Spectral gaps in PEPS:
the possible and the impossible

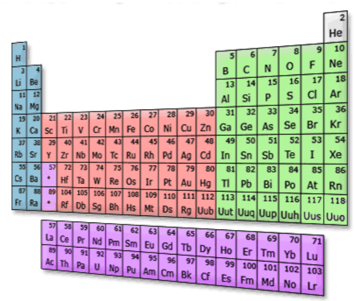
David Pérez-García

Quantum Information Technologies in Madrid

Project 648913

REVIEW ON ANALYTICAL RESULTS ON PEPS.

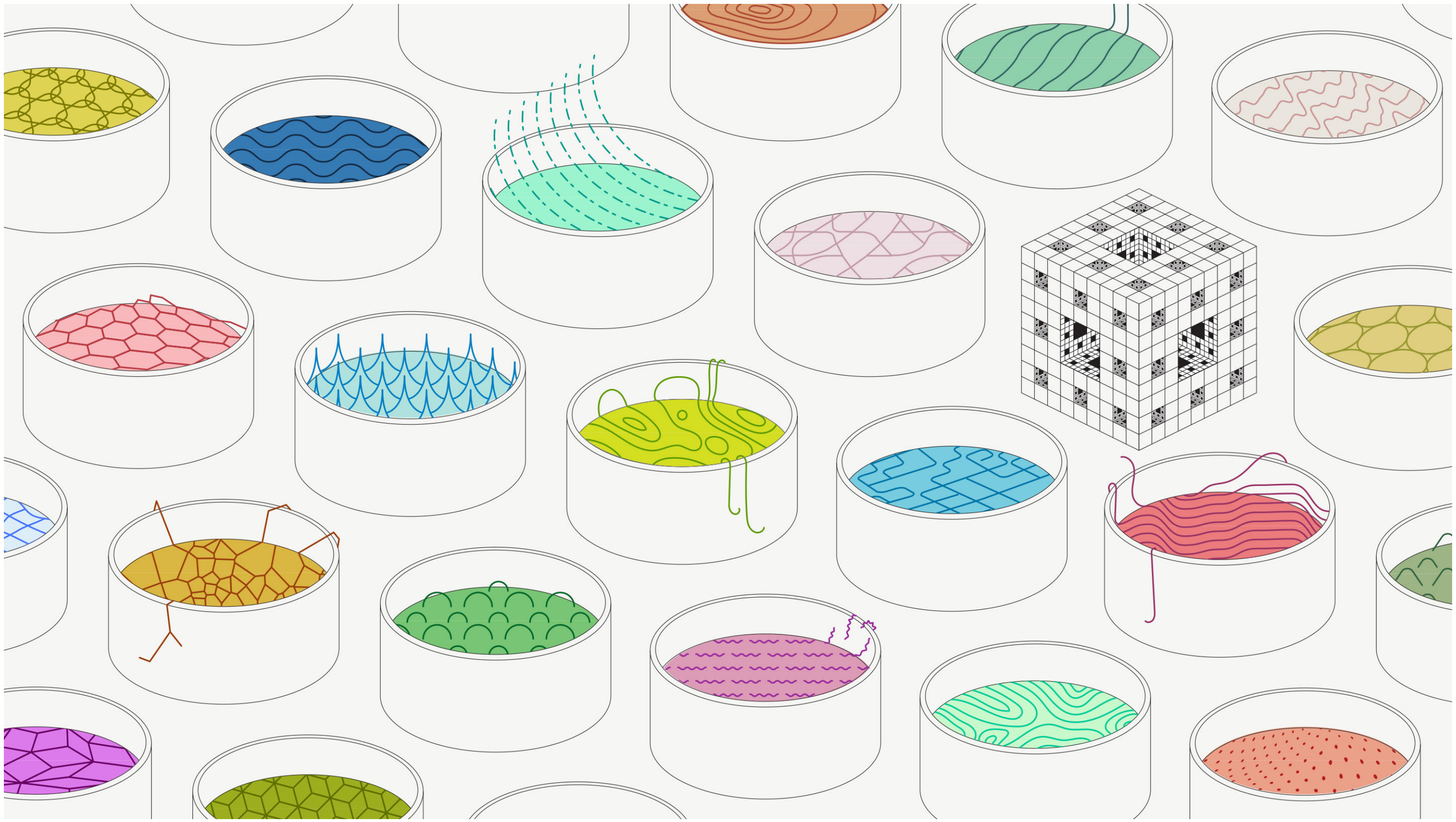
I. Cirac, D Perez-Garcia, N Schuch, F Verstraete, Matrix product states and projected entangled pair states: Concepts, symmetries, and theorems, arXiv preprint arXiv:2011.12127



Motivation:

Classification of topological quantum phases

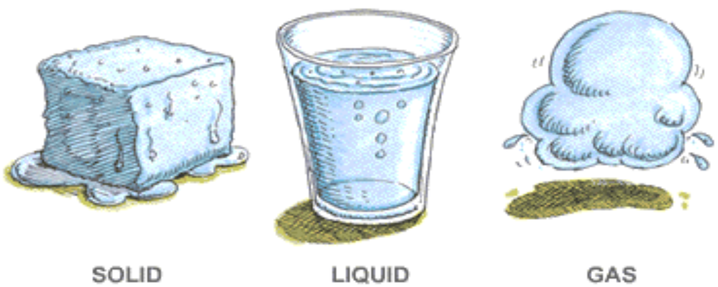
Very nice outreach article in Quanta Magazine



Quantum phases.

What is a phase?

Temperature



Phase transitions

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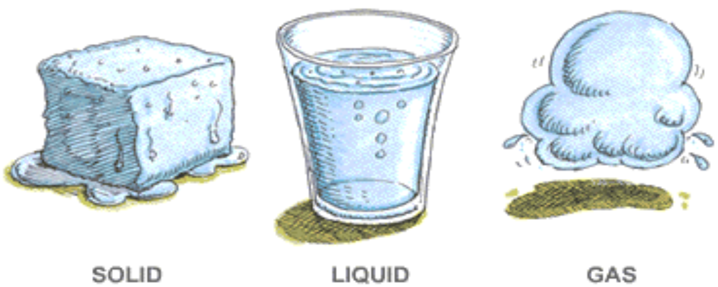
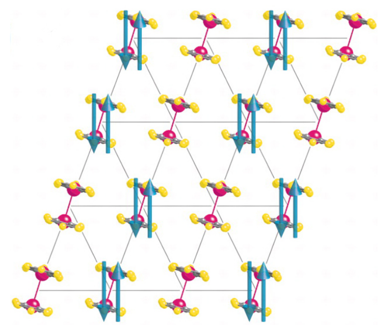


Phase transitions

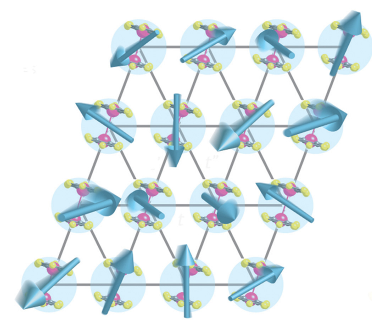
At zero temperature: Quantum phases.

They include very **exotic phenomena:** topological order, superconductivity, spin liquids, etc.

Strength of repulsion terms



Crystal



Spin liquid

Quantum phases.

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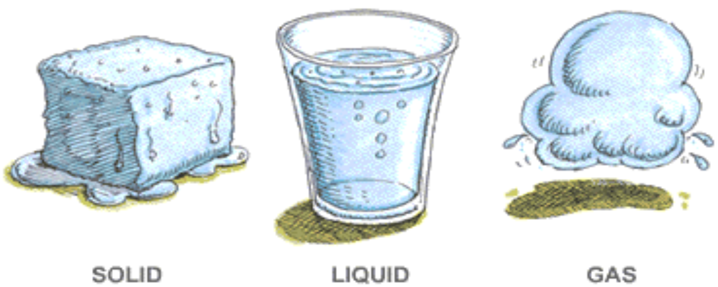

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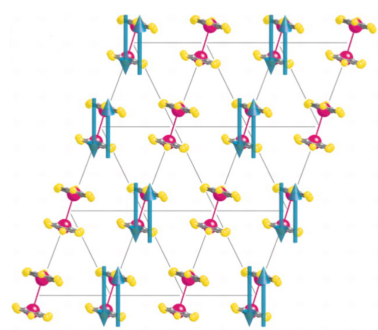

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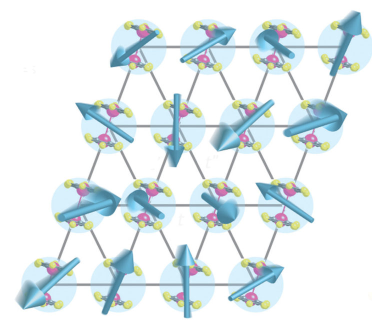
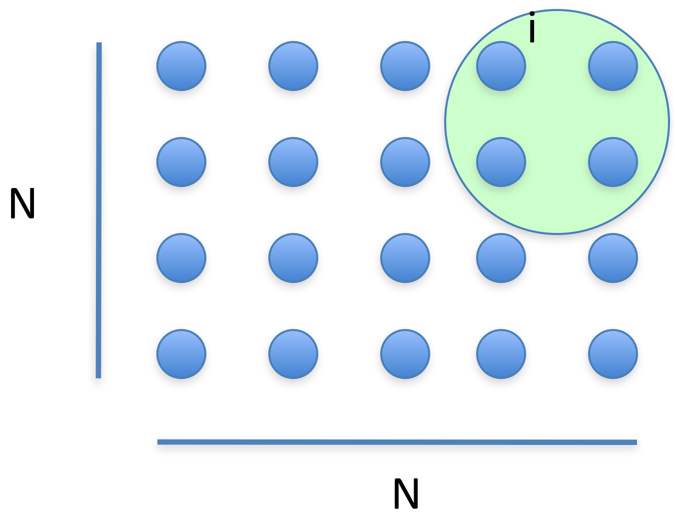


Spin liquid

A **phase** should be something like: “the equivalence class of all states of matter with *similar* properties”

Making it formal. The spectral gap

The spectral gap



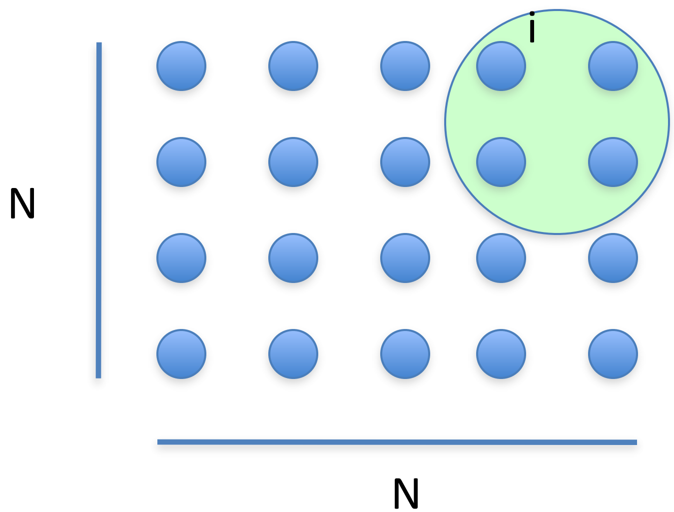
Spin s particles.

Translational invariant finite range interaction

Hamiltonian:

$$H = \sum_i h_i \otimes Id_{\mathrm{rest}}$$

The spectral gap



Spin s particles.

Translational invariant finite range interaction

Hamiltonian:

$$H = \sum_i h_i \otimes Id_{\text{rest}}$$

Spectral gap: $\gamma_N = \lambda_1(N) - \lambda_0(N)$

The system has gap if there exists c>0 such that $\Delta_N > c$ for all N?

Quantum phases

PHASE = an equivalence relation on $\bigcup_r M_{d^r}$

Two systems governed by interactions h^0, h^1 are in the same phase iff there is a smooth path of interactions $[0,1] \ni \alpha \mapsto h^\alpha$ and a constant c>0 s.t the gap $\Delta_N(\alpha)$ of the Hamiltonian $\sum_i h_i^\alpha \otimes Id_{\text{rest}}$ is $\Delta_N(\alpha) > c$ for all N, α

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Two main reasons for this definition:

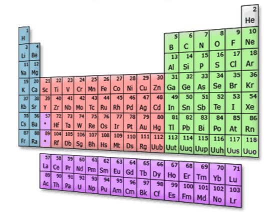
It is stable against small errors in the interactions.

Observable quantities on the *ground state* behave smoothly through the path (no phase transitions).

An approach to the classification of phases with Tensor Networks:

The 1D case

The 1D case



Ingredient 1 (Hastings 2007, Arad et al. 2013): MPS approximate well ground states of 1D Hamiltonians with gap.

Ingredient 2 (Nachtergaele 1995): A lower bound on the spectral gap of certain (parent) Hamiltonians having MPS as ground states

Ingredient 3 (Verstraete et al. 2005, Perez-Garcia et al. 2007): A good description of renormalization transformations in MPS and the structure (phase invariants) of their fixed points.

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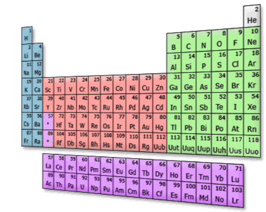

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$|\mathrm{MPS}_1\rangle$ Enough to start with MPS. **Ingredient 1** $|\mathrm{MPS}_2\rangle$

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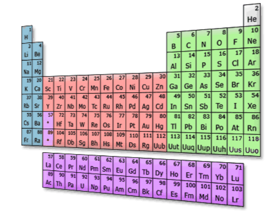

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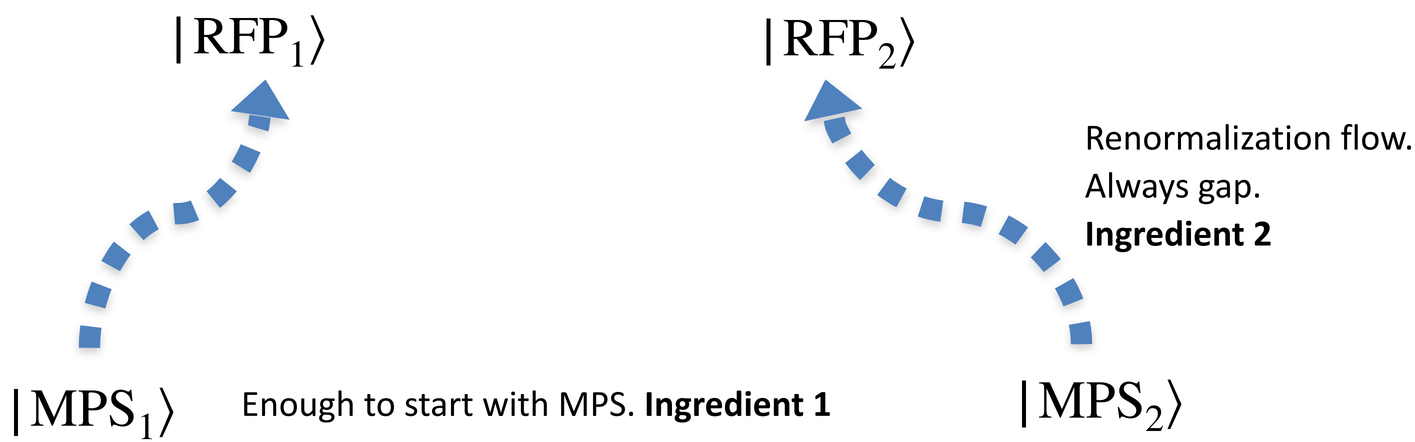

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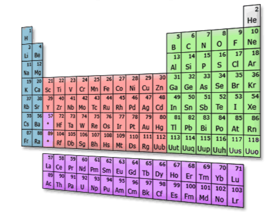

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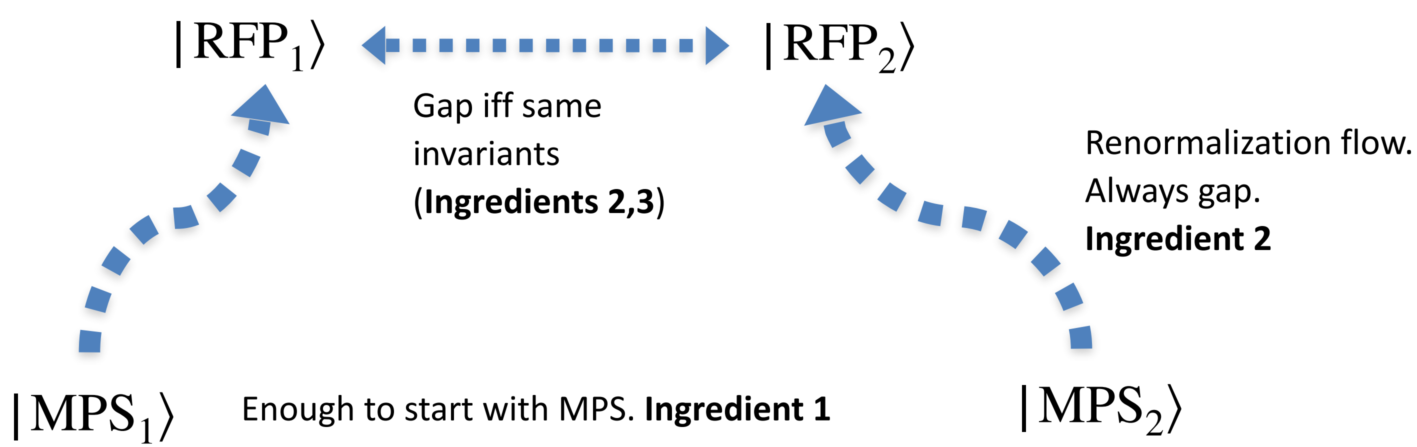

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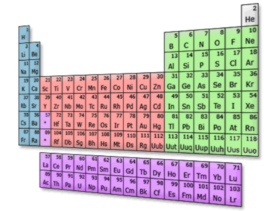


PERIODIC TABLE in 1D:
Phases are indexed by the degeneracy of the ground space

Ingredient 4 (Perez-Garcia et al 2008): A characterization for MPS of global symmetries.

PERIODIC TABLE for Symmetry Protected Topological (SPT) Phases in 1D:
Phases are indexed by the second cohomology group.

Pollmann, Berg, Turner, Oshikawa, Phys. Rev. B. 81, 064439 (2010)
Chen, Gu, Wen, Phys. Rev. B 83, 035107 (2011)
Fidkowski, Kitaev, Phys. Rev. B 83, 075103 (2011)
Schuch, Pérez-García, Cirac, Phys. Rev. B 84, 165139 (2011)
Haegeman, Pérez-García, Cirac, Schuch, Phys. Rev. Lett. 109, 050402 (2012)
Bachman, Nachtergaele, Commun. Math. Phys., 329, 509-544 (2014)

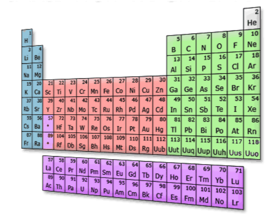
Y. Ogata: Extend this program to the set of Hamiltonians

Y. Ogata, arXiv:1810.01045, arXiv:1908.08621, arXiv:2101.00426

An approach to the classification of phases with Tensor Networks:

Ingredients in the 2D case

The 2D case



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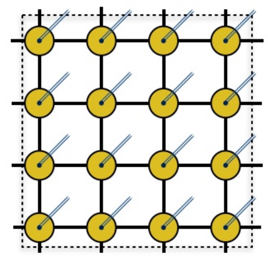

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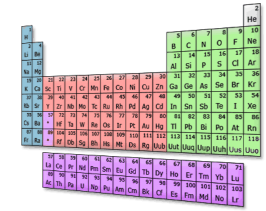

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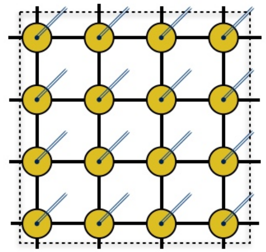

Ingredient 3 (Bultinck et al 2015, Cirac et al 2017, …): RFP are in one to one correspondence with fusion categories.

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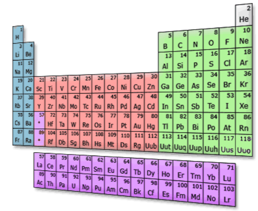

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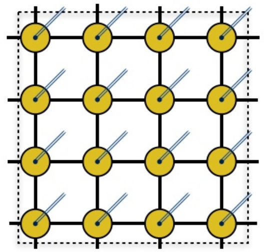

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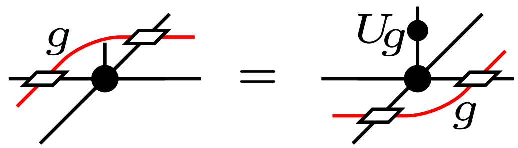

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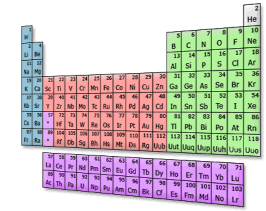

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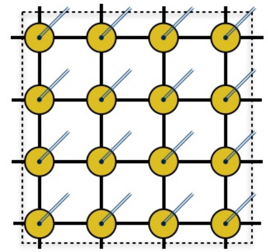

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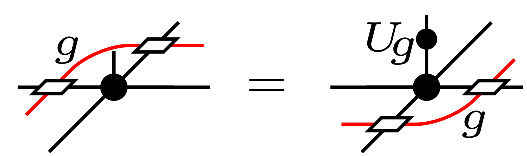


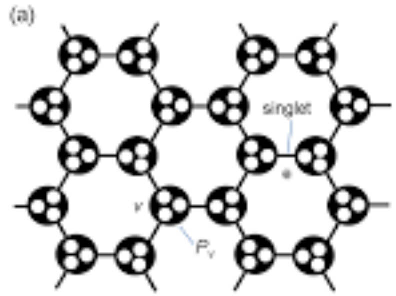
Ingredient 2 (gap) ¿?   THIS IS THE MAIN GOAL OF THESE LECTURES

SPECTRAL GAP:

THE IMPOSSIBLE

The gap problem is notoriously hard, even for PEPS

Example 1: AKLT



Non-topological (**gapped**) spin liquid
Universal for measurement based quantum computation

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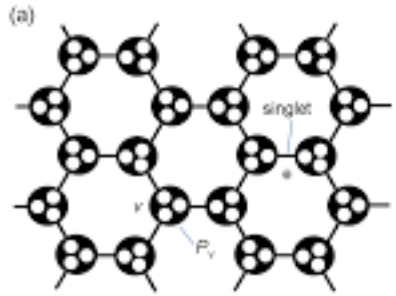

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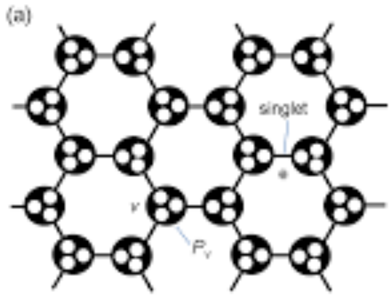

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Example 2: RVB state

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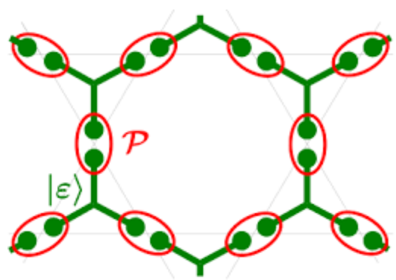
Proposed to explain high-Tc superconductivity



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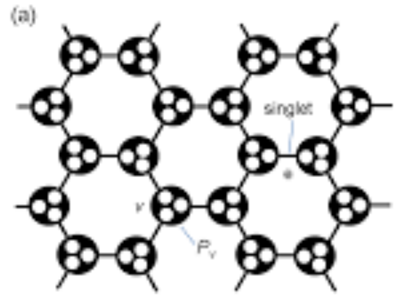

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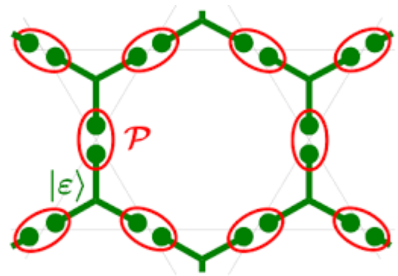

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Schuch et al, Phys. Rev. B 86, 115108 (2012)

The spectral gap problem is officially hard, even for PEPS

Theorem 1 (Cubitt et al 2015, Bausch et al 2020): The existence of spectral gap is an UNDECIDABLE problem, even for 1D systems.

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Theorem 2 (Scarpa et al 2020): The existence of spectral gap is an UNDECIDABLE problem, even for parent Hamiltonians of PEPS.

Indeed many properties are UNDECIDABLE in a PEPS. Example: existence of on-site symmetries.

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THESE ARE WORST CASE STATEMENTS

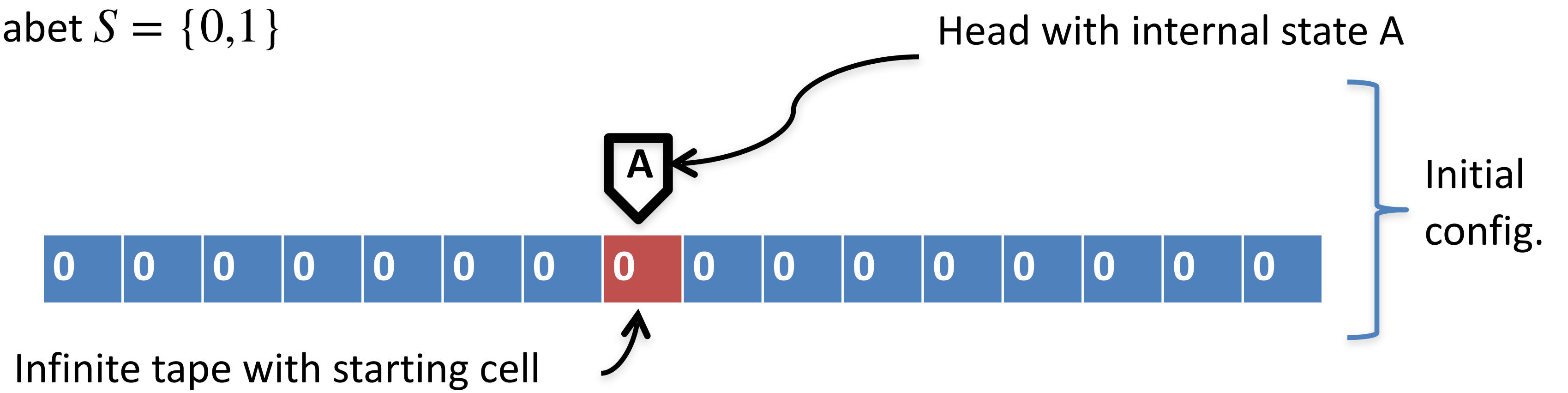
Let’s get a bit more formal.

Undecidability. Turing Machines

Turing Machines

Finite number of internal states $Q = \{A, B, C, \ldots\} \cup \{H = \text{ halting state}\}$

Finite alphabet $S = \{0,1\}$



Instructions $\delta : Q \times S \to Q \times S \times \{L, R\}$

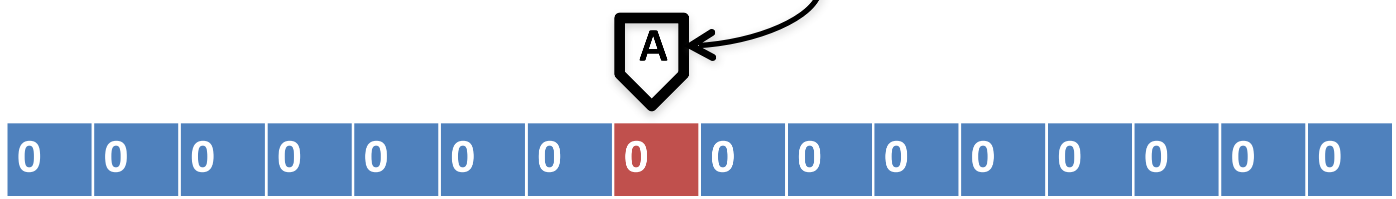
Turing Machines ⇔ Natural numbers

Turing Machines

Finite number of internal states $Q = \{A, B, C, \ldots\} \cup \{H = \text{ halting state}\}$

Finite alphabet $S = \{0,1\}$

Head with internal state A



Initial config.

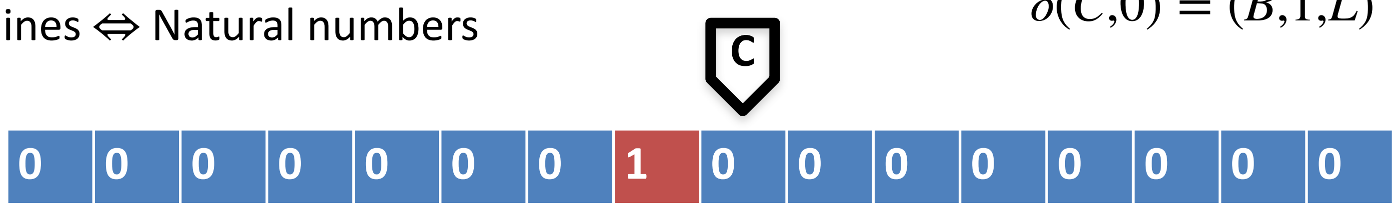
Infinite tape with starting cell

Instructions $\delta : Q \times S \to Q \times S \times \{L, R\}$

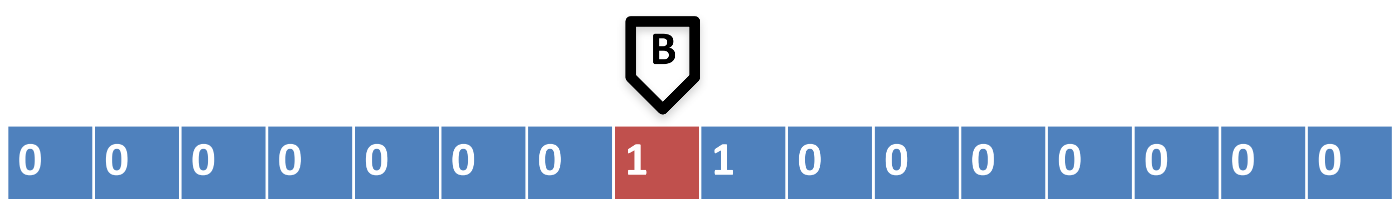
E.g. $\delta(A,0) = (C,1,R)$

$\delta(C,0) = (B,1,L)$

Turing Machines ⇔ Natural numbers



Time =1

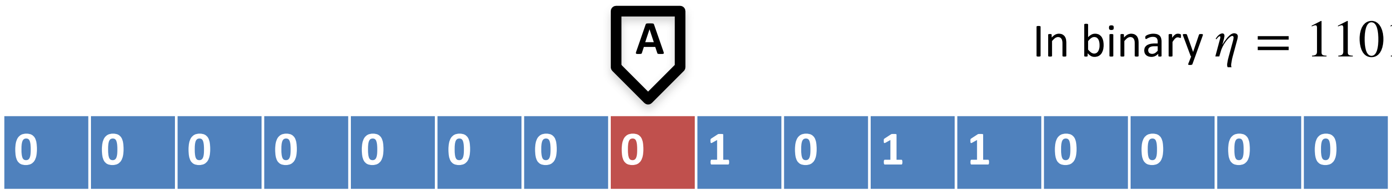


Time =2

The halting problem of a TM

A TM **halts on input** η if it eventually enters the halting state

In binary $\eta = 1101$



We say simply that a TM **halts** if it halts on input 0.

Halting problem: Given a TM, does it halt?

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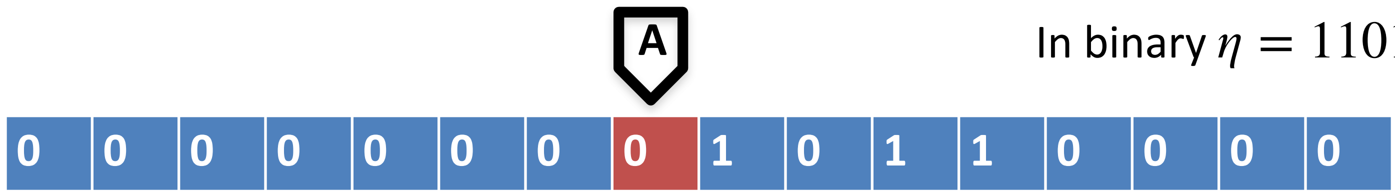

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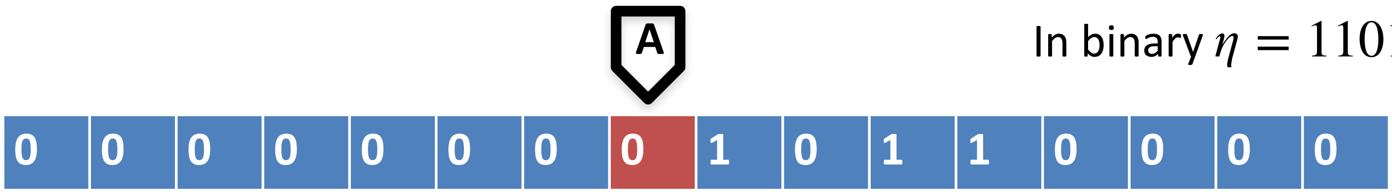

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Theorem (1936, Turing): There exists a TM M, called *universal* (UTM), so that it halts on input η iff the TM=η halts on input 0.

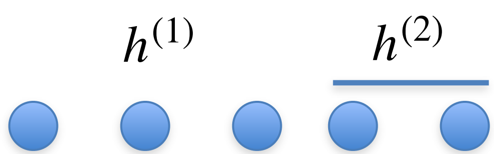
Corollary: There is no algorithm that on input a natural number η, decides whether the UTM halts or not on input η.

Undecidability of the spectral gap in 1D

Given a natural number η, $|\eta|$ denotes the number of digits in the binary expansion $\eta = \eta_1\eta_2\ldots\eta_{|\eta|}$

$\phi(\eta) := 0.\eta_1 1\eta_2 1\ldots 1\eta_{|\eta|} \in [0,1]$

Theorem:



$$\begin{cases} h^{(1)}(\eta) = A_1 + \beta\,(2^{-2|\eta|}A_2 + A_3) \\ h^{(2)}(\eta) = B_1 + \beta\,(2^{-2|\eta|}B_2 + e^{i\pi\phi(\eta)}B_3 + e^{i\pi 2^{-2|\eta|}}B_4 + B_5 + h.c.) \end{cases}$$

Only dependency on η

$$H(\eta) = \sum_{i=1}^{N-1} h^{(2)}_{i,i+1}(\eta) + \sum_{i=1}^{N} h^{(1)}_i(\eta)$$

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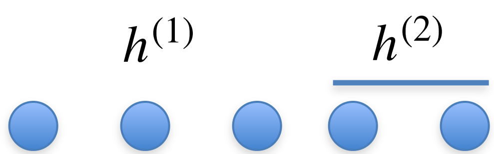
Theorem:

We construct explicit matrices A_i, B_j so that **for all** rational $0 < \beta < 1$ the interactions



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Only dependency on η

have norm ≤ 1 and the Hamiltonian

$$H(\eta) = \sum_{i=1}^{N-1} h^{(2)}_{i,i+1}(\eta) + \sum_{i=1}^{N} h^{(1)}_i(\eta)$$

1.- Has gap ≥ 1 and unique product state for all N, if the **UTM does not halt on input** η

2. Has spectrum = $\mathbb{R}$ when $N \to \infty$ if the **UTM halts on input** η

Undecidability of the spectral gap in 1D

Given a natural number η, $|\eta|$ denotes the number of digits in the binary expansion $\eta = \eta_1\eta_2\ldots\eta_{|\eta|}$

$\phi(\eta) := 0.\eta_1 1 \eta_2 1 \ldots 1 \eta_{|\eta|} \in [0,1]$

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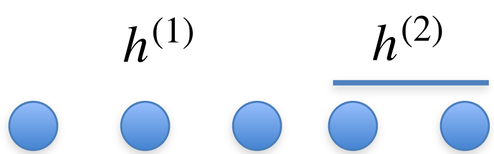

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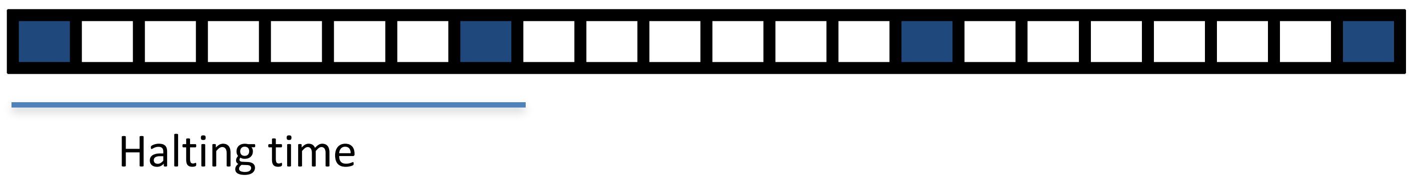
Key ingredient: **very long** periods (or a very exotic spin chain)

For each Turing machine, there exists a nearest neighbor interaction in 1D whose ground state has the following periodic structure:



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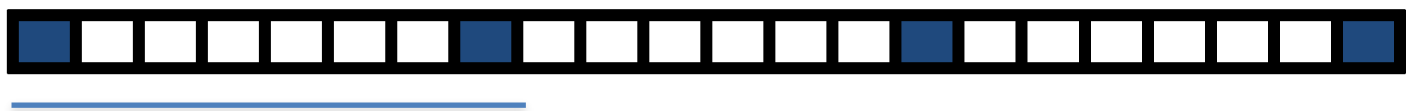

Halting time

↑

Grows faster than any computable function, e.g.

$$2^{2^{2^n}}$$

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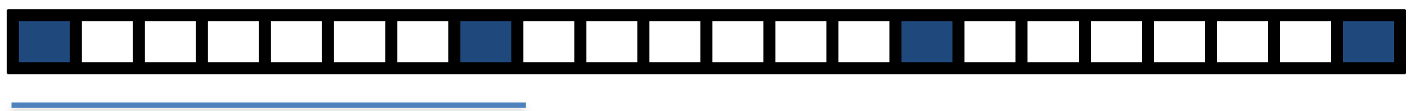

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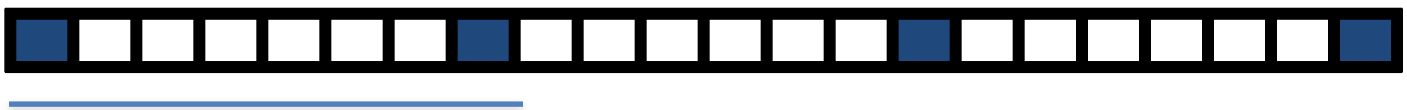

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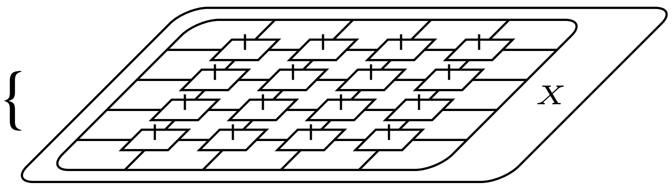
Innocent looking, but …

Halting time > 10^{35000}

Undecidability of the spectral gap in PEPS

Associated to any PEPS there is a **parent** Hamiltonian

$G_R = \{$  $: X\}$

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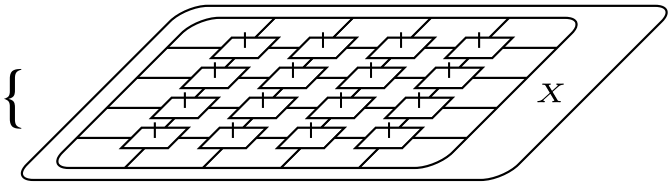
$G_R = \{$ 

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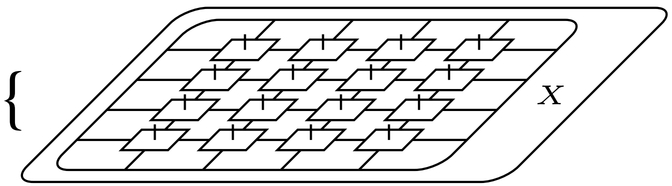
$G_R = \{$ 

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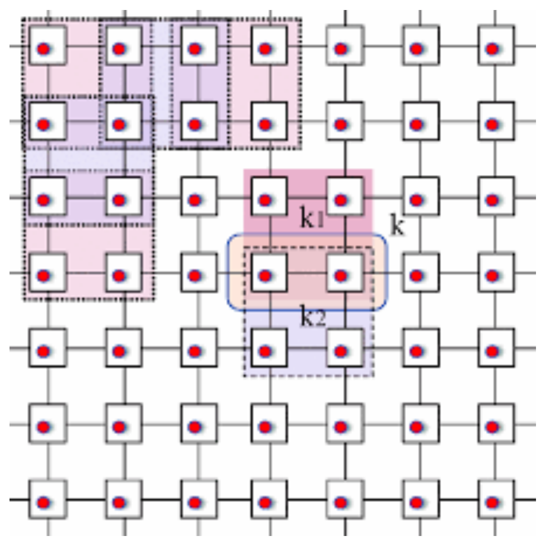
$$h_R = P_{G_R}^{\perp}$$

$$H = \sum_R h_R$$



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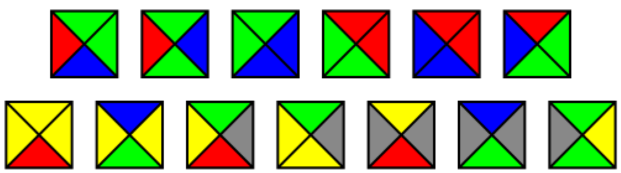

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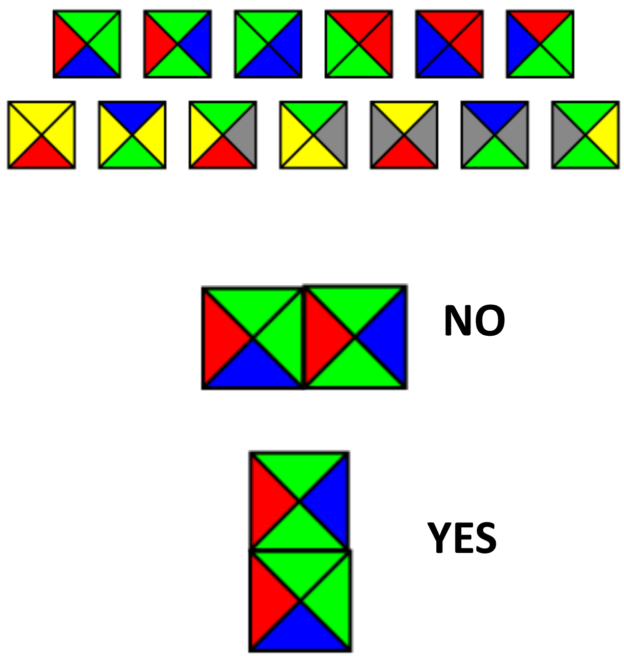

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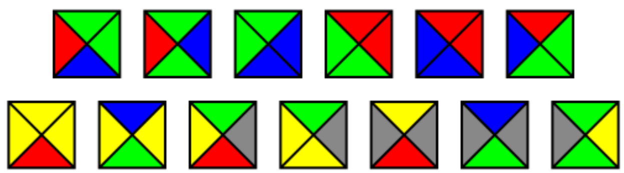
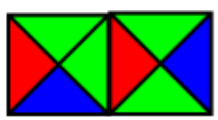
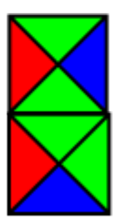


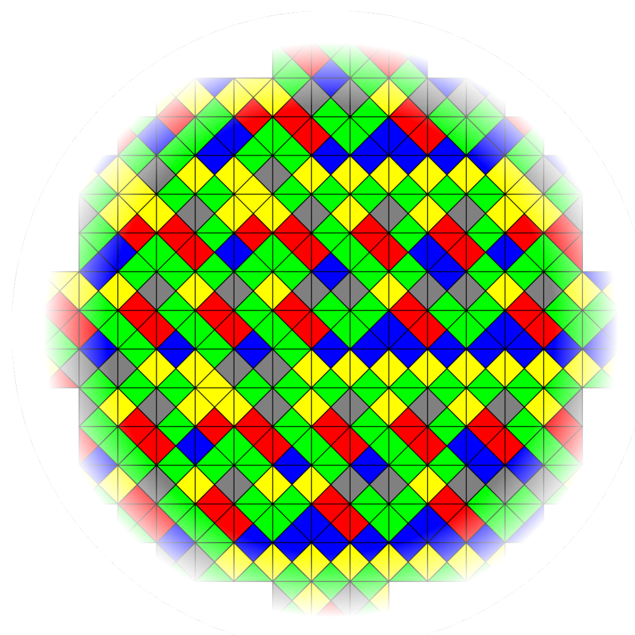
Can one tile the plane? It is undecidable (Berger 66)



NO

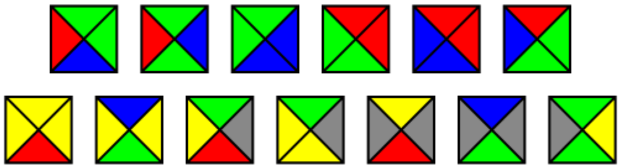


YES



Undecidability of the spectral gap in PEPS

Bond dimension = number of colors



Tiles ⟷ Coefficients of the tensor = 1
(Rest = 0)

Undecidability of the spectral gap in PEPS

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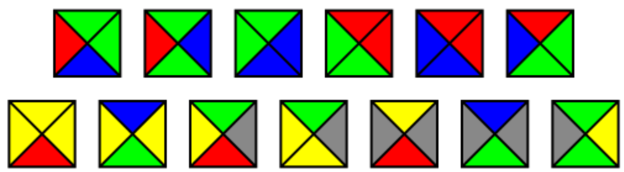

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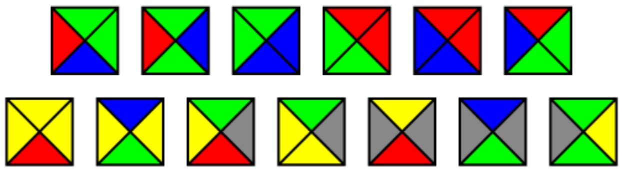

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Verstaete et al 2016, PRL 96, 220601

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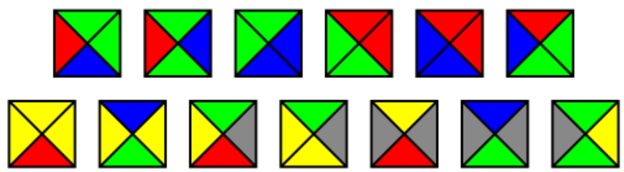

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Consider the tensor $A \oplus B \otimes C$

Gapped if and only if there is valid tiling

Appendix: Ingredient 3

RFP are in one to one correspondence with fusion categories

(J.I. Cirac, DPG, N. Schuch, F. Verstraete. Annals of Physics 2017)

Renormalization Fixed Points (RFP)

PEPS fulfill a very strong area law: isometric bulk-boundary mapping (Poilblanc et al 2013). Bulk properties can be understood at the boundary.

More next week !!!

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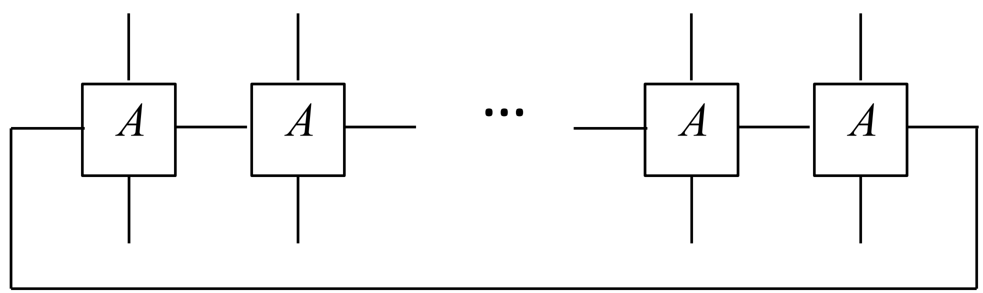
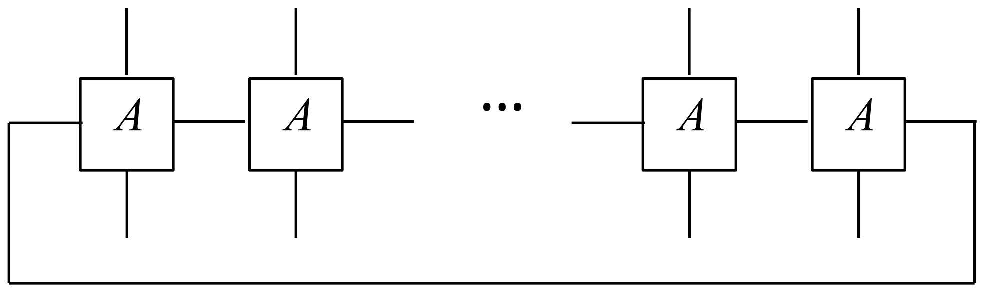
If a PEPS is a RFP, its boundary state is an exact Matrix Product Density Operator (MPDO), which in turn will be a RFP.



Which MPDO are RFP?

RFP = no correlation lengths in the system

RFP MPDOs



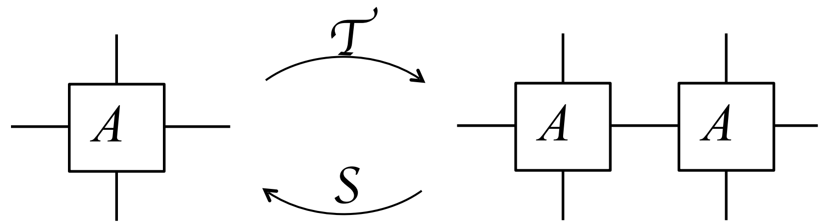
RFP = no correlation lengths in the system

Type 1: Correlation functions are independent of the distance between the observables.

Type 2: Saturation of the area law for the mutual information.

RFP MPDOs

"Theorem": A MPDO is a RFP if and only if there exist two quantum channels T and S so that



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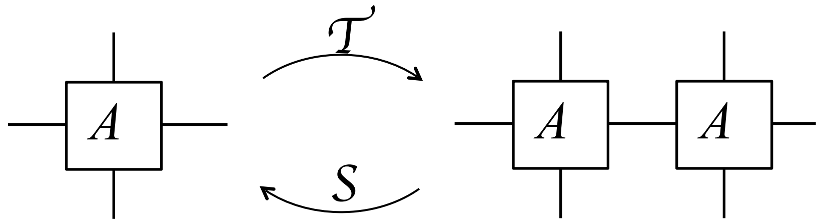
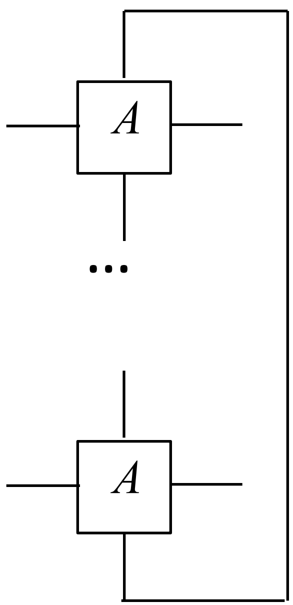


$$O_A^N =$$



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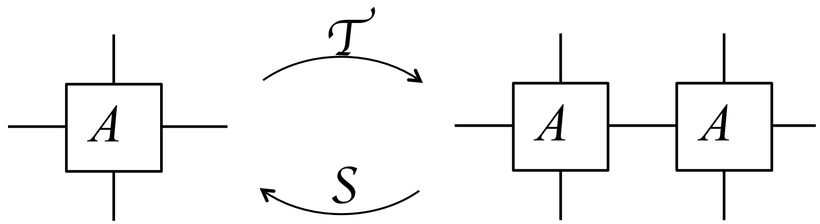

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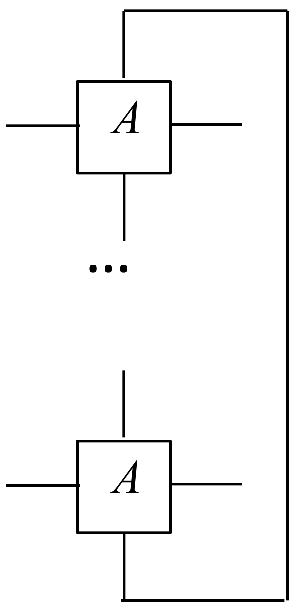

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Verstraete's group

One obtains a fusion category. The RFP MPDO is exactly the boundary theory associated to the corresponding string-net model.

More next week!