Benasque, February 16, 2021

Stability of the bulk gap
How to prove it when gapless edge modes cannot be avoided

Bruno Nachtergaele (UC Davis)

joint work with

Robert Sims (U Arizona) and Amanda Young (MCQST-TUM)

Outline

- Gapped ground state phases and the bulk gap
- Infinite lattice systems
- The Bravyi-Hastings-Michalakis (BHM) strategy in infinite volume
- Assumptions and stability result for the bulk gap
- Concluding remarks

Joint work with Bob Sims and Amanda Young:

1. Quasi-locality bounds for quantum lattice systems. I. Lieb-Robinson bounds, quasi-local maps, and spectral flow automorphisms, arXiv:1810.02428, J. Math. Phys **60**, 061101 (2019).
2. — II . Perturbations of frustration-free spin models with gapped ground states, arXiv:2010.15337.
3. —, Stability of the bulk gap for frustration-free topologically ordered quantum lattice systems, arXiv:2102.07209 (since today!)
4. planned final installment: lattice fermion systems. Outline: Lieb-Robinson bounds, the spectral flow, and stability for lattice fermion systems, Contemporary Mathemathics, **717** 2018 (QMATH13 Atlanta).

Gapped ground state phases

The term phase here refers to an open region in a space of Hamiltonians in which there are no phase transitions and where the ground states have qualitatively similar properties.

More precisely: two Hamiltonians, $H(0)$ and $H(1)$ are in the same phase, if there exists a differentiable interpolation $[0,1] \ni s \mapsto H(s)$ such that $H(s)$ has a gap $\geq \gamma > 0$, for all $s \in [0,1]$, and uniformly in the system size. (Chen-Gu-Wen 2011).

Possible closing of the gap makes sense only for infinite systems, while the Hamiltonians are, a priori, well-defined only for finite systems.

For example, consider system on $\mathbb{Z}^\nu$ that have a well-defined thermodynamic limit and with Hamiltonians H_Λ, for finite $\Lambda \subset \mathbb{Z}^\nu$ of the form

$$H_\Lambda(s) = \sum_{X \subset \Lambda} \Phi(X, s), \quad \text{for example} \quad \Phi(X, s) = (1-s)\Phi_0(X) + s\Phi_1(X).$$

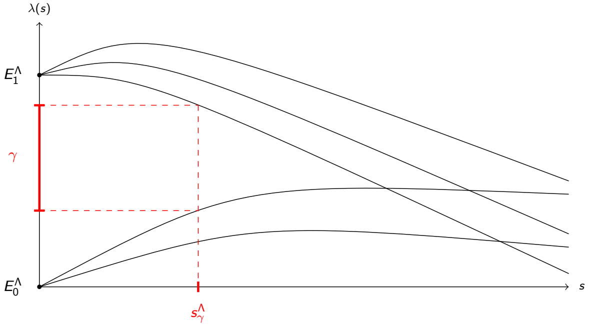
and require that the interaction Φ are sufficiently short-range.

For symmetry protected (topological) (SPT) phases one requires that $H(s)$ has a given symmetry for all s (Pollmann et al, 2012).

Stability of the spectral gap for finite systems

For a model with a gap above the ground state to represent a gapped phase, the gap should be stable under a broad class of perturbations.

$$H_\Lambda(s) = H_\Lambda(0) + sV_\Lambda$$



The spectral gap of $H_\Lambda(s)$ above the 'ground state' is at least γ for all $0 \leq s \leq s_\gamma^\Lambda$.

Stability means that there is a Λ-independent lower bound for s_γ^Λ.

Stability of the spectral gap for infinite systems

We start from the observation that the Heisenberg dynamics of observables in the bulk is not sensitive to boundary effects:

$$\frac{d}{dt}A(t) = i\lim_{\Lambda\to\Gamma}[H_\Lambda(s), A(t)] = i\delta_s(A(t)), A(0) \text{ any local observable,}$$

with Γ the infinite lattice, and $\delta_s(A)$ is the commutator

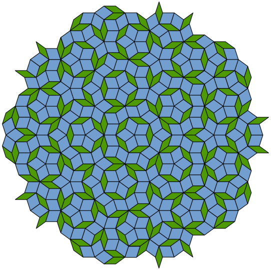
$$\delta_s(A) = [H(s), A] = [\sum_{x\in\Gamma} h_x + s\sum_{X\subset\Gamma}\Phi(X), A].$$

We assume that Γ is finite-dimensional in the sense that there is $\nu > 0$ for which

$$\#\text{ sites in any ball of radius } n \ \leq cn^\nu,\ \text{ for all } n \geq 1.$$

We think of Γ as boundary-less, i.e. describing the bulk.

For finite $\Lambda \subset \Gamma$, the gap of $H_\Lambda(s)$ may vanish with increasing Λ, due to edge states.



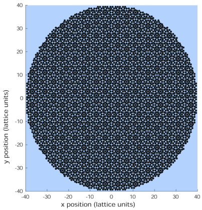


FIG. 2. Round section of the quasilattice, of radius 40 lattice units.

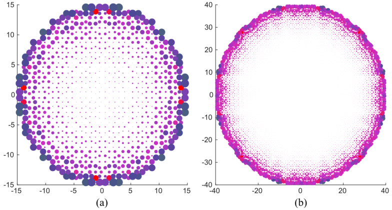


FIG. 7. (a) The radius is 15, with eigenvalue 0.018. (b) The radius is 40 with eigenvalue 0.007.

Figure: Penrose tiling. Ammann-Beenker tiling. Edges state or not? (T. Loring, J. Math. Phys. **60**, 081903 (2019))

Infinite-system Hilbert space

Assume the ground state is unique for the infinite system: there is a unique thermodynamic limit of any sequence of ground states, for any choice of boundary conditions.

$$\langle A\rangle_{\Lambda_n} \to \langle A\rangle_{\Gamma}, \quad A \text{ local}.$$

Fact (GNS representation): there is a Hilbert space $\mathcal{H}$ in which the ground state of the infinite system is given by a vector, say, $|1\rangle$.

General vectors in $\mathcal{H}$ are of the form $|A\rangle$ (or limits thereof).

The (quasi-local) observables $A, B, \ldots$ act as follows:

$$\tilde{A}|1\rangle = |A\rangle, \text{ and } \tilde{B}|A\rangle = |BA\rangle.$$

Here, the $\tilde{A}, \tilde{B}$ denote a representation of the observables. This representation is an isomorphism.

The inner product is

$$\langle B|A\rangle = \langle 1|B^* A|1\rangle = \langle B^* A\rangle_{\Gamma}.$$

Infinite-system Hamiltonian and bulk spectrum

Our main object of interest is the Hamiltonian on $\mathcal{H}$. One gets it from the Heisenberg dynamics:

$$\tilde{A}(t) = e^{itH}\tilde{A}e^{-itH}, \text{ and setting } H|1\rangle = 0$$

implies the definition

$$H|A\rangle = HA|1\rangle = [H, A]|1\rangle = i\,\frac{d}{dt}\bigg|_{t=0} e^{itH}\tilde{A}e^{-itH}|1\rangle = i\,\frac{d}{dt}\bigg|_{t=0}\tilde{A}(t)|1\rangle.$$

H is an unbounded self-adjoint operator. Its spectrum are the energies of excitations above the ground state in the bulk.

Perturbations in finite regions

$$H(\Lambda, s) = H + s\tilde{V}_\Lambda, \quad V_\Lambda = \sum_{X\subset\Lambda}\Phi(X).$$

We want to estimate the gap of $H(\Lambda, s)$ uniformly in the perturbation region Λ.

Relative form bounded perturbations

Theorem (persistence of gaps)

Let H a densely defined self-adjoint operator on $\mathcal{H}$ suppose V is a self-adjoint operator with $\mathrm{dom}(H) \subset \mathrm{dom}(V)$. Suppose there exist a constant $\beta \in [0,1)$, such that

$$|\langle \psi, V\psi \rangle| \leq \beta \langle \psi, H\psi \rangle, \text{ for all } \psi \in \mathrm{dom}(H).$$

Then, if $a < b \in \mathbb{R}$, such that $(a,b) \cap \mathrm{spec}(H) = \emptyset$, then

$$((1+s\beta)a, (1-s\beta)b) \cap \mathrm{spec}(H+sV) = \emptyset, \text{ for all } |s| \leq 1.$$

Bravyi-Hastings-Michalakis strategy: combine with quasi-adiabatic unitary transformation (Bravyi-Hastings-Michalakis 2010-11, Michalakis-Zwolak 2013)

$$U_\Lambda^*(s)(H + s\tilde{V}_\Lambda)U_\Lambda(s) = H + s\tilde{W}_\Lambda(s) + E(\Lambda, s)\mathbb{1},$$

with $E(\Lambda, s)$ the ground state energy of $H(\Lambda, s) = H + s\tilde{V}_\Lambda$, and

$$\tilde{W}_\Lambda(s) = \sum_{x \in \Lambda, n \geq 1} \tilde{\Phi}(x, n, s),$$

where $\tilde{\Phi}(x,n,s)$ satisfies the condition $\tilde{\Phi}(x,n,s)P_{b_x(n)} = 0$, and $\|\Phi(x,n,s)\| \leq Ce^{-a' n^{\theta'}}$, for $|s| \leq s_\gamma^\Lambda$.

Assumptions

Assumptions on the unperturbed model:

- finite-range and frustration-free: $H = \sum_x h_x$.
- bulk gap: $(0, \gamma_0) \cap \operatorname{spec} H = \emptyset$.
- finite-volume gaps do not vanish too fast: for $H_{\Lambda_n} := \sum_{x, \operatorname{supp} h_x \subset \Lambda_n} h_x$, for suitable Λ_n with $\operatorname{diam} \Lambda_n \sim n$,

$$\operatorname{gap}(H_{\Lambda_n}) \geq \frac{\gamma_1}{n^\alpha}, \alpha \geq 0.$$

- LTQO: next slide

Assumption on the perturbation:

Assume $\Phi(x) \neq 0$ only for $X = b_x(n)$, a ball of radius n centered at $x \in \Gamma$ and that

- the perturbation has stretched exponential decay or faster:

$$\|\Phi(x, n)\| \leq \|\Phi\| e^{-an^\theta}, \quad \theta \in (0, 1], a > 0.$$

Local Topological Quantum Order (LTQO)

Define P_Λ to be the projection onto the unperturbed ground states in finite or infinite volume Λ. $b_x(n)$ denotes a ball of radius n centered at $x \in \Gamma$.

Definition (Indistinguishability radius)

For any $\Omega : \mathbb{R} \to [0, \infty)$ non-increasing function with $\lim_{r\to\infty} \Omega(r) = 0$, the indistinguishability radius of H_Λ at $x \in \Lambda$ is the largest integer $r_x^\Omega(\Lambda) \leq \mathrm{diam}(\Lambda)$ such that for all integers $0 \leq k \leq n \leq r_x^\Omega(\Lambda)$ and all observables $A \in \mathcal{A}_{b_x^\Lambda(k)}$,

$$\|P_{b_x^\Lambda(n)} A P_{b_x^\Lambda(n)} - \langle A \rangle_\Lambda P_{b_x^\Lambda(n)}\| \leq |b_x^\Lambda(k)| \|A\| \Omega(n-k).$$

Here,

$$\langle A \rangle_\Lambda = \frac{\mathrm{Tr} P_\Lambda A}{\mathrm{Tr} P_\Lambda}.$$

This expresses that an observable in $b_x^\Lambda(k)$ cannot distinguish the ground states in $b_x^\Lambda(n)$, if $n-k$ is sufficiently large. As a consequence, the possibly many different ground states will not split under the influence of the perturbation A.

For stability of the bulk gap we require $r_x^\Omega(\Gamma) = \infty$ for a fucntion Ω that decays at least as a power law with sufficiently large exponent. This by itself implies that the ground states have a unique thermodynamic limit.

Proof of relatively boundedness and, hence, gap stability

Theorem (Michalakis-Zwolak 2013, N-Sims-Young 2021)

For all $\psi \in \mathrm{dom}\, H(\Lambda, s)$, $\left|\langle\psi|\tilde{W}_\Lambda(s)\,\psi\rangle\right| \;\le\; \beta\langle\psi|H\,\psi\rangle$, *with*

$$\beta = C\|\Phi\|\gamma_1^{-1}\sum_{n\ge 1} n^{\nu+\alpha}\sqrt{n\Omega(n)}.$$

Theorem (N-SIms-Young 2021)

Under the assumptions above, if a frustration-free model H *on* Γ *has bulk gap* $\gamma_0 > 0$, *we have*

$$\mathrm{gap}(H + s\sum_{X\subset\Gamma}\Phi(X)) \ge (1-|s|\beta)\gamma_0.$$

Concluding Remarks

- infinite system setting simplifies defining and studying the bulk gap
- infinite system setting also useful to study boundary system
- the frustration-free condition should not be needed