# Supersymmetric Lattice Models

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Benasque – 17-19 February 2020

Lecture 3/3

# Twelve Easy Pieces

1. Teaser
2. N=2 susy – Witten index
3. $M_1$ model in 1D: Witten index, spectra, CFT connection
4. $M_1$ model: scaling form of 1-pt functions from CFT
5. $M_k$ models: Witten index, CFT
6. $M_k$ models off criticality → massive (integrable) QFT
7. $M_1$ model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$
8. PH symmetric model and coupled fermion chains
9. **Superfrustration on ladder: zig-zag, Nicolai, $Z_2$ Nicolai**
10. **Superfrustration on 2D grids**
11. **Back to 1D $M_1$ model: kink dynamics**
12. **Towards realization with Rydberg-dressed cold atoms**

# Twelve Easy Pieces

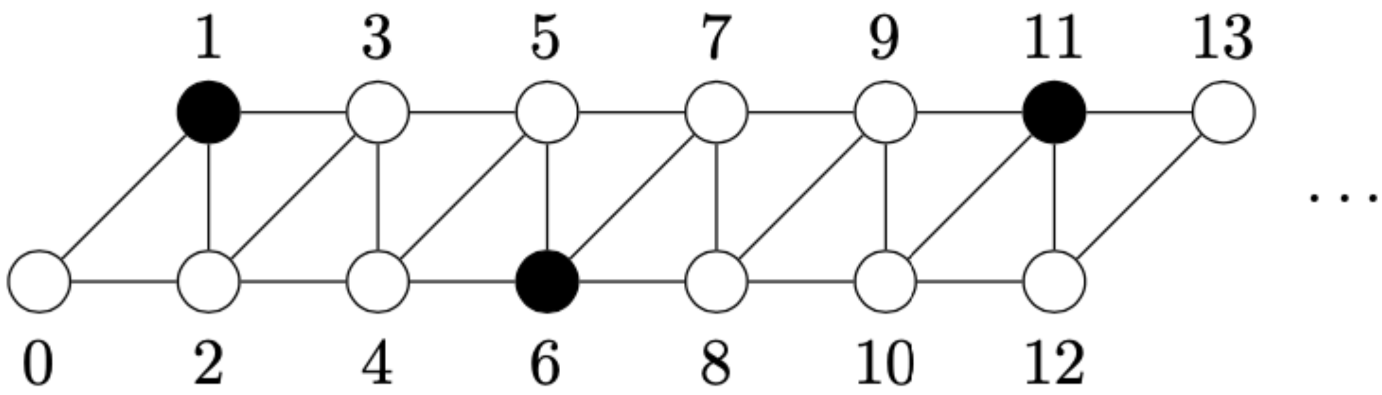
9. **Superfrustration on ladder: zig-zag, Nicolai, $Z_2$ Nicolai**

10. Superfrustration on 2D grids

11. Back to 1D $M_1$ model: kink dynamics

12. Towards realization with Rydberg-dressed cold atoms

# Zig zag ladder – proliferation of susy groundstates

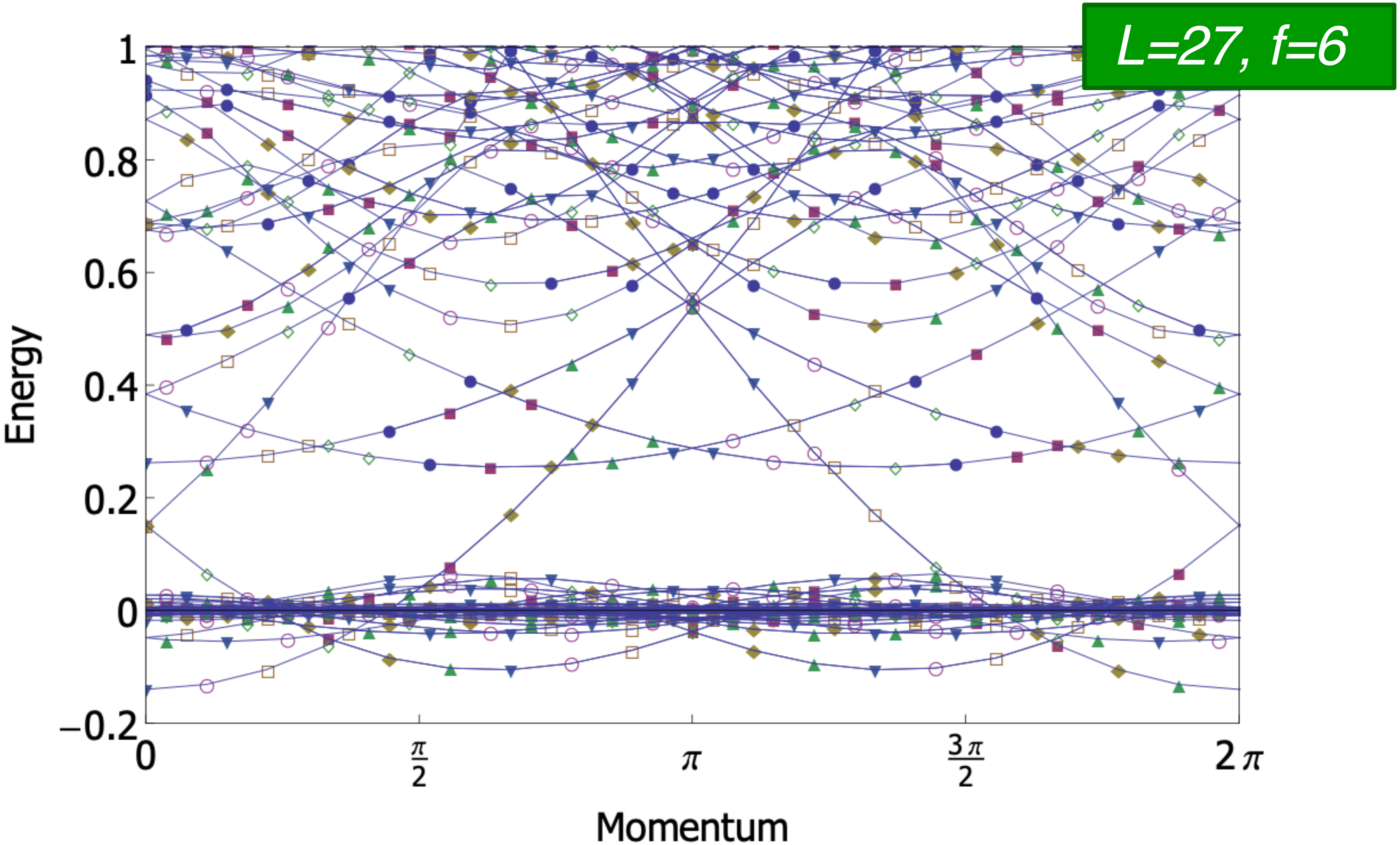



$$P_N(z) = zP_{N-4}(z) + zP_{N-5}(z)$$

$$P_N(z) = \sum_{f \in \mathbb{Z}} \left( \binom{f}{N-4f+2} + \binom{f}{N-4f+1} + \binom{f}{N-4f} \right) z^f.$$

La-Shadrin-KjS 2019

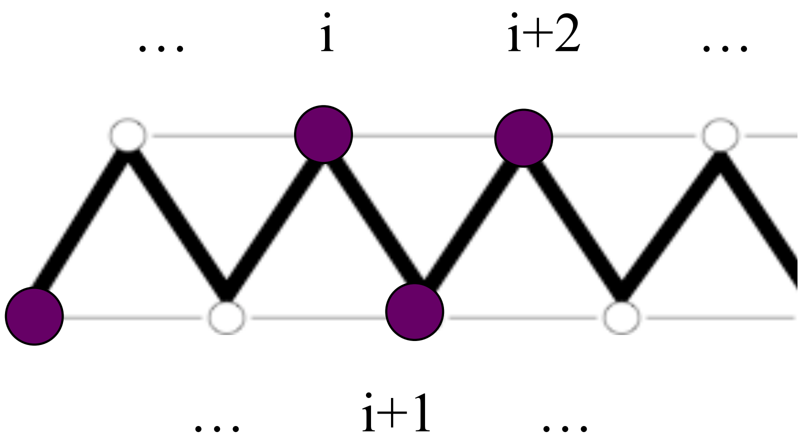
# Zig zag ladder – critical groundstates at 2/9 filling




Huijse, 2010

# Models with $Q$ cubic in $c_i$

**$Z_2$ Nicolai model**




$$Q^{\mathbb{Z}_2} = \sum_i c_i c_{i+1} c_{i+2}$$

Sannomiya-Katsura-Nakayama 2017

***N*=2 susy SYK model**

$$Q^{\mathrm{SYK}} = \sum_{ijk} J_{ijk} c_i c_j c_k$$

random couplings on full graph

Fu-Gaiotto-Maldacena-Sachdev 2017

# Ground state counting

## $Z_2$ Nicolai model

The number of $E$=0 states (OBC)

| $N$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| deg. | 6 | 12 | 20 | 36 | 64 | 112 | 200 | 352 | 624 | 1104 | 1952 |

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

They are generated by the recursion:

$$a_n = 2a_{n-2} + 2a_{n-3}, \qquad a_0 = 1,\ a_1 = 2,\ a_2 = 4$$

$$Z \sim (1.77)^N$$

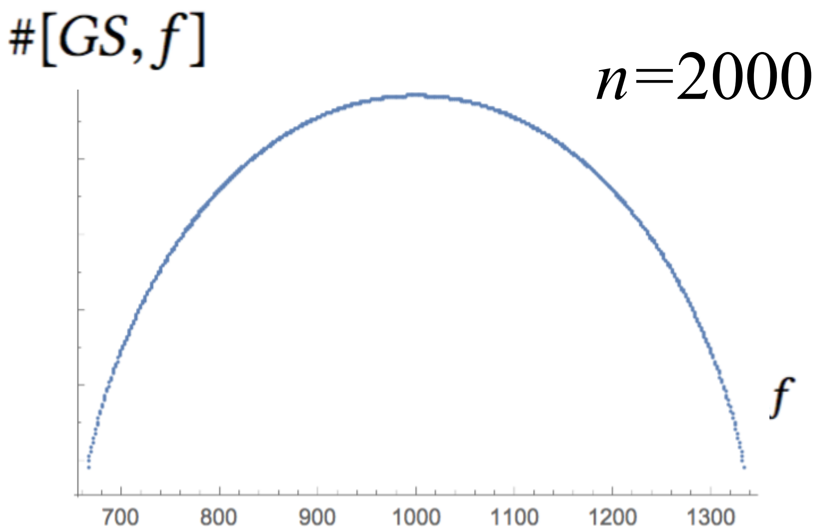
slide H. Katsura in talk at UvA, Nov 2017

# Ground state counting – recent results

**$Z_2$ Nicolai model**

generating function

$$P_n^{\mathbb{Z}_2}(z) := \sum_f \#[GS, f]\, z^f$$




**Theorem 2.2.1.** *The polynomials $P_n^{\mathbb{Z}_2}(z)$, $n \geq 3$, can be determined by the recursion*

$$P_n^{\mathbb{Z}_2}(z) = 2zP_{n-2}(z) + (z + z^2)P_{n-3}^{\mathbb{Z}_2}(z)$$

*with the initial values given by*

$$P_0^{\mathbb{Z}_2}(z) := 1, \qquad P_1^{\mathbb{Z}_2}(z) := 1 + z \quad \textit{and} \quad P_2^{\mathbb{Z}_2}(z) = 1 + 2z + z^2.$$

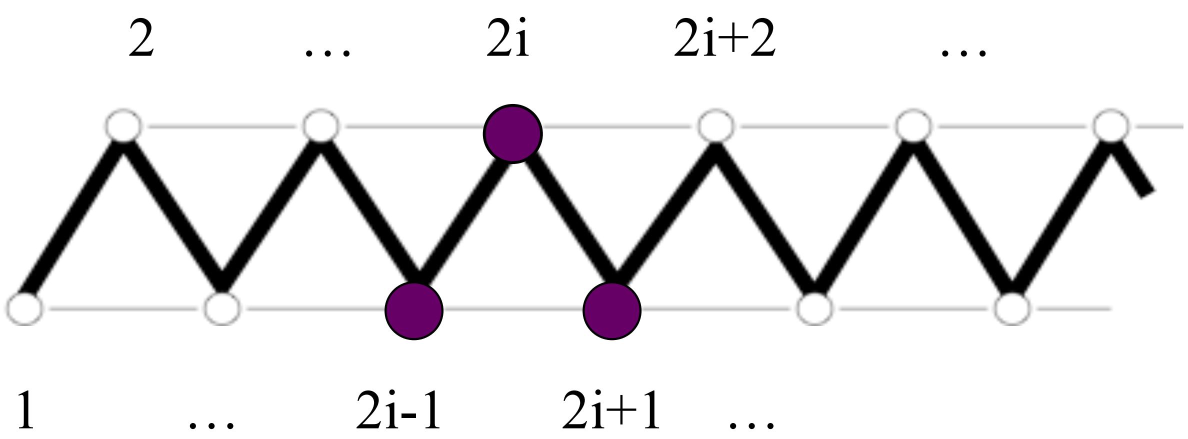
conjecture in Sannomiya-Katsura-Nakayama 2017

proof in La-Shadrin-KjS 2019

# Nicolai model

**supercharge**

$$Q^{\text{Nic}} = \sum_i c_{2i-1} c_{2i}^{\dagger} c_{2i+1}$$




Nicolai 1976

# Ground state counting

## Nicolai model

■ Number of $E$=0 ground states ($N$: odd, OBC)

| $N$ | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|
| deg. | 6 | 20 | 64 | 208 | 672 | 2176 |

Grows exponentially with system size!

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Sorry, but the terms do not match anything in the table.

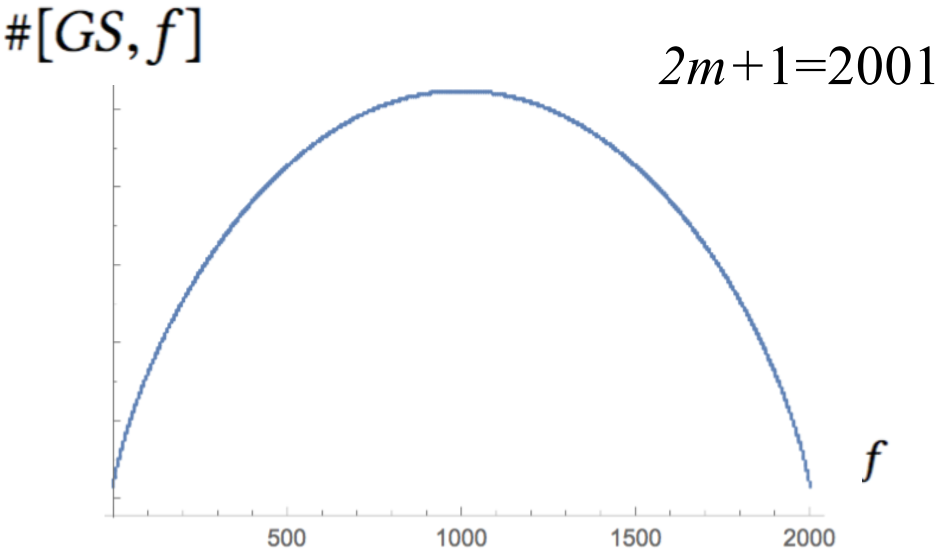
slide H. Katsura in talk at UvA, Nov 2017

# Ground state counting

**Nicolai model**

generating function

$$P_{2m+1}(z) := \sum_f \#[GS,f]\, z^f$$




**Theorem 2.1.1.** *The polynomials* $P_{2m+1}(z)$, $m \geq 3$, *can be determined by the recursion*

$$P_{2m+1}(z) = (1 + z^2)P_{2m-1}(z) + (z + 2z^2 + z^3)P_{2m-3}(z)$$

*with the initial values given by*

$$P_3(z) = 1 + 2z + 2z^2 + z^3 \quad \textit{and} \quad P_5(z) = 1 + 3z + 6z^2 + 6z^3 + 3z^4 + z^5.$$

La-Shadrin-KjS 2019

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10. **Superfrustration on 2D grids**

11. Back to 1D $M_1$ model: kink dynamics

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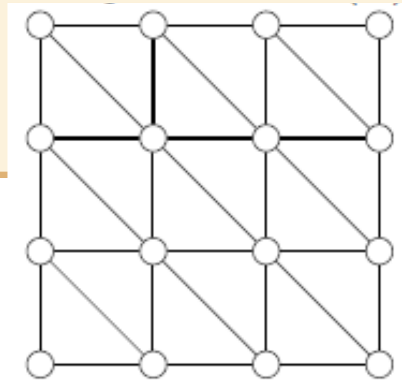
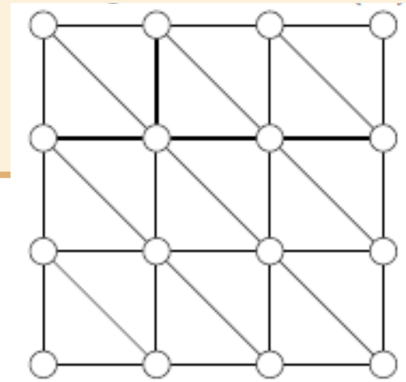
# $M_1$ model, 2D triangular lattice



Witten index for $N{\times}M$ sites with periodic BC

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | -3 | -5 | 1 | 11 |
| 3 | 1 | -5 | -2 | 7 | 1 |
| 4 | 1 | 1 | 7 | -23 | 11 |
| 5 | 1 | 11 | 1 | 11 | 36 |

# $M_1$ model, 2D triangular lattice



Witten index for $N{\times}M$ sites with periodic BC

| | 1 | 2 | 3 | 4 | 5 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | -3 | -5 | 1 | 11 | -5 | 57 |
| 3 | 1 | -5 | -2 | 7 | 1 | -2 | -65 |
| 4 | 1 | 1 | 7 | -23 | 11 | -29 | -279 |
| 5 | 1 | 11 | 1 | 11 | 36 | 811 | -1064 |
| 6 | 1 | 9 | -14 | 25 | -49 | 1462 | -4911 |
| 7 | 1 | -13 | 1 | -69 | 211 | -7055 | 5237 |
| 8 | 1 | -31 | 31 | 193 | -349 | -28517 | 50849 |
| 9 | 1 | -5 | -2 | -29 | 881 | 31399 | 313315 |
| 10 | 1 | 57 | -65 | -279 | -1064 | 313315 | 950592 |
| 11 | 1 | 67 | 1 | 859 | 1651 | 499060 | 2011307 |
| 12 | 1 | -47 | 130 | -1295 | -589 | -2573258 | -3973827 |
| 13 | 1 | -181 | 1 | -77 | -1949 | -10989458 | -49705161 |
| 14 | 1 | -87 | -257 | 3641 | 12611 | 4765189 | -232675057 |
| 15 | 1 | 275 | -2 | -8053 | -32664 | 134858383 | -702709340 |

\`superfrustration’

van Eerten 2005

# Proliferation of susy ground states

**ground state counting problem**

- unsolved for most $D>1$ lattices (including triangular)

**important clue**

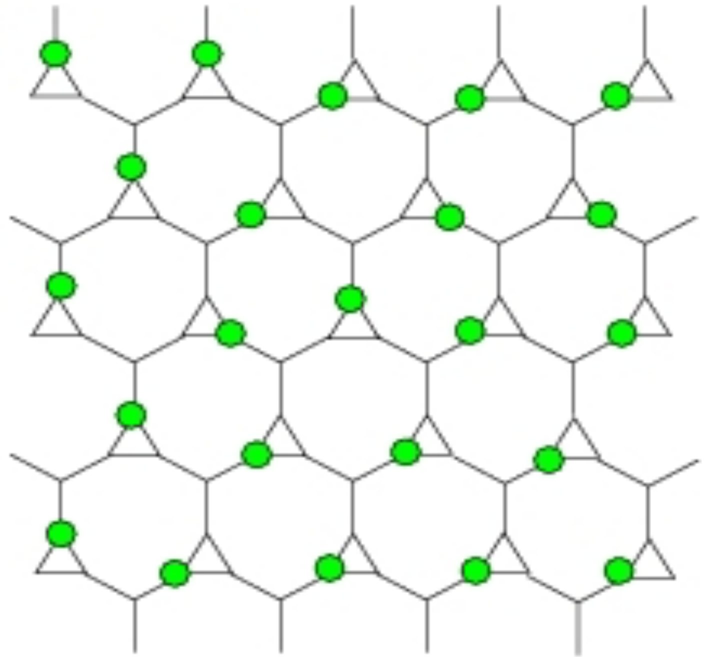
- gs counting problem equivalent to determining (dimension of) the homology of the operator $Q$
- can use methods from math literature (spectral sequences & tic-tac-toe lemma, homological perturbation lemma) to make progress

Fendley-KjS 2005, Huijse-KjS 2010

# Proliferation of susy ground states

**GS counting problem**

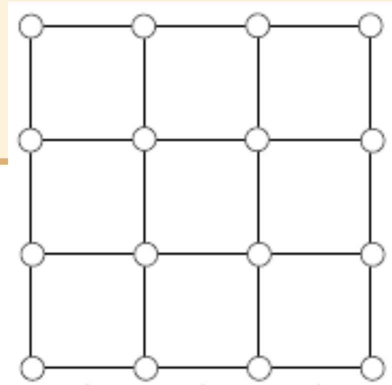
- solved for 2D enneagon-triangle lattice (a.k.a. martini-lattice)
- all GS at filling 1/4
- GS counting ←→ dimer coverings of hexagonal lattice



$$\frac{S_{\mathrm{GS}}}{N} = \frac{1}{\pi}\int_0^{\pi/3} d\theta \, \ln[2\cos(\theta)] = 0.16153\ldots$$

Fendley-KjS 2005

# $M_1$ model, 2D square lattice



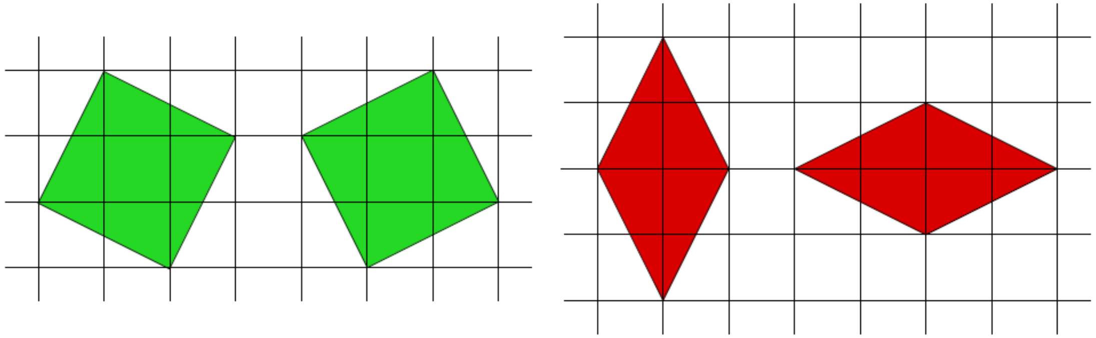
Witten index for $N \times M$ sites with periodic BC

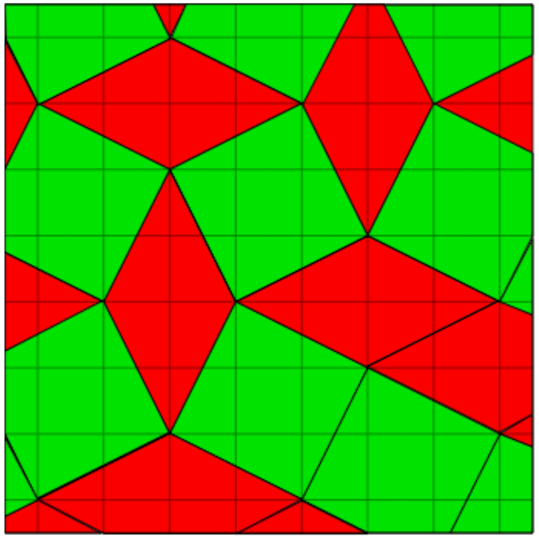
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | −1 | 1 | 3 | 1 | −1 | 1 | 3 | 1 | −1 | 1 | 3 | 1 | −1 | 1 | 3 | 1 | −1 | 1 | 3 |
| 3 | 1 | 1 | 4 | 1 | 1 | 4 | 1 | 1 | 4 | 1 | 1 | 4 | 1 | 1 | 4 | 1 | 1 | 4 | 1 | 1 |
| 4 | 1 | 3 | 1 | 7 | 1 | 3 | 1 | 7 | 1 | 3 | 1 | 7 | 1 | 3 | 1 | 7 | 1 | 3 | 1 | 7 |
| 5 | 1 | 1 | 1 | 1 | −9 | 1 | 1 | 1 | 1 | 11 | 1 | 1 | 1 | 1 | −9 | 1 | 1 | 1 | 1 | 11 |
| 6 | 1 | −1 | 4 | 3 | 1 | 14 | 1 | 3 | 4 | −1 | 1 | 18 | 1 | −1 | 4 | 3 | 1 | 14 | 1 | 3 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −27 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 3 | 1 | 7 | 1 | 3 | 1 | 7 | 1 | 43 | 1 | 7 | 1 | 3 | 1 | 7 | 1 | 3 | 1 | 47 |
| 9 | 1 | 1 | 4 | 1 | 1 | 4 | 1 | 1 | 40 | 1 | 1 | 4 | 1 | 1 | 4 | 1 | 1 | 76 | 1 | 1 |
| 10 | 1 | −1 | 1 | 3 | 11 | −1 | 1 | 43 | 1 | 9 | 1 | 3 | 1 | 69 | 11 | 43 | 1 | −1 | 1 | 13 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 3 | 4 | 7 | 1 | 18 | 1 | 7 | 4 | 3 | 1 | 166 | 1 | 3 | 4 | 7 | 1 | 126 | 1 | 7 |
| 13 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −51 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 1 | −1 | 1 | 3 | 1 | −1 | −27 | 3 | 1 | 69 | 1 | 3 | 1 | 55 | 1 | 451 | 1 | −1 | 1 | 73 |
| 15 | 1 | 1 | 4 | 1 | −9 | 4 | 1 | 1 | 4 | 11 | 1 | 4 | 1 | 1 | 174 | 1 | 1 | 4 | 1 | 11 |

Fendley-KjS-van Eerten 2005

# $M_1$ model, 2D square lattice

Witten index related to rhombus tilings of the lattice





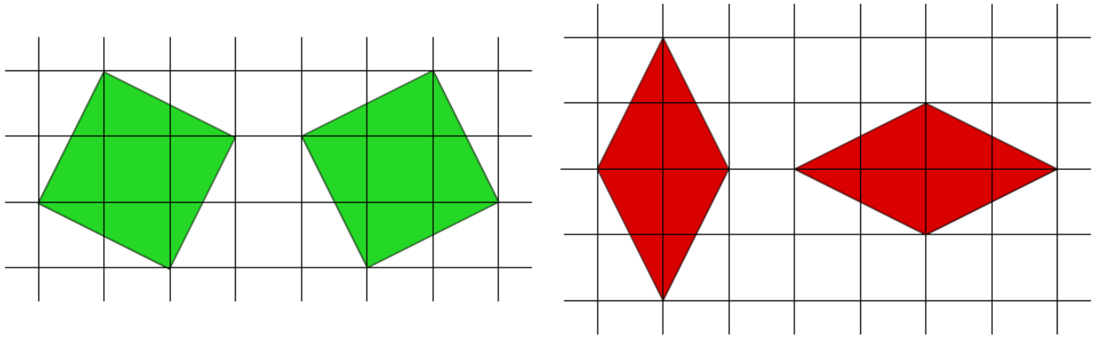
Theorem [Jonsson 2005]

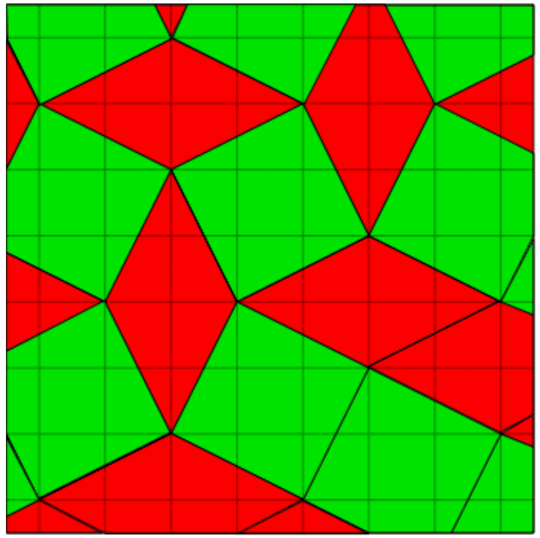
$$W_{\vec{u},\vec{v}} = t_{even} - t_{odd} - (-1)^{d_-}\,\theta_{d_-}\,\theta_{d_+}$$

with $d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$ , $\theta_{3p} = 2$ , $\theta_{3p\pm1} = -1$

# $M_1$ model, 2D square lattice

Number of gs related to rhombus tilings of the lattice, with $F = N_t$





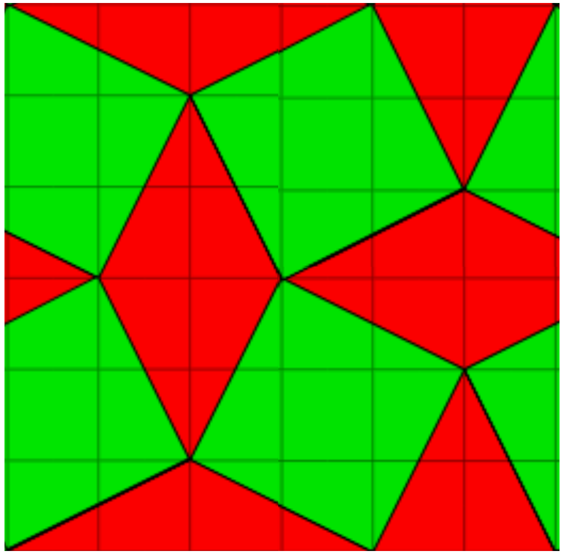
Theorem [Jonsson, Fendley, Huijse-KjS 2009]

$$\# \text{ GS } = t_{even} + t_{odd} - (-1)^{(\theta_m+1)p}\theta_{d_-}\theta_{d_+}$$

with $d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$ , $\theta_{3p} = 2$ , $\theta_{3p\pm1} = -1$

# $M_1$ model, 2D square lattice

**Example: square lattice 6x6**



- 18 tilings with $N_t$=8
- correction term equals -4

⇒ 14 groundstates with f=8, filling 2/9

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→ work in progress

# $M_1$ model on general graph - challenge

**Challenge:**

For each of the other platonic solid graphs, find the number of *E=0* susy groundstates and, for each of them, the number of fermions

**Reward**

A chocolate bar for the first correct result @ c.j.m.schoutens@uva.nl

# $M_1$ model on general graph - challenge

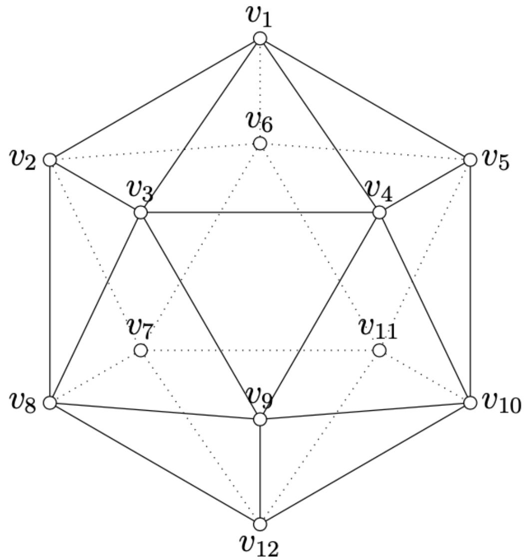
**Super frustrated lattice models**

Tristan Kuen

December 18, 2015

Bachelors project mathematics and physics

Supervisors: prof. dr. Kareljan Schoutens, dr. Raf Bocklandt




**Proposition 4.** *The icosahedron has six two-particle and one three-particle ground states.*

$$
\begin{aligned}
|\phi_3\rangle = & |1,7,9\rangle + |1,9,11\rangle + |1,11,8\rangle + |1,8,10\rangle + |1,10,7\rangle \\
& + |7,5,9\rangle + |9,2,11\rangle + |11,4,8\rangle + |8,6,10\rangle + |10,3,7\rangle \\
& + |5,2,9\rangle + |2,4,11\rangle + |4,6,8\rangle + |6,3,10\rangle + |3,5,7\rangle \\
& + |2,5,12\rangle + |4,2,12\rangle + |6,4,12\rangle + |3,6,12\rangle + |5,3,12\rangle \,.
\end{aligned}
$$