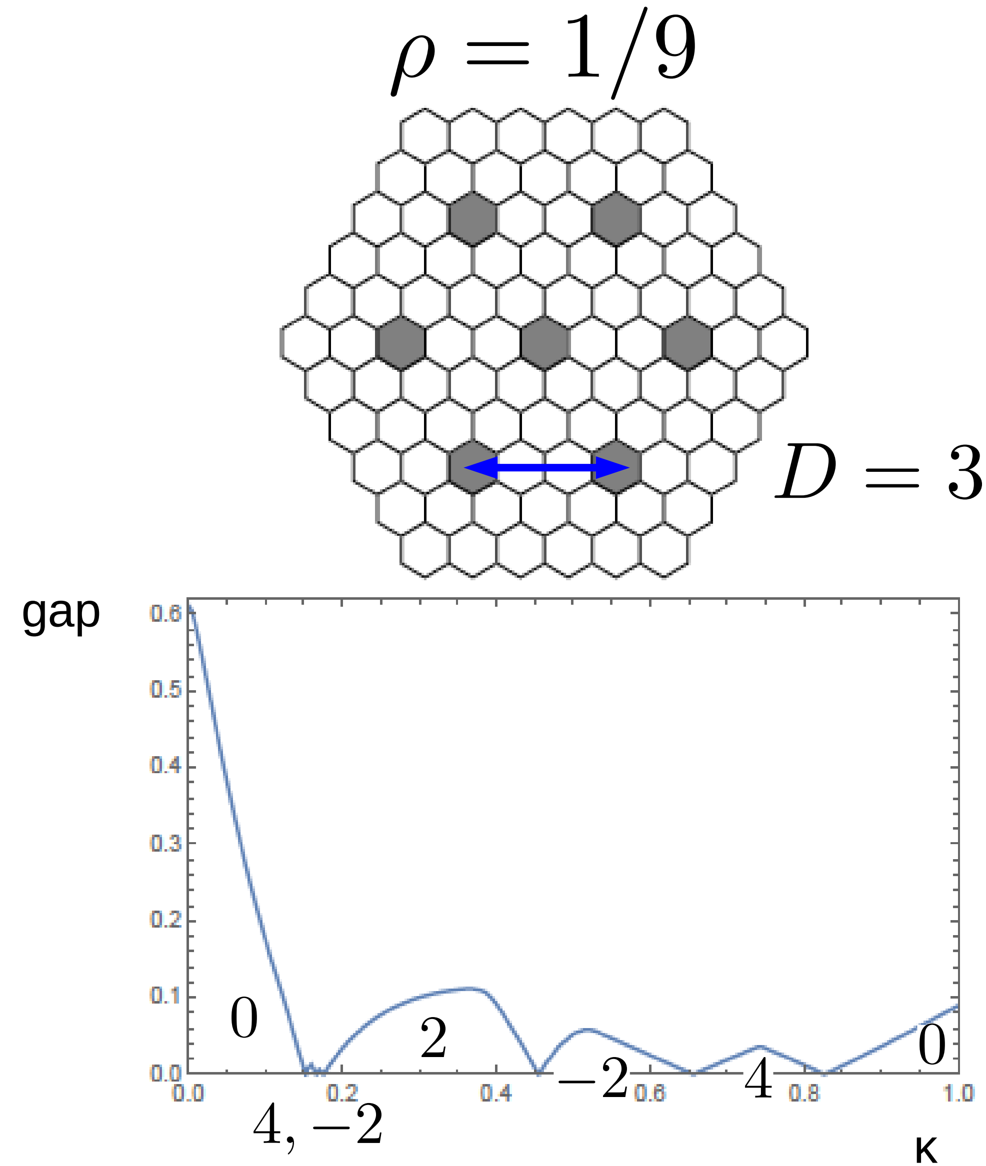
Periodic system in the thermodynamic limit

Gap versus κ

Chern number versus κ

Nucleated abelian phases $\rho=1/D^2$



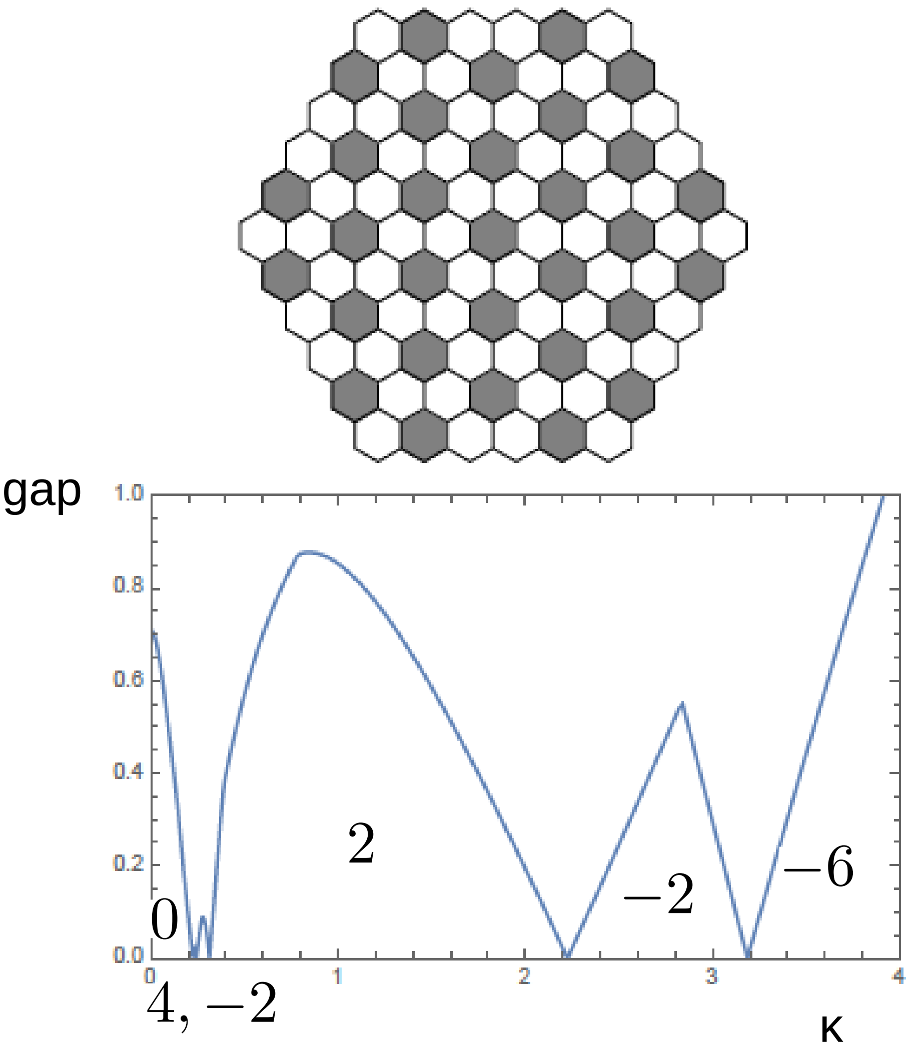


Also vortex-full gives ν = -2,2

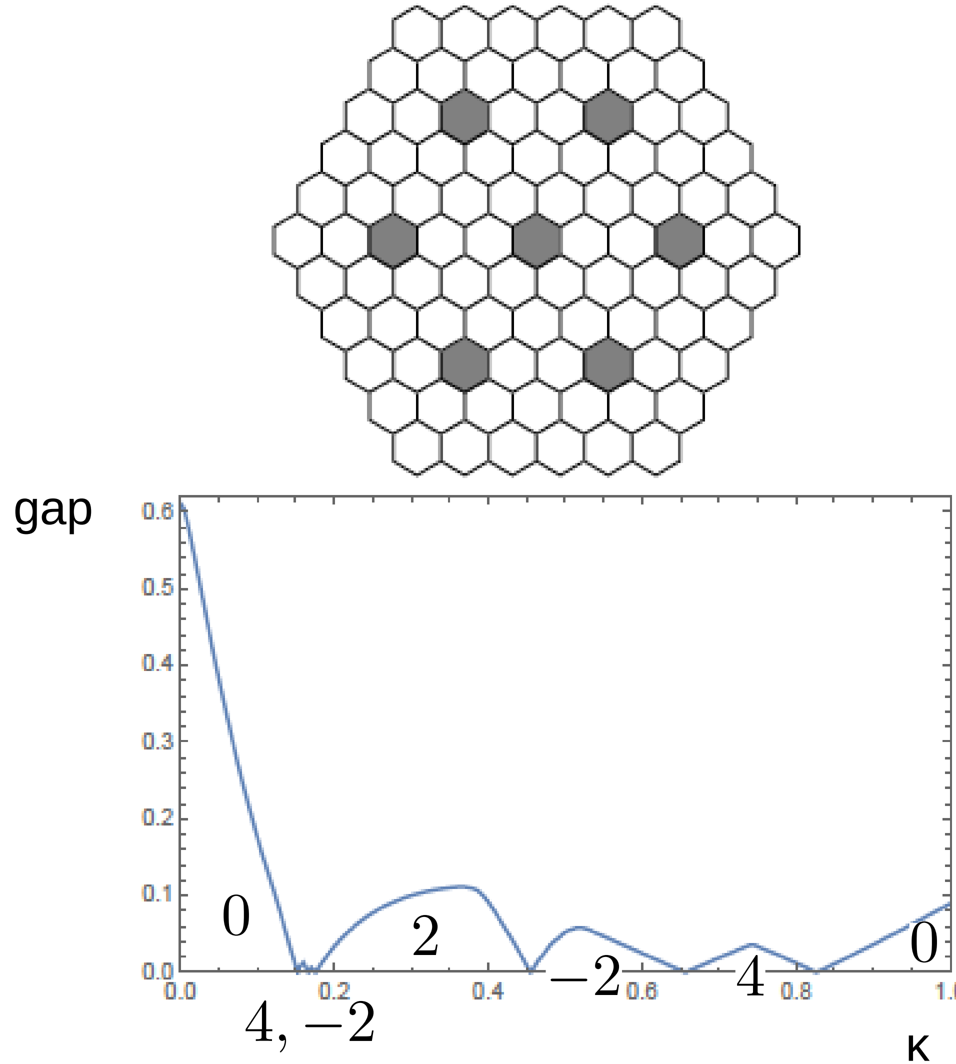
In agreement with Lahtinen, Ludwig, Pachos, Trebst 2012

Gapped for isotropic J & κ=0 : ν=0

$\rho = 1/3$



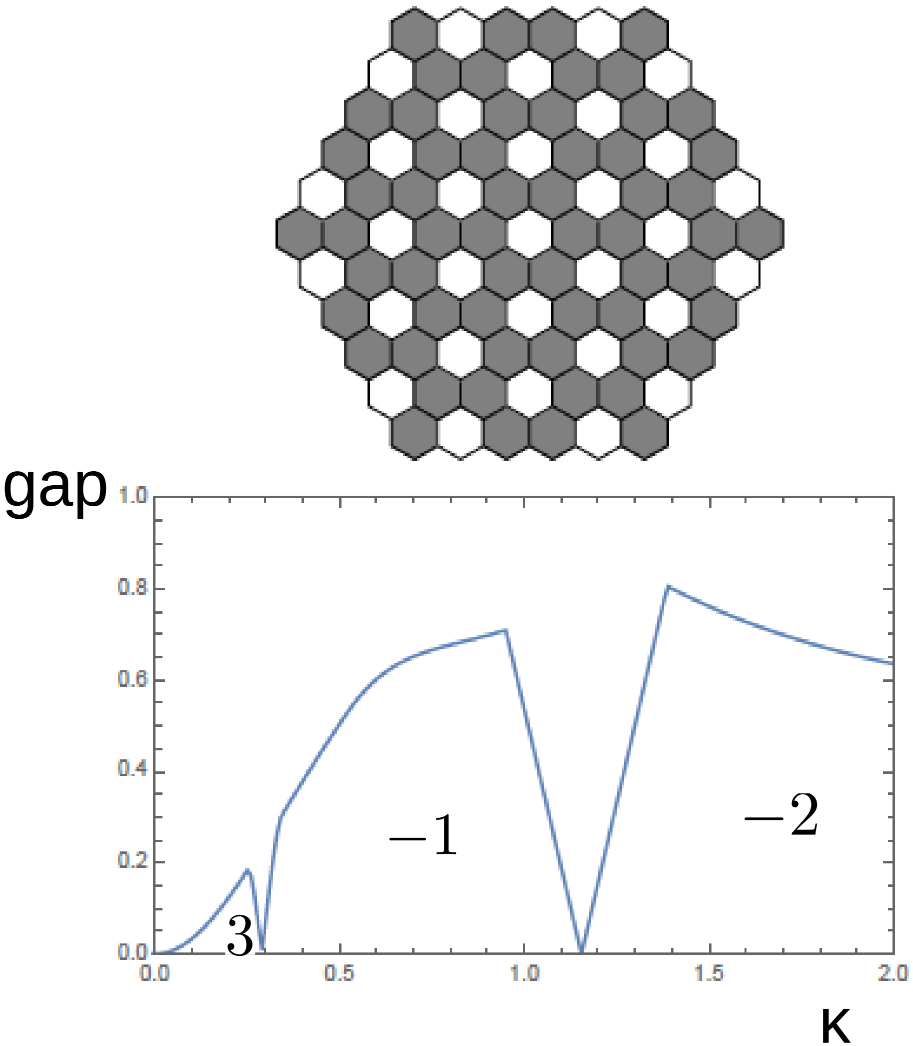
$\rho = 1/9$



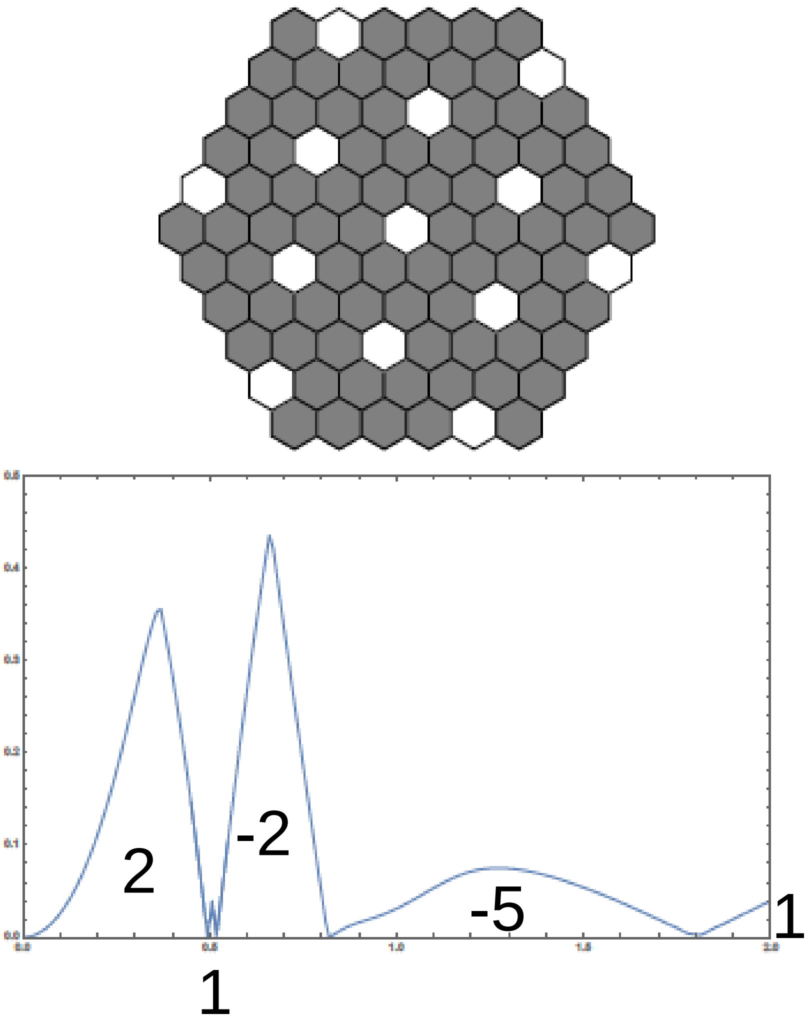
Gapped by vortex lattice if 1/ρ=0 mod 3: Dirac points nesting. Kamfor, Dusuel, Schmidt, Vidal, 2011

Non-abelian phases beyond ν=±1

$\rho = 2/3$



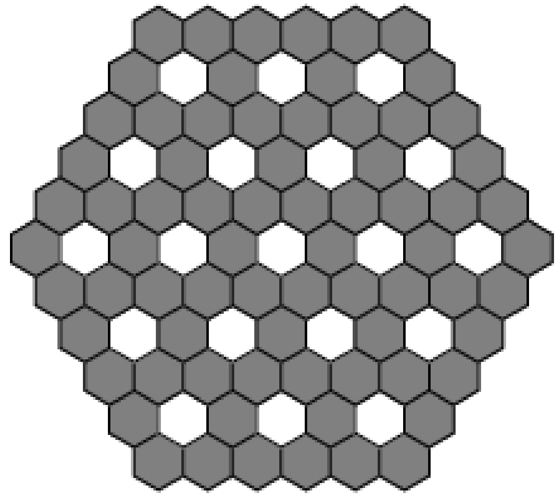
$\rho = 6/7$

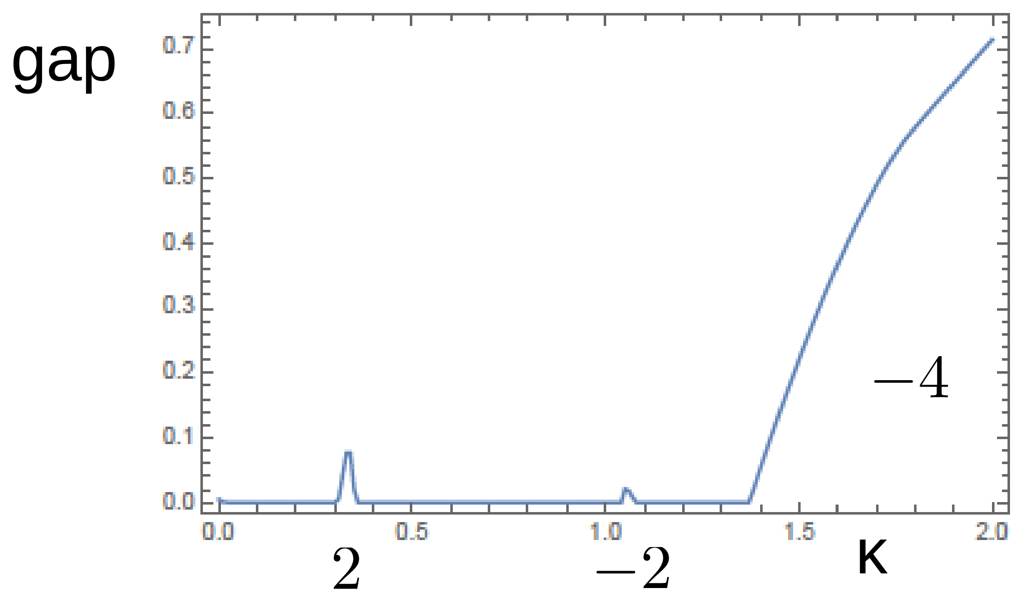


Dual triangular lattices
(triangular lattice of white vortices)

Extended gapless phases at finite κ

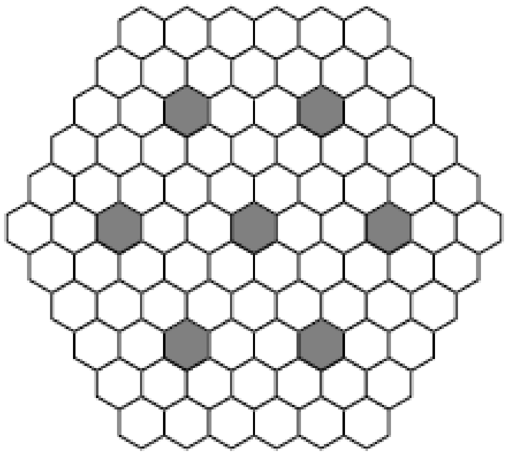
$$\rho = 3/4$$





Close to vortex free (direct)

Vortex density
ρ=1/n



Pattern of
Chern numbers

1/3*	1/4	1/7	1/9*	1/12*	1/13	1/16	1/19	1/21*	1/25
0	4	2	0	0	2	2	2	0	2
4	0	-2	4	4	-2	-2	-2	4	-2
-2	4	2	-2	8	4	4		-2	4
2	-8	-2	2	-4	0	0		2	0
-2		-14	-2	-8	4	4		-2	4
-6			4	-2	-2				-2
			0	-6	2				2
			4		-2				-2
			-2						4
									0
									4

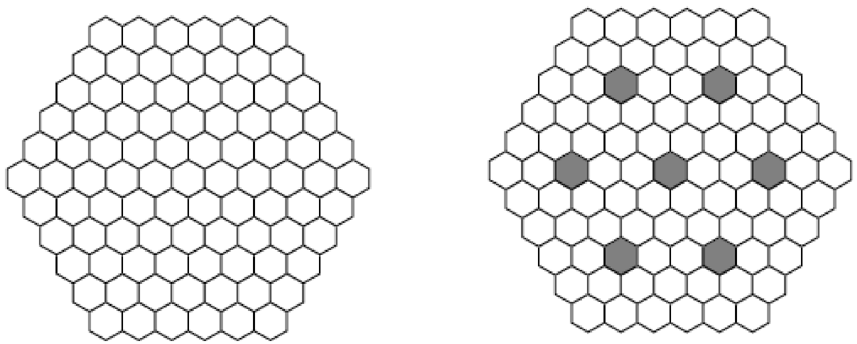
Increasing κ

Black family : flux per triangle = π/2, i.e. ρ=odd/n
Only even Chern numbers, changes by 4,6,8 or 12
Stars: gapped at κ=0 i.e. Chern=0 (TRS) when denominator is multiple of 3 (nesting, see Kamfor et al. 11)

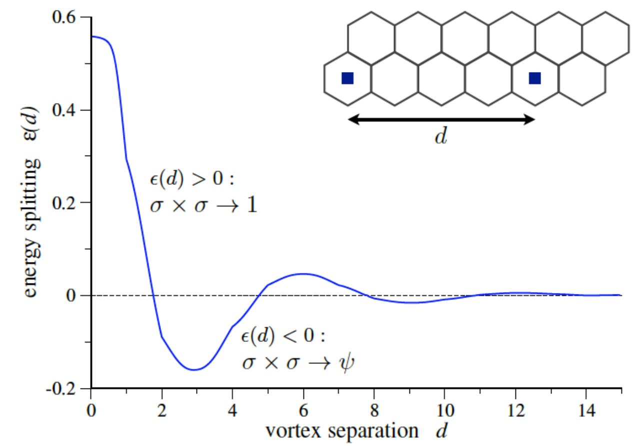
Effective model : ingredients

- Close to zero energy : localized modes in vortices (either MF or DF), well separated from bulk modes (large bulk gap, dilute vortex lattice)
- Hopping between vortices
- Broken time-reversal symmetry
- Conservation of fermionic parity

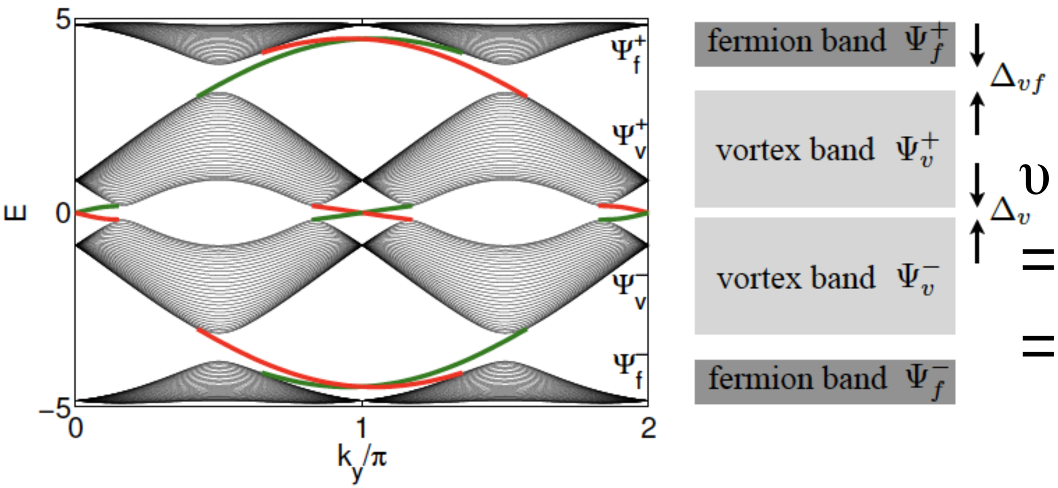
Direct effective model: MF on triangular lattice t_1 & $t_{\sqrt{3}}$



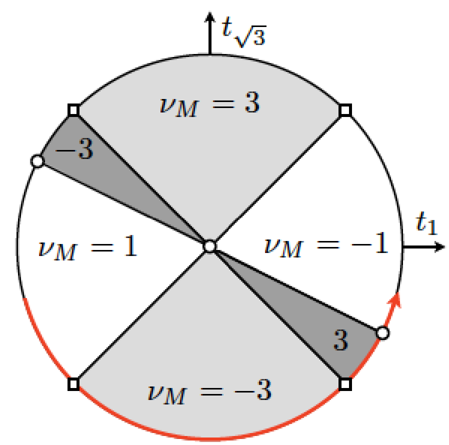
Vortex-free background



$$H = \sum_{i,j} i t_{ij} \gamma_i \gamma_j$$

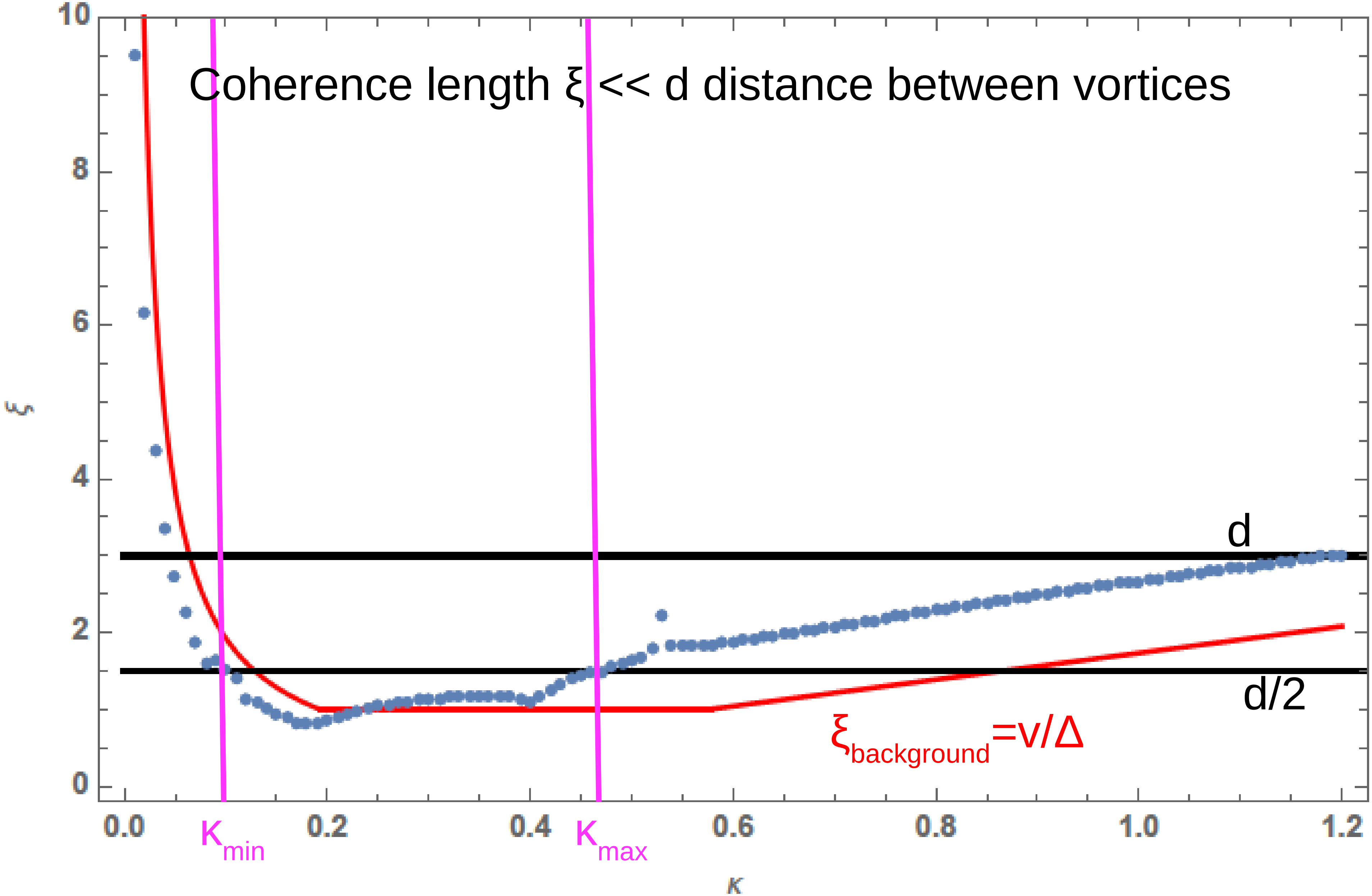


$\upsilon = \upsilon_f + \upsilon_v$
$= 1 + \upsilon_v$
$= -2,0,2,4$

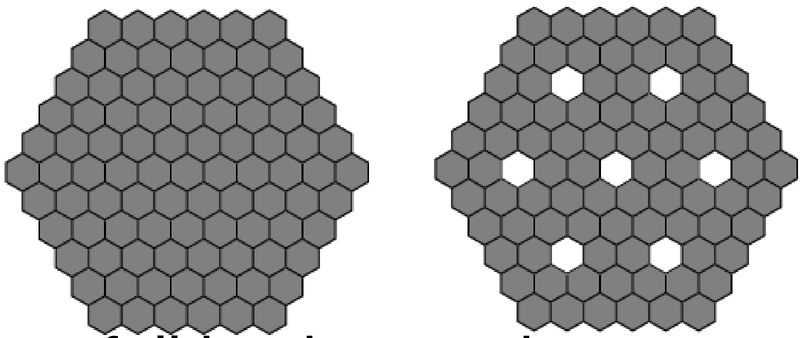


Lahtinen, Ludwig, Pachos & Trebst 2012

MF on triangular lattice: validity

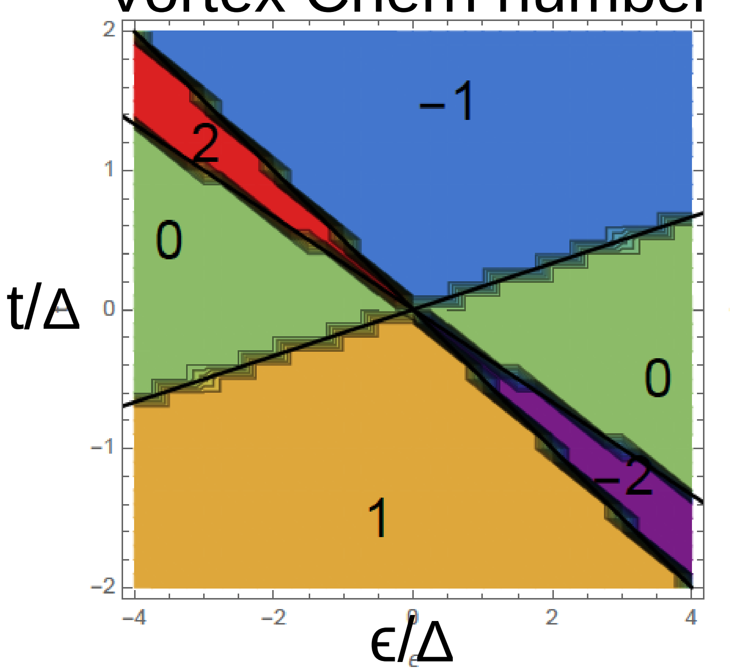


Dual effective model: p_x+ip_y SC on triangular lattice with 0 or π flux



Vortex-full background

Vortex Chern number



Bogolubov-de Gennes Hamiltonian

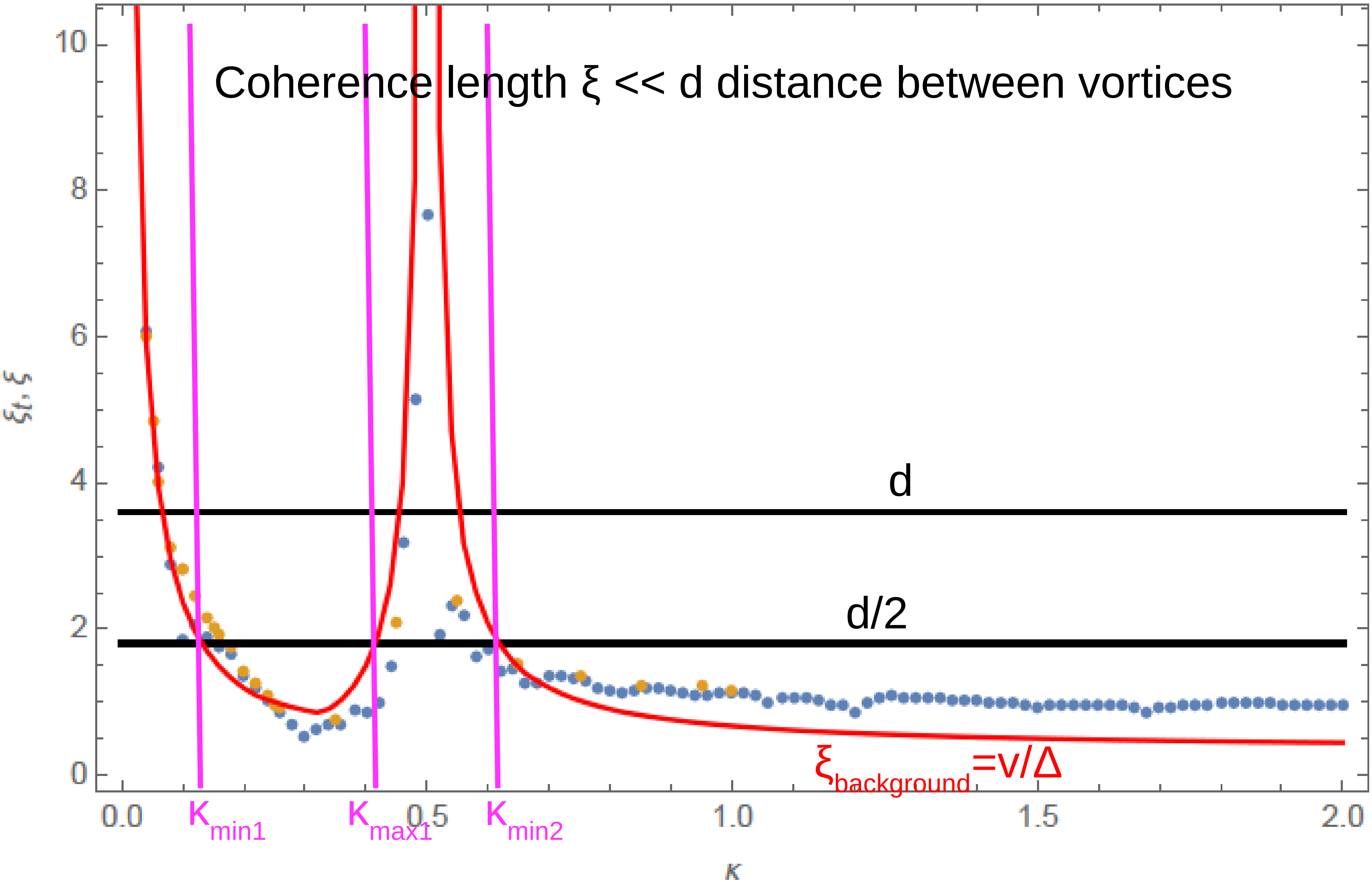
$$H(\mathbf{k}) = \begin{pmatrix} H_0(\mathbf{k}) & \Delta(\mathbf{k}) \\ \Delta(\mathbf{k})^* & -H_0(-\mathbf{k}) \end{pmatrix}$$

$$H_0(\mathbf{k}) = \epsilon - 2\sum_j t_j \cos \mathbf{k}.\mathbf{a}_j$$

$$\Delta(\mathbf{k}) = 2\mathrm{i}\Delta \sum_j (\mathbf{u}_x + \mathrm{i}\mathbf{u}_y).\mathbf{a}_j \sin \mathbf{k}.\mathbf{a}_j$$

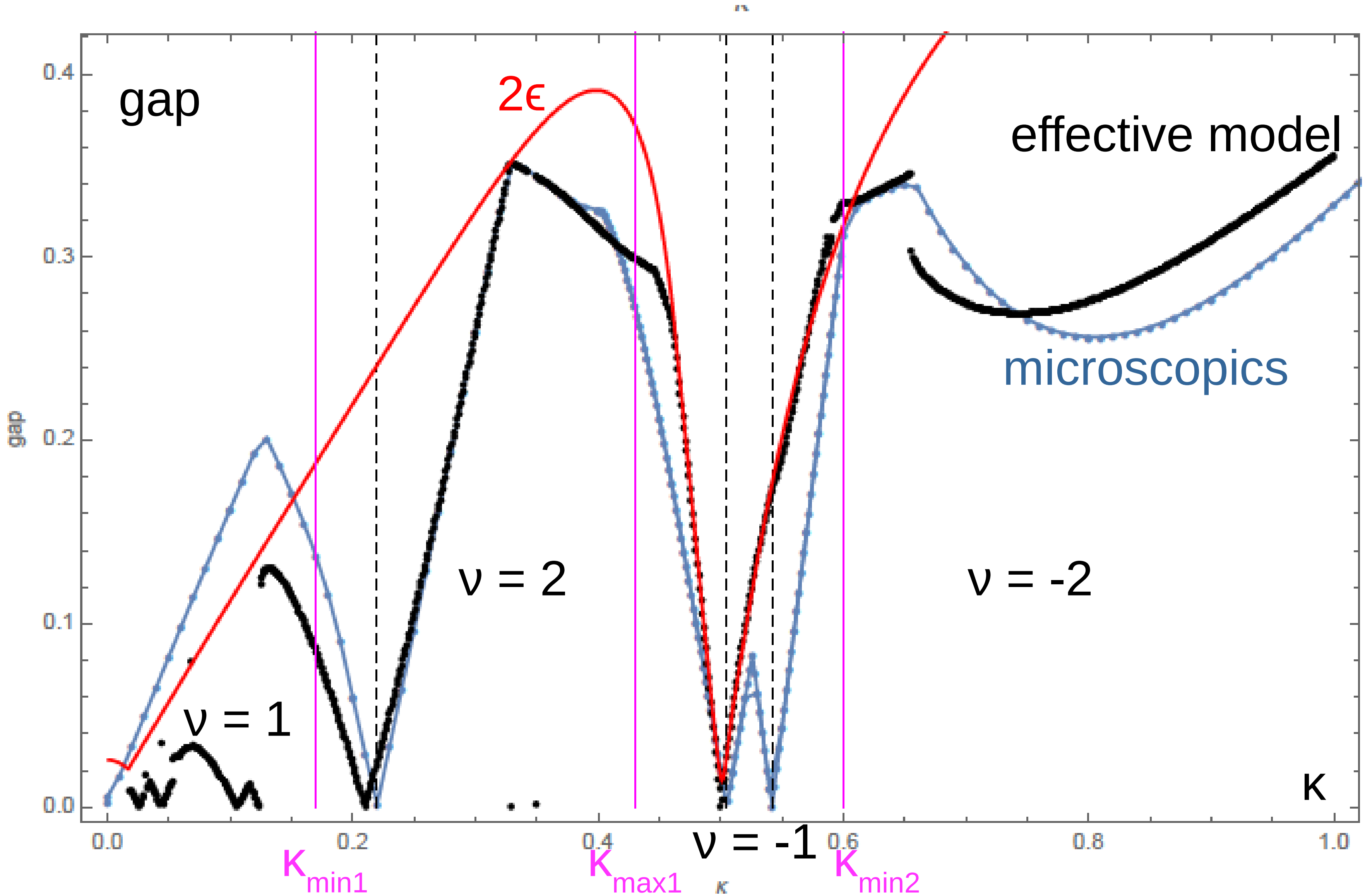
$\upsilon = \upsilon_f + \upsilon_v = \pm 2 + \upsilon_v = 0, \pm 1, \pm 2, \pm 3, \pm 4$

DF on triangular lattice: validity



Gap: effective model VS microscopics

Red family : ρ = 12/13

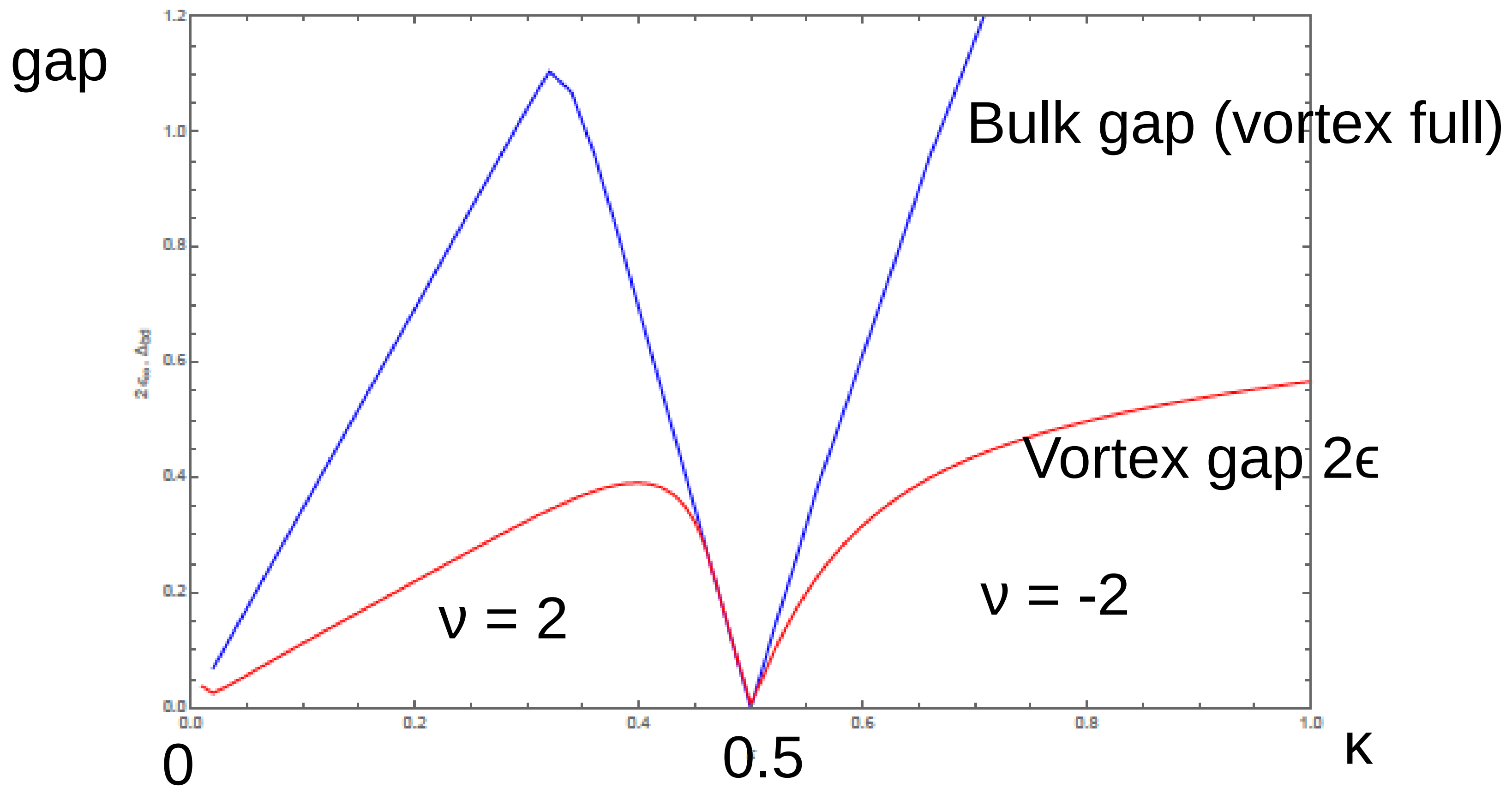


Dilute limit: $\rho = (n-1)/n \rightarrow 1$

Isolated white vortices with $\pm\epsilon$, gap = 2ϵ

$$\upsilon = \upsilon_f + \upsilon_v = \pm 2 + 0$$

Vortex-full background dominates (except at $\kappa \approx 0$ and 0.5)



Conclusion

- All Chern numbers should be accessible in the Kitaev honeycomb model by vortex pattern engineering. In particular modulo 16.
- Triangular vortex lattice:

  Chern = 0;±1;±2;±3;±4;±5;±6;8 but not ±7
- Kitaev honeycomb model (2D topological superconductors): relation btw topological bands (Chern numbers) and topological order (anyons)