Combining Tensor Networks and Monte Carlo for Lattice Gauge Theories

Entanglement in Strongly Correlated Systems, Benasque

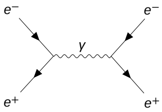
21st of February 2020 | *Patrick Emonts*, E. Zohar, M. C. Banuls, I. Cirac | MPI of Quantum Optics

MPQ

Why do we need Lattice Gauge Theories?

QED

$$\mathcal{L}_{QED} = i\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - e\overline{\Psi}\gamma_{\mu}A^{\mu}\Psi - m\overline{\Psi}\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$



Small coupling

$$\alpha_{QED} = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

QCD

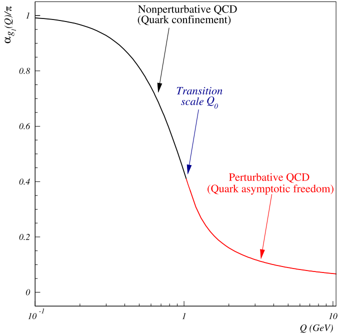


Image adapted from Alexandre Deur, Stanley J. Brodsky, and Guy F. de Téramond, 2016, *Progress in Particle and Nuclear Physics*

Path integral formalism in QFT

pure QED

$$S_{QED}[A_\mu] = -\frac{1}{4}\int \mathrm{d}x^\alpha F_{\mu\nu}(x_\alpha)F^{\mu\nu}(x^\alpha) = \int \mathrm{d}x^\alpha \partial_\mu A_\nu(x^\alpha)\partial^\nu A^\mu(x^\alpha)$$

vacuum expectation value

$$\langle\Omega|O[A_\mu]|\Omega\rangle = \frac{\int \mathcal{D}A O[A_\mu] e^{iS_{QED}[A_\mu]}}{\int \mathcal{D}A e^{iS_{QED}[A_\mu]}}$$

Problems

- ✖ Numerator oscillating
- ✖ Integration measure ill-defined

Wick rotation

Shift to imaginary time

$$t \rightarrow -i\tau$$

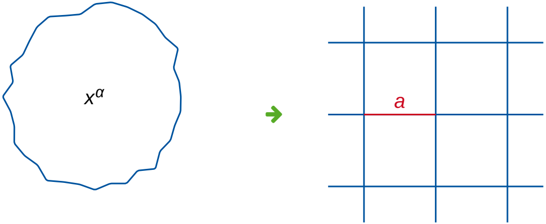
Change of metric from Minkowski to Euclidean

$$e^{iS_M} = e^{i \int \mathrm{d}x_M^\alpha \mathcal{L}(x_M^\alpha)} \longrightarrow e^{-\int \mathrm{d}x_E^\alpha \mathcal{L}(x_E^\alpha)} = e^{-S_E}$$

Problems

- ✔ Numerator converging
- ✖ Integration measure ill-defined

Discretization: Lattice Gauge Theory



$$A_\mu \rightarrow U_\mu = e^{iaA_\mu}$$

Find the lattice action $\tilde{S}_E$ that agrees with S_E in the continuum limit of vanishing a

$$\tilde{S}_E[U] \rightarrow S_E[A](a \rightarrow 0)$$

Kenneth G. Wilson, 1974, *Physical Review D*

Vacuum expectation value in the action formalism

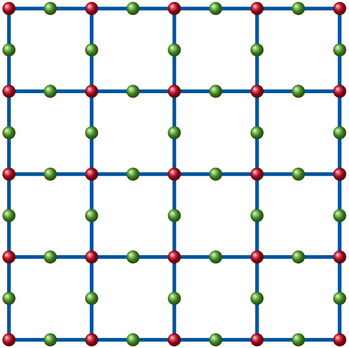
Vacuum expectation value

$$\langle O[U] \rangle = \frac{\int \mathcal{D}U O[U] e^{-S_E[U]}}{\int \mathcal{D}U e^{-S_E[U]}} \text{ with } \mathcal{D}U = \prod_{x^\alpha} \mathrm{d}U_\mu(x^\alpha)$$

Problems

- ✔ Numerator converging
- ✔ Integration with the Haar measure

Lattice Systems



Hilbert space

$$\mathcal{H} \subset \mathcal{H}_{\text{gauge fields}} \otimes \mathcal{H}_{\text{fermions}}$$

A general state

$$|\Psi\rangle = \int \mathcal{D}\mathcal{G}\, |\mathcal{G}\rangle \left|\Psi_F(\mathcal{G})\right\rangle$$

with $\mathcal{D}\mathcal{G} = \prod_{\mathbf{x},k} dg(\mathbf{x},k)$

Erez Zohar and J. Ignacio Cirac, 2018, *Physical Review D*
Patrick Emonts and Erez Zohar, 2020, *SciPost Physics Lecture Notes*

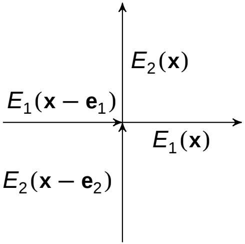
Gauss law

Gauss law

$$\sum_k \left(E_k(\mathbf{x}) - E_k(\mathbf{x} - \mathbf{e}_i)\right) |\text{phys}\rangle = 0 \quad \forall \mathbf{x}$$

Classical analogue in (cont.) electrodynamics

$$\nabla \cdot \mathbf{E} = 0$$



Expectation value of an Observable

Assume that O acts only on the gauge field and is diagonal in the group element basis:

$$\begin{aligned}\langle O\rangle &= \frac{\langle\Psi|O|\Psi\rangle}{\langle\Psi|\Psi\rangle}\\ &= \frac{\int \mathcal{D}\mathcal{G}\,\langle\mathcal{G}|\,O\,|\mathcal{G}\rangle\,\big\langle\Psi_F(\mathcal{G})\big|\Psi_F(\mathcal{G})\big\rangle}{\int \mathcal{D}\mathcal{G}'\,\big\langle\Psi_F(\mathcal{G}')\big|\Psi_F(\mathcal{G}')\big\rangle}\\ &= \int \mathcal{D}\mathcal{G}\mathcal{F}_O(\mathcal{G})p(\mathcal{G})\end{aligned}$$

with $p(\mathcal{G}) = \dfrac{\big\langle\Psi_F(\mathcal{G})\big|\Psi_F(\mathcal{G})\big\rangle}{\int \mathcal{D}\mathcal{G}'\big\langle\Psi_F(\mathcal{G}')\big|\Psi_F(\mathcal{G}')\big\rangle} = \dfrac{\big\langle\Psi_F(\mathcal{G})\big|\Psi_F(\mathcal{G})\big\rangle}{Z}$

The rest of this talk

Expectation value

$$\langle O \rangle = \int \mathcal{D}\mathcal{G} \mathcal{F}_O(\mathcal{G}) p(\mathcal{G})$$

$$\text{with } p(\mathcal{G}) = \frac{\langle \Psi_F(\mathcal{G}) | \Psi_F(\mathcal{G}) \rangle}{Z}$$

TODO List

1. How do we construct $|\Psi_F(\mathcal{G})\rangle$?
2. How do we efficiently calculate $p(\mathcal{G})$?
3. Are those states useful?

Creation of the fermionic state

Desirable properties

- $|\Psi\rangle$ fulfills the Gauss law
- $\left|\Psi_F(\mathcal{G})\right\rangle$ allows efficient calculations of
  - the norm
  - expectation values

Definition of Ψ

$$|\Psi\rangle = \int \mathcal{D}\mathcal{G}\, |\mathcal{G}\rangle \left|\Psi_F(\mathcal{G})\right\rangle$$

Choice for $\left|\Psi_F(\mathcal{G})\right\rangle$

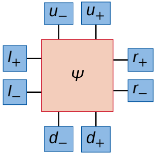
We construct $\left|\Psi_F(\mathcal{G})\right\rangle$ with a tensor network.

Patrick Emonts and Erez Zohar, 2020, *SciPost Physics Lecture Notes*

Definition of Modes

Gauss law in terms of our modes

$$G_0 = E_r - E_l + E_u - E_d$$
$$= r_+^\dagger r_+ - r_-^\dagger r_- - l_+^\dagger l_+ + l_-^\dagger l_- + u_+^\dagger u_+ - u_-^\dagger u_- - d_+^\dagger d_+ + d_-^\dagger d_-$$



Definition of positive and negative modes

a: $\{l_+, r_-, u_-, d_+\}$ (neg. modes)
b: $\{l_-, r_+, u_+, d_-\}$ (pos. modes)

Erez Zohar et al., 2015, *Annals of Physics*

Creating a fermionic state

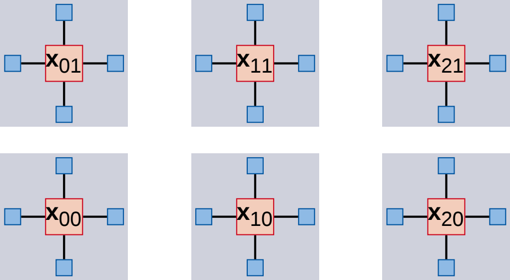
The state

$$|\psi_0\rangle = \langle \Omega_V| \qquad\qquad\qquad\qquad |\Omega\rangle$$

Creating a fermionic state

The state

$$|\psi_0\rangle = \langle \Omega_V| \qquad \prod_{\mathbf{x}} \mathcal{A}(\mathbf{x}) \, |\Omega\rangle$$

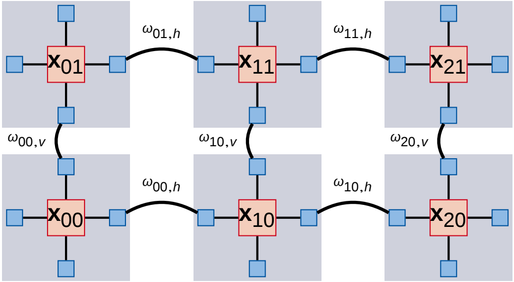


$$\mathcal{A}(\mathbf{x}) = \exp\left(\sum_{ij} T_{ij} a_i^\dagger(\mathbf{x}) b_j^\dagger(\mathbf{x}) \right)$$

Creating a fermionic state

The state

$$|\psi_0\rangle = \langle \Omega_V | \prod_{\mathbf{x},k} \omega(\mathbf{x},k) \prod_{\mathbf{x}} \mathcal{A}(\mathbf{x}) |\Omega\rangle$$



$$\mathcal{A}(\mathbf{x}) = \exp\left(\sum_{ij} T_{ij} a_i^\dagger(\mathbf{x}) b_j^\dagger(\mathbf{x})\right)$$

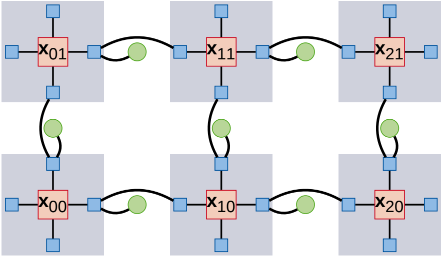
$$\begin{aligned} \omega(\mathbf{x},k) &= \omega_k(\mathbf{x}) \Omega_k(\mathbf{x}) \omega_k^\dagger(\mathbf{x}) \\ \omega_0(\mathbf{x}) &= \exp\left(l_+^\dagger(\mathbf{x}+\mathbf{e}_1) r_-^\dagger(\mathbf{x})\right) \\ &\quad \exp\left(l_-^\dagger(\mathbf{x}+\mathbf{e}_1) r_+^\dagger(\mathbf{x})\right) \end{aligned}$$

Moving towards local symmetry

Lattice Gauge theory

We demand a local symmetry

$$\sum_{\mathbf{x}} G(\mathbf{x})\,|\Psi\rangle = 0 \rightarrow G(\mathbf{x})\,|\Psi\rangle = 0$$

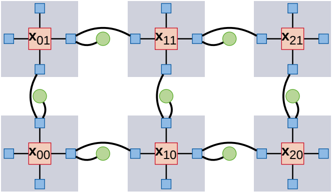


Erez Zohar et al., 2015, *Annals of Physics*
Erez Zohar and Michele Burrello, 2016, *New Journal of Physics*

Local symmetry – The state

Substitution

$$
\begin{aligned}
r_{\pm}^{\dagger}(\mathbf{x}) &\to e^{\pm i\theta(\mathbf{x})} r_{\pm}^{\dagger}(\mathbf{x}) \\
u_{\pm}^{\dagger}(\mathbf{x}) &\to e^{\pm i\theta(\mathbf{x})} u_{\pm}^{\dagger}(\mathbf{x})
\end{aligned}
$$



Fermionic state

Fermionic state

$$|\psi(\mathcal{G})\rangle = \langle\Omega_V| \prod_{\mathbf{x}} \omega(\mathbf{x}) \prod_{\mathbf{x}} \mathcal{U}_{\Phi(\mathbf{x})} \prod_{\mathbf{x}} A(\mathbf{x}) |\Omega\rangle$$

- ✔ Gauge invariance of $|\Psi\rangle$ by constructing $\Psi(\mathcal{G})$
- ✔ Obeys all demanded symmetries
- ? Efficient to calculate with

Is $|\Psi_F(\mathcal{G})\rangle$ special?

The fermionic state $|\Psi_F(\mathcal{G})\rangle$

$$|\Psi_F(\mathcal{G})\rangle = \langle\Omega_v| \prod_{\mathbf{x}} \omega(\mathbf{x}) \prod_{\mathbf{x}} \mathcal{U}_{\Phi(\mathbf{x})} \prod_{\mathbf{x}} \mathcal{A}(\mathbf{x}) |\Omega\rangle$$

$$\mathcal{A}(\mathbf{x}) = \exp\left(\sum_{ij} T_{ij} a_i^\dagger(\mathbf{x}) b_j^\dagger(\mathbf{x}) \right)$$

$$\omega(x) = \omega_0(\mathbf{x})\omega_1(\mathbf{x})\Omega(\mathbf{x})\omega_1^\dagger(\mathbf{x})\omega_0^\dagger(\mathbf{x})$$
$$\omega_0(\mathbf{x}) = \exp\left(l_+^\dagger(\mathbf{x}+\mathbf{e}_1) r_-^\dagger(\mathbf{x})\right) \exp\left(l_-^\dagger(\mathbf{x}+\mathbf{e}_1) r_+^\dagger(\mathbf{x})\right)$$
$$\omega_1(\mathbf{x}) = \exp\left(d_+^\dagger(\mathbf{x}+\mathbf{e}_2) u_-^\dagger(\mathbf{x})\right) \exp\left(d_-^\dagger(\mathbf{x}+\mathbf{e}_2) u_+^\dagger(\mathbf{x})\right)$$

Gaussian States

Definition

Fermionic Gaussian states are represented by density operators that are exponentials of a quadratic form in Majorana operators.

$$\rho = K \exp\left(-\tfrac{i}{4}\gamma^T G \gamma\right)$$

Covariance matrix

Covariance matrix for a state Φ:

$$\Gamma_{ab} = \tfrac{i}{2}\left\langle [\gamma_a, \gamma_b] \right\rangle = \tfrac{i}{2}\frac{\langle \Phi | [\gamma_a, \gamma_b] | \Phi \rangle}{\langle \Phi | \Phi \rangle}$$

Sergey Bravyi, 2005, *Quantum Inf. and Comp.*

Calculating the Norm and the Observables

$$|\psi(\mathcal{G})\rangle = \langle\Omega_V| \underbrace{\prod_{\mathbf{x}} \omega(x) \prod_{\mathbf{x}} \mathcal{U}_{\Phi(\mathbf{x})}}_{\rightsquigarrow \Gamma_{\text{in}}(\mathcal{G})} \underbrace{\prod_{\mathbf{x}} A(\mathbf{x})}_{\rightsquigarrow \Gamma_M} |\Omega\rangle$$

$$\Gamma^M_{i,j} = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}$$

A Physical-Physical correlations

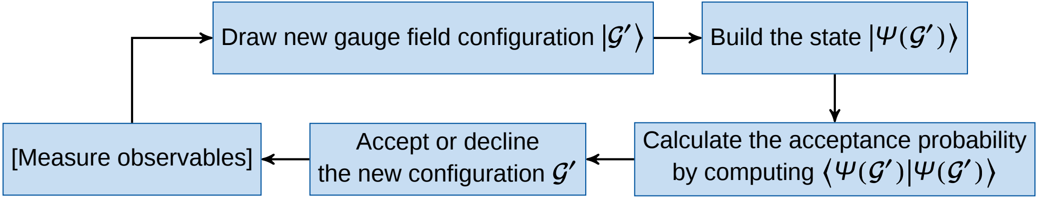
B Physical-Virtual correlations

C Virtual-Virtual correlations

Norm

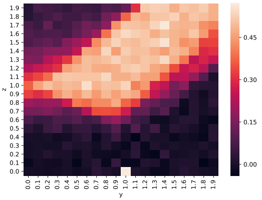
$$\langle\psi(\mathcal{G})|\psi(\mathcal{G})\rangle = \sqrt{\det\left(\frac{1-\Gamma_{\text{in}}(\mathcal{G})M_D}{2}\right)}$$

The whole framework

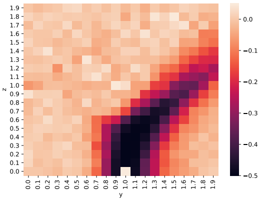


Results for $\mathbb{Z}_3$

Wilson loop

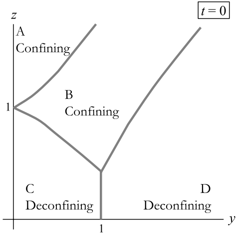


Polyakov loop



Transfer matrix Calculation

Erez Zohar et al., 2015, *Annals of Physics*

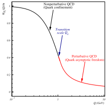


Different phases

We can model different phases with our variational Ansatz for the state.

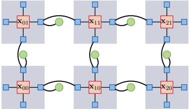
Conclusion and Outlook

- We need Lattice Gauge Theories



- A Hamiltonian approach shows promising possibilities (time evolution, finite μ)

- The GGPEPS Ansatz shows confined and non-confined phases



- Formulation of a variational minimization procedure for the energy
- Optimization of the Monte Carlo procedure for the sampling

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