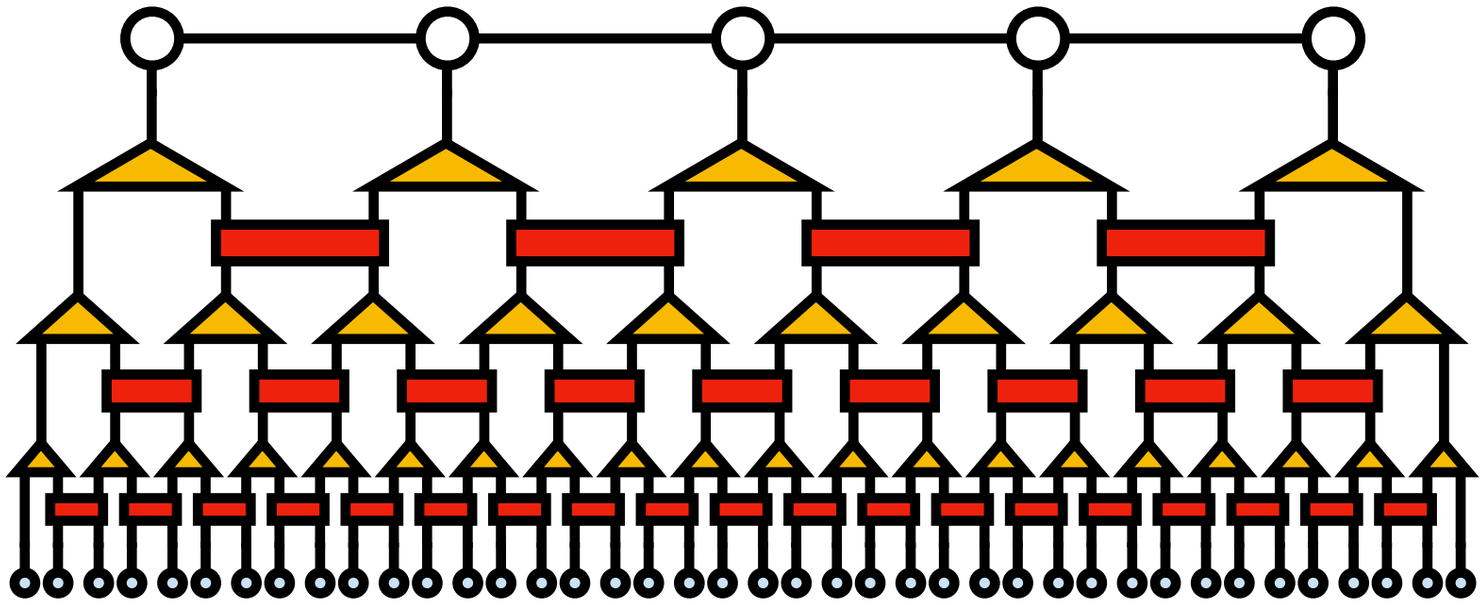
Tensor Networks for Machine Learning: Architectures, Algorithms, and Applications



E.M. Stoudenmire

Feb 2021 - Benasque SCS

FLATIRON INSTITUTE
SIMONS FOUNDATION

Talk Advertisement:

Prof. Jens Eisert will speak about tensor networks in machine learning today at 9pm CET (3pm EST)

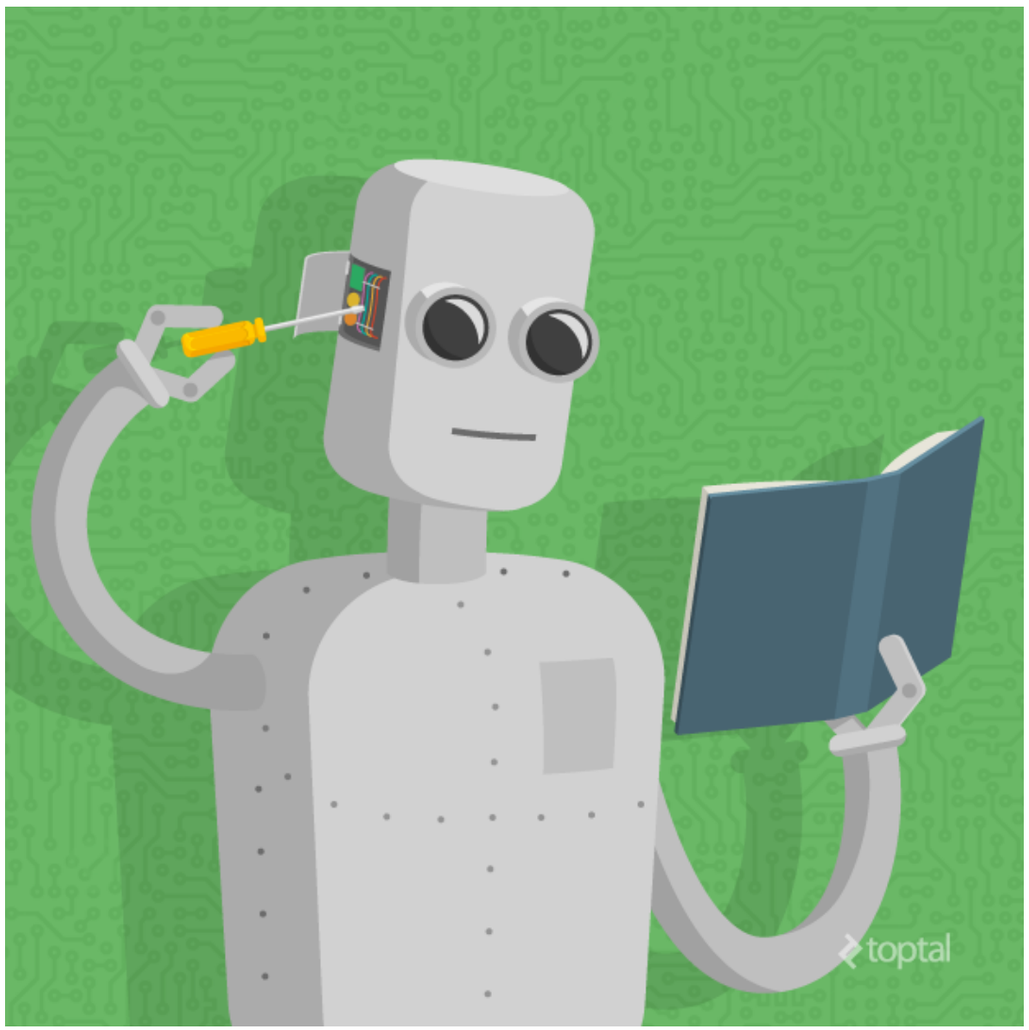
"Tensor networks as a data structure in probabilistic modelling and for learning dynamical laws from data"

Zoom Meeting ID: ***923 9226 6655****, Password:* ***919332***

Screen shot this if interested

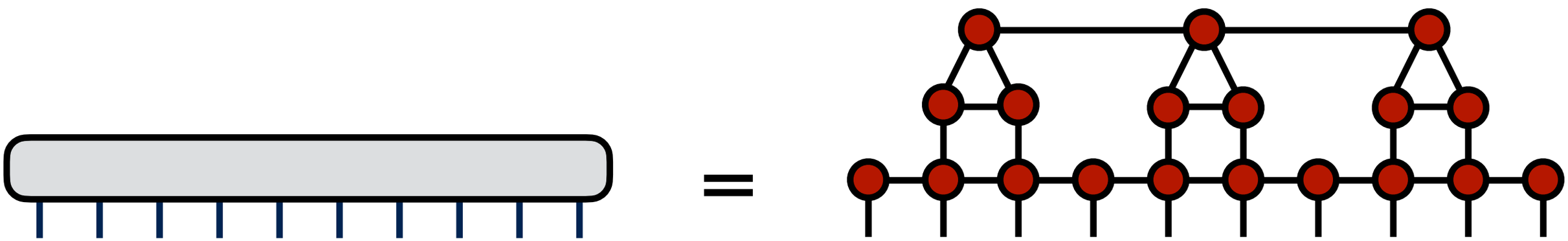
Last Time...

Introduced machine learning: system which given more data performs increasingly well at a task



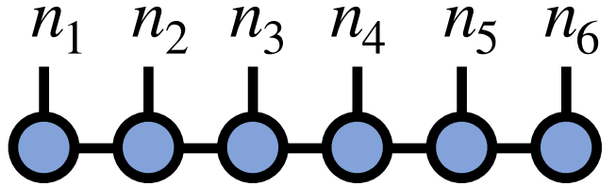
Last Time...

Introduced tensor networks: factorization of N-index tensor into contraction of many small tensors



Last Time...

Introduced tensor network machine learning models with weights as tensor network (entering linearly)

$$f(x_1, x_2, \ldots, x_6) = \sum_{\mathbf{n}}$$ 

$$x_1^{n_1} x_2^{n_2} \cdots x_6^{n_6}$$

Last Time...

Discussed applications to both real-world and physics-specific machine learning tasks

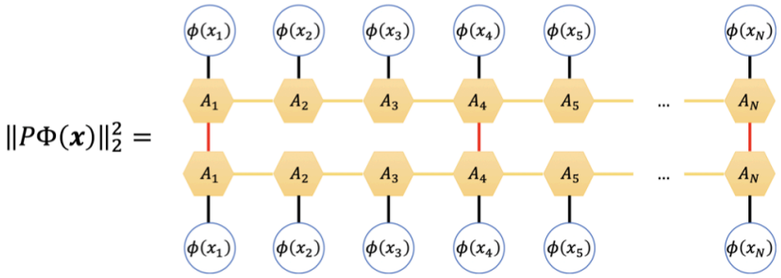


Table 3: Mean AUROC scores (in %) and standard errors on ODDS datasets.

Dataset	OC-SVM	IF	GOAD	DAGMM	TNAD
Wine	**60.0**	46.0 ± 8.4	48.2 ± 24.7	51.7 ± 19.3	$\mathbf{97.3 \pm 4.5}$
Glass	**62.0**	57.2 ± 1.6	53.5 ± 13.6	52.5 ± 12.9	$\mathbf{81.8 \pm 7.3}$
Thyroid	98.8	$\mathbf{99.0 \pm 0.1}$	95.8 ± 1.3	88.8 ± 6.8	$\mathbf{99.0 \pm 0.1}$
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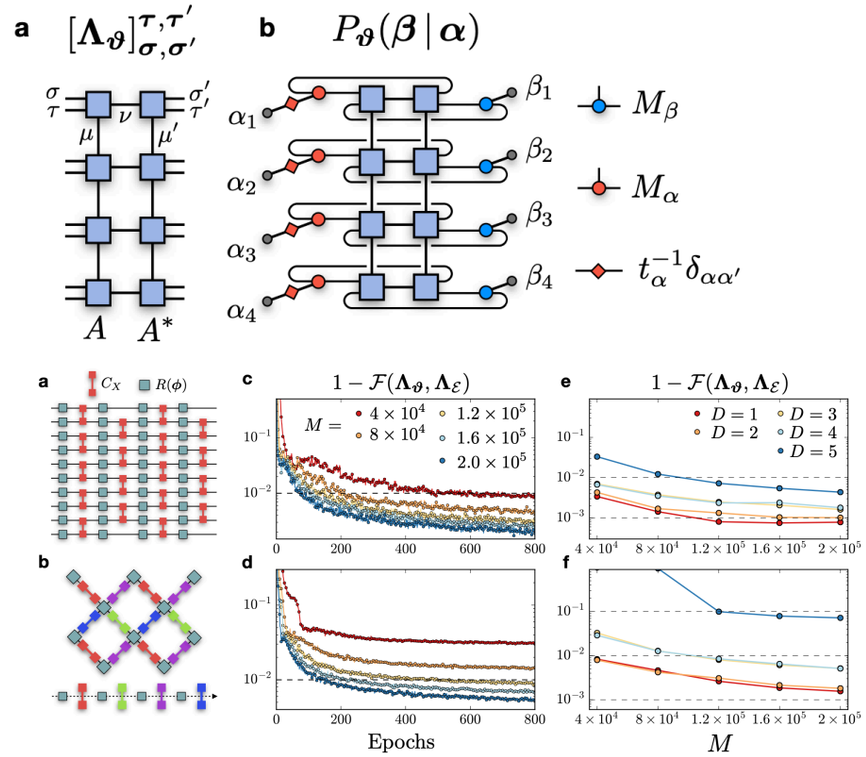
Forest	**97.7**	71.7 ± 2.6	64.6 ± 4.7	60.9 ± 8.9	$\mathbf{98.8 \pm 0.6}$

Anomaly Detection with Tensor Networks

arxiv:2006.02516



Quantum process tomography with ... tensor networks

arxiv:2006.02424

Outline of Today's Talk

Review of tensor network machine learning

Architectures beyond MPS

Direct learning algorithm for training MPS tensor network (for generative modeling)

Theoretical prediction of generalization performance

Typical Ingredients of Machine Learning

Dataset represented as vectors $\{\mathbf{x}_j\}$

Labels are numbers or vectors $\{\mathbf{y}_j\}$

Model function $f_W(\mathbf{x})$

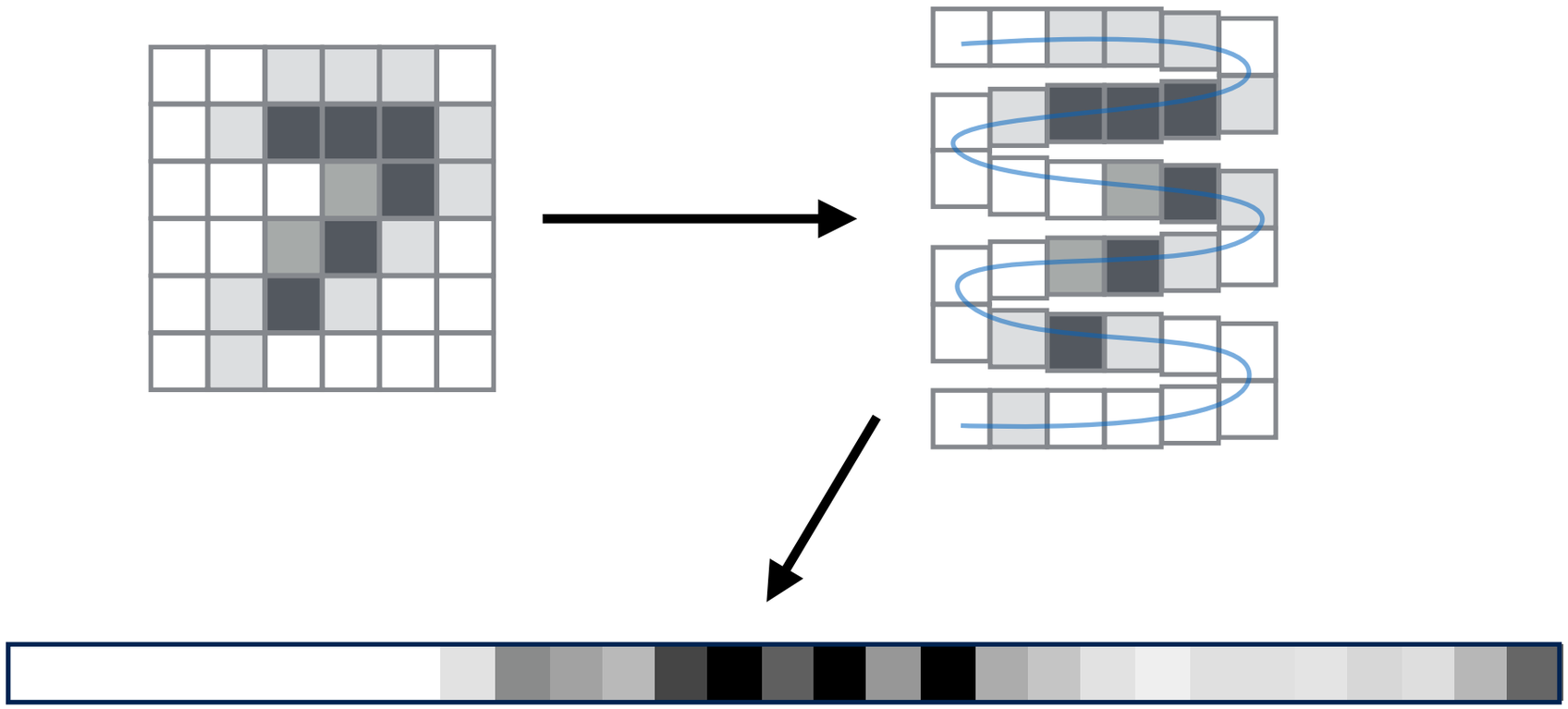
with adjustable weight parameters W

Choose weights to minimize cost function

$$C = \sum_j \Xi(\, f_W(\mathbf{x}_j),\, \mathbf{y}_j)$$

Typical Ingredients of Machine Learning

Dataset represented as vectors $\{\mathbf{x}_j\}$



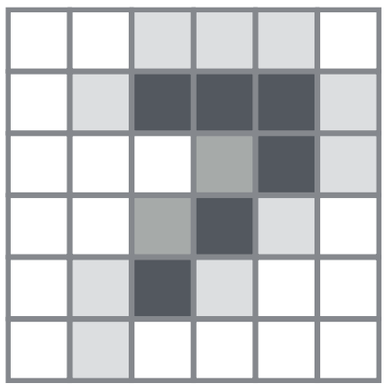
$\mathbf{x} =$ { 0.0, 0.0, 0.1, 0.1, 0.1, 0.0, 0.1, 0.7, 0.7, 0.7, 0.2, ...

Typical Ingredients of Machine Learning

Dataset represented as vectors $\{\mathbf{x}_j\}$

Labels are numbers (or vectors) $\{y_j\}$

Task: distinguish 7's from 8's



$\mathbf{x} =$ { 0.0, 0.0, 0.1, 0.1, 0.1, 0.0, 0.1, 0.7, 0.7, 0.7, 0.2, ...

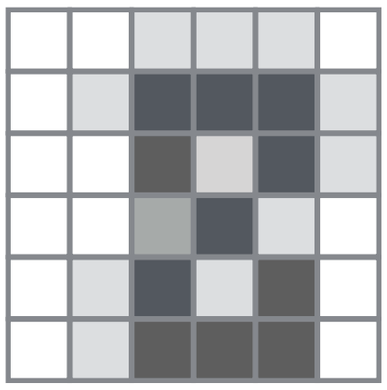
$y =$ +1

Typical Ingredients of Machine Learning

Dataset represented as vectors $\{\mathbf{x}_j\}$

Labels are numbers (or vectors) $\{y_j\}$

Task: distinguish 7's from 8's



$\mathbf{x} =$ { 0.0, 0.0, 0.1, 0.1, 0.1, 0.0, 0.1, 0.7, 0.7, 0.7, 0.2, ...

$y =$ -1

Typical Ingredients of Machine Learning

Model function $f_W(\mathbf{x})$

with adjustable weight parameters W

For example, *linear classifier* model

$$f_{\mathbf{W}}(\mathbf{x}) = \mathbf{W} \cdot \mathbf{x} + W_0$$

just a dot product of weights with input

Works suprisingly well!

Typical Ingredients of Machine Learning

Choose weights to minimize cost function,
for example *squared error* cost function:

$$C = \frac{1}{N}\sum_{j=1}^{N}\left(f_W(\mathbf{x}_j) - y_j\right)^2$$

$$= \frac{1}{N}\sum_{j=1}^{N}\left(\mathbf{W}\cdot\mathbf{x}_j - y_j\right)^2$$

$$y_j = \begin{cases} +1 & \mathbf{x}_j \in \text{7's} \\ -1 & \mathbf{x}_j \in \text{8's} \end{cases}$$

Typical Ingredients of Machine Learning

Can optimize weights using *gradient descent*

$$C = \frac{1}{N}\sum_{j=1}^{N}\left(\mathbf{W}\cdot\mathbf{x}_j - y_j\right)^2$$

$$-\frac{dC}{dW_n} \;=\; \frac{2}{N}\sum_{j=1}^{N}\left(y_j - \mathbf{W}\cdot\mathbf{x}_j\right)x_j^n \;=\; \Delta^n$$

$$W^n \to W^n + \alpha\Delta^n \qquad\qquad \alpha = \text{learning rate}$$

Project: program this in Julia or Python

Adding Features

What if a linear classifier is not enough?

$$f_{\mathbf{W}}(\mathbf{x}) = \mathbf{W} \cdot \mathbf{x} + W_0$$

$$= W_0 + W_1 x_1 + W_2 x_2 + W_3 x_3 + \ldots$$

Adding Features

What if a linear classifier is not enough?

$$f_{\mathbf{W}}(\mathbf{x}) = \mathbf{W} \cdot \mathbf{x} + W_0$$

$$= W_0 + W_1 x_1 + W_2 x_2 + W_3 x_3 + \ldots$$

Can make extended linear classifier as follows:

$$f_{\mathbf{W},\mathbf{V}}(\mathbf{x}) = W_0 + W_1 x_1 + W_2 x_2 + W_3 x_3 + \ldots$$

$$+ V_{12} x_1 x_2 + V_{13} x_1 x_3 + V_{23} x_2 x_3 + \ldots$$

Think of $\mathbf{W} \oplus \mathbf{V}$ as weight vector of this 'linear' model

Manifestly non-linear function of $\mathbf{x}$

Adding Features

Why not keep going, adding third-order $x_1 x_2 x_3$ and fourth-order $x_3 x_5 x_7 x_9$ terms etc. ?

Arrive at model with *weight tensor*

$$f(x_1, x_2, \ldots, x_N) = \sum_{\mathbf{n}} W_{n_1 n_2 \cdots n_N} \; x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}$$

Non-linear function of $\mathbf{x}$

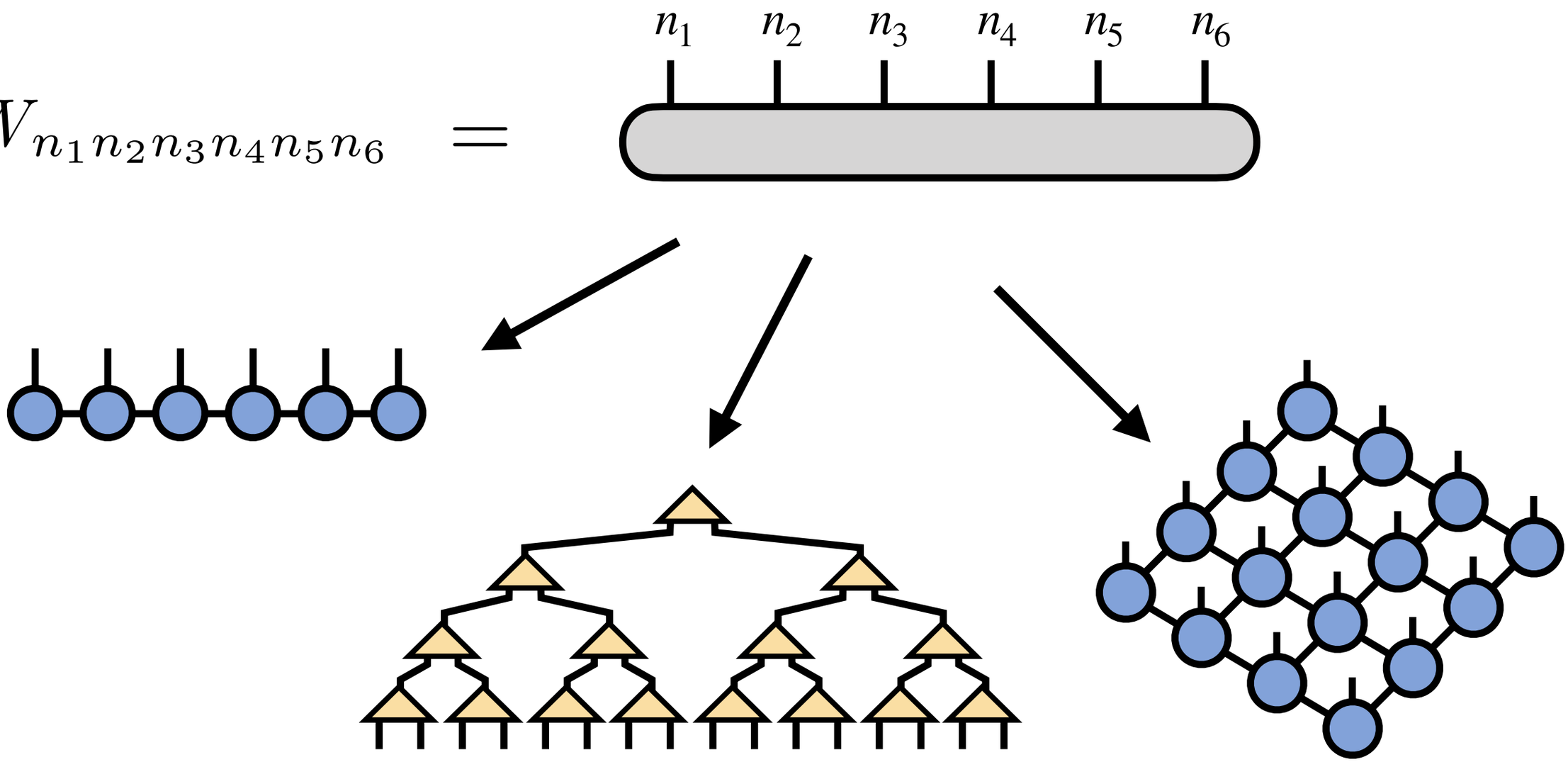
Weights enter *linearly* into model

Novikov, Trofimov, Oseledets, arxiv:1605.03795

Stoudenmire, Schwab, arxiv:1605.05775

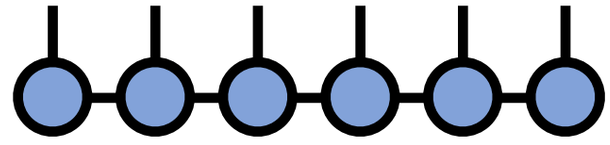
By factorizing exponentially big weight tensor

$$W_{n_1 n_2 n_3 n_4 n_5 n_6} =$$



Into a tensor network, model becomes efficiently trainable & we will see other advantages

Architectures Beyond Matrix Product States

$$W_{n_1 n_2 n_3 n_4 n_5 n_6} =$$ 

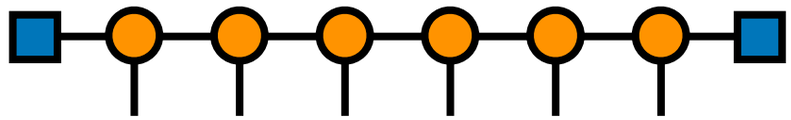
Matrix product state (MPS)
weights already a powerful representation

- **expressive**, especially for one-dimensional correlations
- multiple optimization **algorithms** & strategies
- best understood network **theoretically**

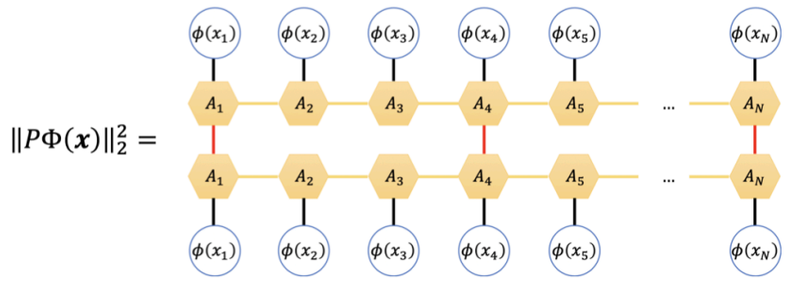
But worthwhile to explore other tensor networks –
let's briefly see why...

Since 2016, tensor network machine learning now successfully "ported" to other tensor net architectures

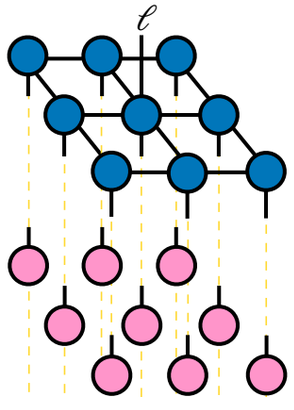
Infinite MPS



Locally purified states

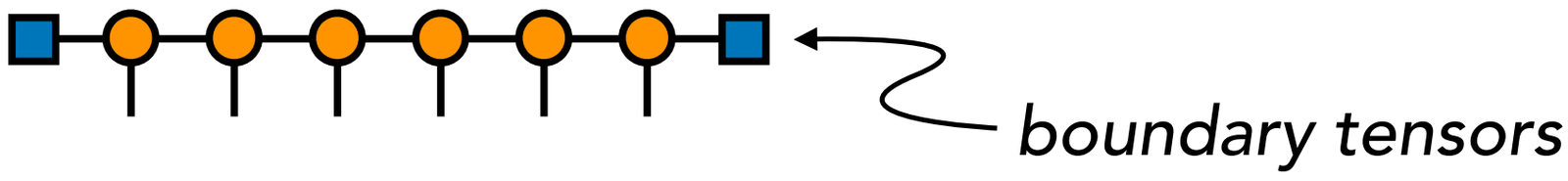


PEPS



Infinite MPS

Miller, Rabusseau, Terilla, "Tensor Networks for Probabilistic Sequence Modeling", arxiv:2003.01039



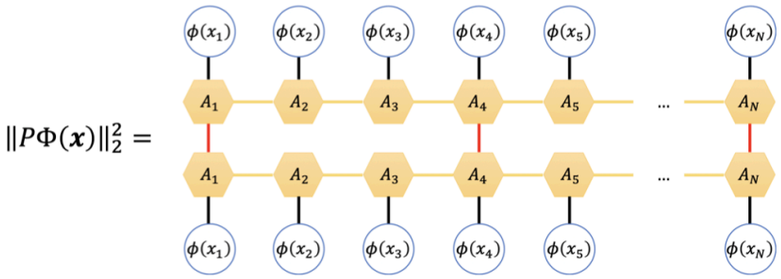
- used to generate model langauges with various grammars
- very few parameters and parallel optimization
- superior results to LSTM in many cases, equal in most others
- can *generalize* from training on shorter sequences to correct results on longer sequences (so really learning the grammar)

Miller, Rabusseau, Terilla, arxiv:2003.01039

Locally purified states

Anomaly Detection with Tensor Networks

arxiv:2006.02516



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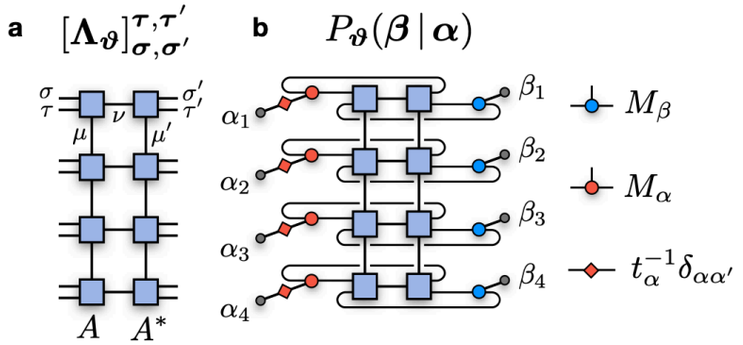
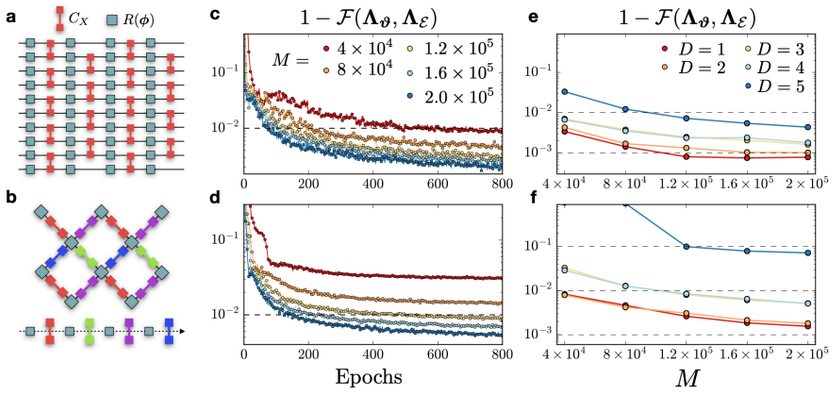
Novel anomaly detection framework

Results better than neural networks for tabular data

Quantum process tomography with ... tensor networks

arxiv:2006.02424

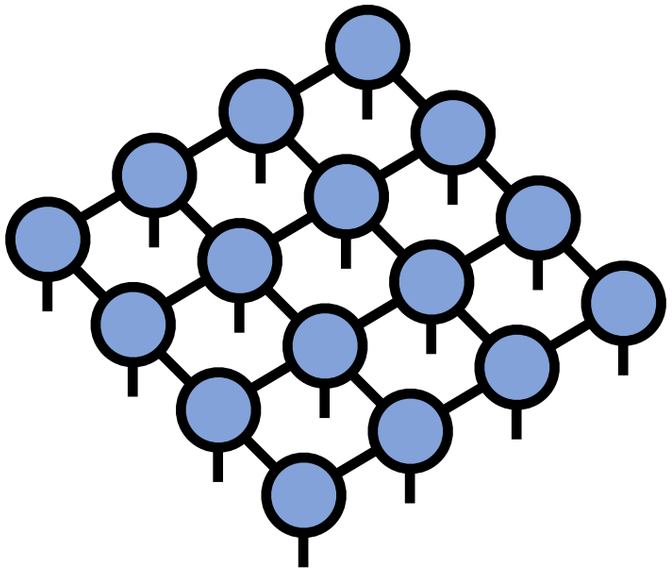




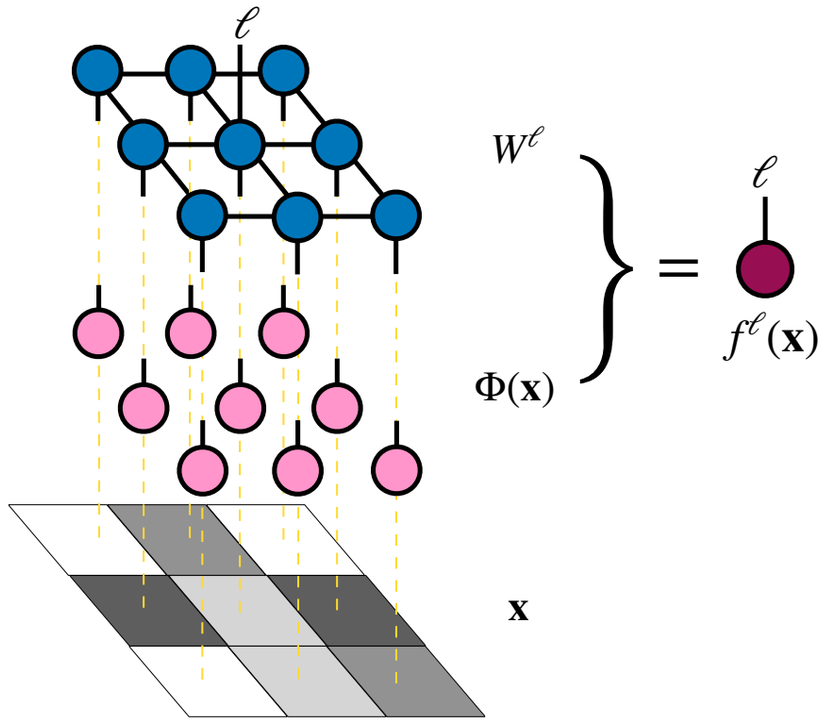
Scalable learning of noisy quantum "channels" or processes from experiments

PEPS (Projected Entangled Pair States)

Cheng, Wang, Zhang, "Supervised Learning with PEPS" arxiv:2009.09932

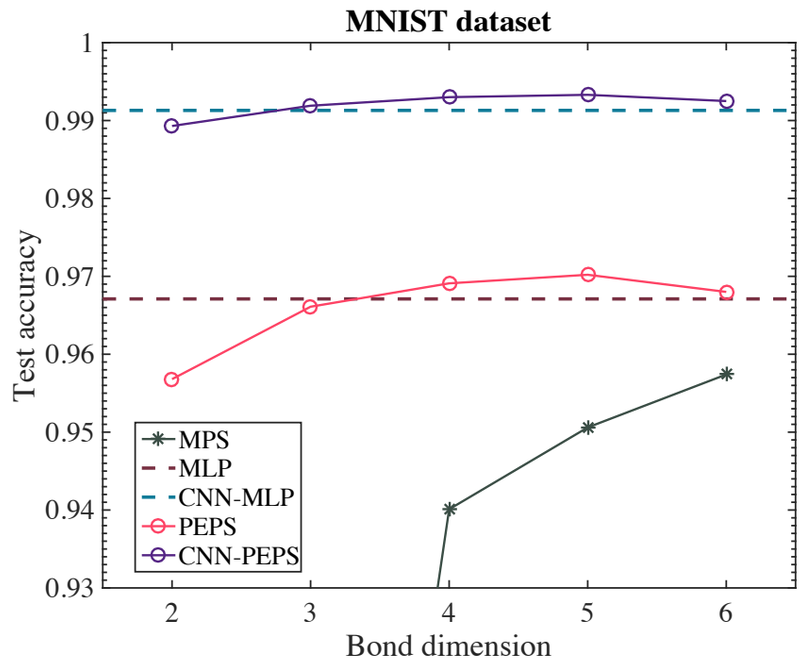


PEPS = 2D analogue of MPS



Framework for learning

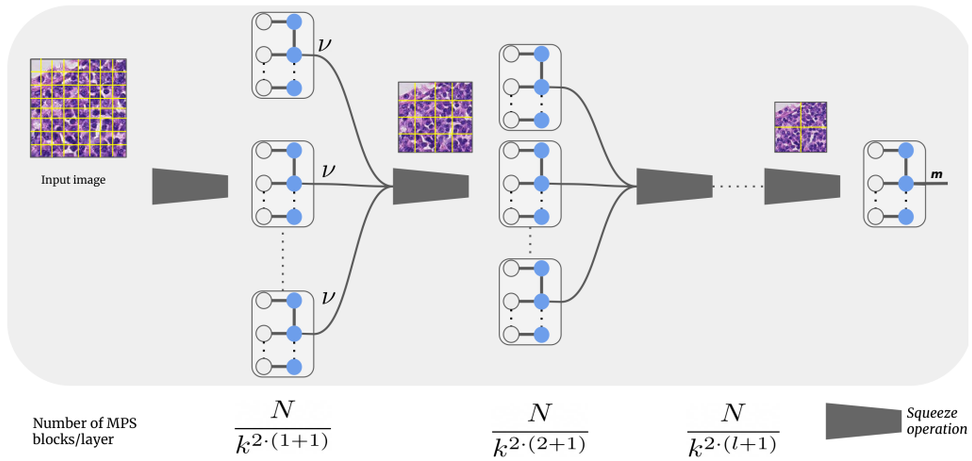
- comparable results to NN with fewer parameters
- starting to see overfitting at larger D?
- convolutional layers help (91% test correct on FashionMNIST using CNN+PEPS!)
- cost to train is high



Architectures & Applications

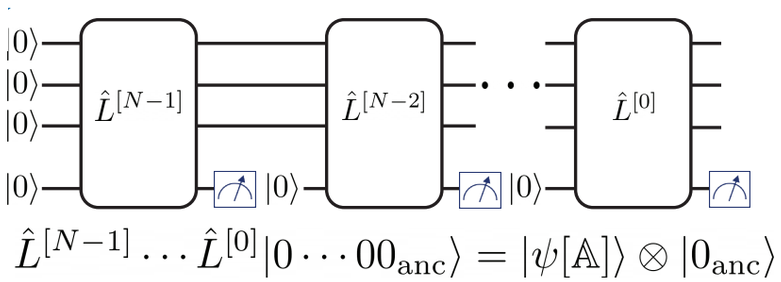
Many other applications & architectures could be mentioned:

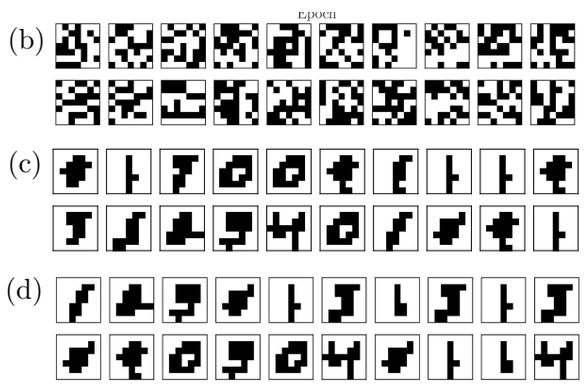
- Selvan, Dam, "Tensor Networks for Medical Image Classification"



arxiv:2011.06982

- Wall et al. "Generative machine learning with tensor networks: benchmarks on near-term quantum computers" arxiv:2010.03641





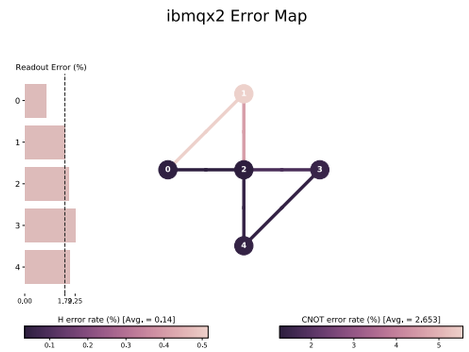


FIG. 4. *Exemplar NISQ hardware architecture.* The qubit layout

Algorithms for Optimizing Tensor Network Models

Exploring *performance* of tensor networks is interesting

But also seek

theory of training algorithms

theory of generalization

understanding which models best for certain data

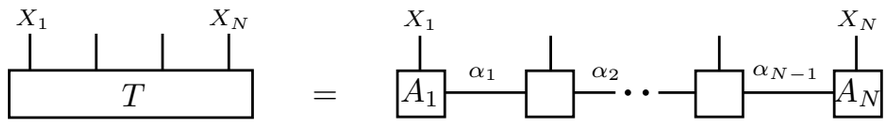
....

even if it means trading away performance

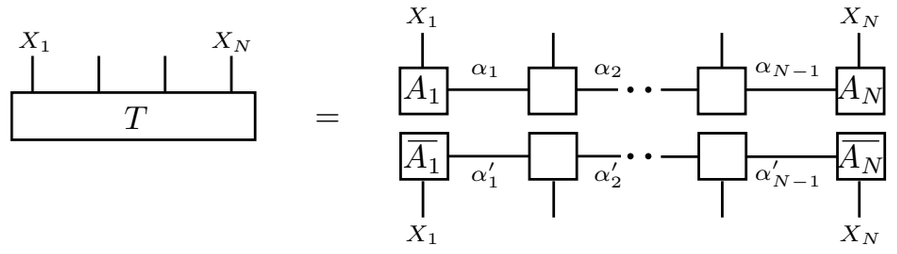
Theory is Already Being Developed

Theory of expressive power of tensor architectures:

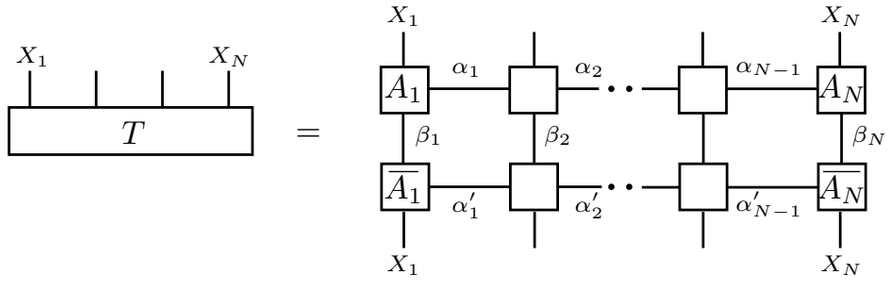
TT/MPS



Born Machine (BM)



Locally Purified State (LPS)



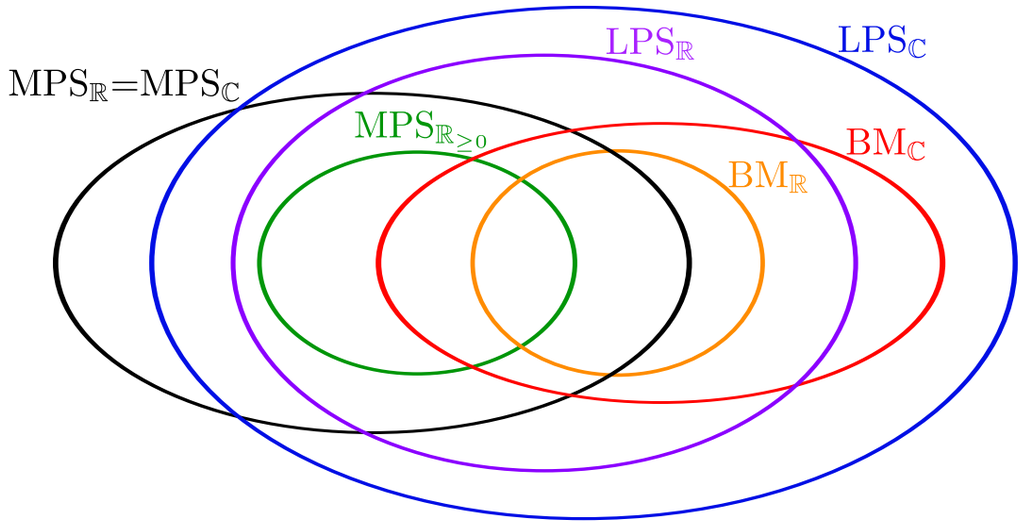


Table 2: Results of Proposition 2

	TT-rank$_{\mathbb{R}}$	TT-rank$_{\mathbb{R}_{\geq 0}}$	Born-rank$_{\mathbb{R}}$	Born-rank$_{\mathbb{C}}$	puri-rank$_{\mathbb{R}}$	puri-rank$_{\mathbb{C}}$
TT-rank$_{\mathbb{R}}$	=	<	<, >	<, >	<, >	<, >
TT-rank$_{\mathbb{R}_{\geq 0}}$	>	=	<, >	<, >	>	>
Born-rank$_{\mathbb{R}}$	<, >	<, >	=	>	>	>
Born-rank$_{\mathbb{C}}$	<, >	<, >	<	=	<, >	>
puri-rank$_{\mathbb{R}}$	<, >	<	<	<, >	=	>
puri-rank$_{\mathbb{C}}$	<, >	<	<	<	<	=

I. Glasser, R. Sweke, N. Pancotti, J. Eisert, I. Cirac, arxiv:1907.03741

Plan for This Section

Let's study training algorithms more systematically

Use synthetic data (even-parity dataset) to make evaluation of model well-posed

Contrast two training algorithms

Theoretical prediction of generalization performance of second algorithm

For the synthetic data set, we consider *even-parity data set*

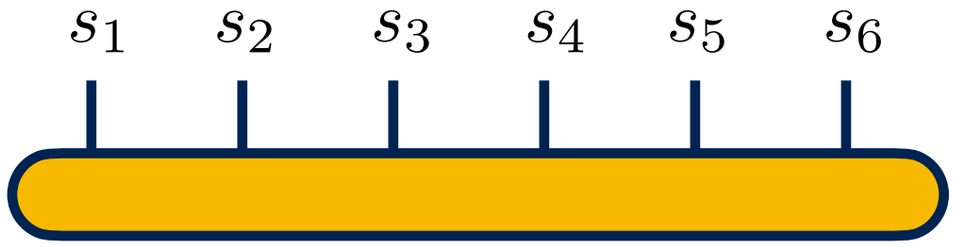
Bit strings of length L with even number of 1 bits

Even-parity data set (L = 10):

$$
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & & & & \vdots & & & & &
\end{array}
$$

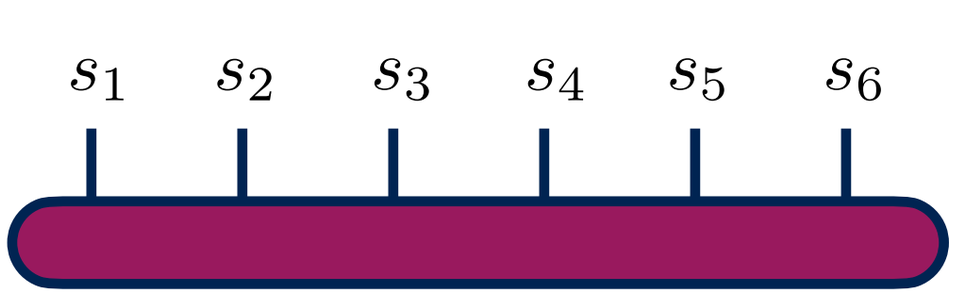
Two ways of representing probability distribution through a tensor:

$$p(s_1, s_2, s_3, s_4, s_5, s_6) =$$



1-norm formalism

$$p(s_1, s_2, s_3, s_4, s_5, s_6) = \left|\right.$$

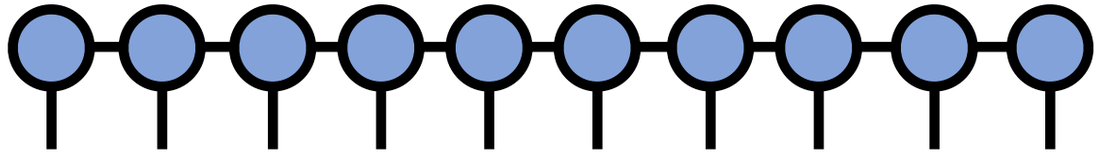


$$\left.\right|^2$$

2-norm formalism

Advantages & disadvantages of each
(will use **2-norm** in what follows)

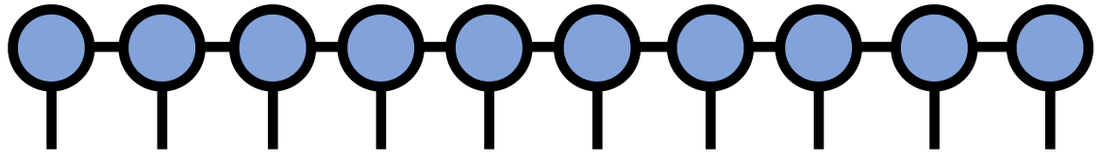
Seek an MPS which generates this data with uniform probability



$$
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & & & & \vdots & & & & &
\end{array}
$$

Entire even-parity dataset can be fit by
MPS of **bond dimension 2**

So we already know model is in the right class

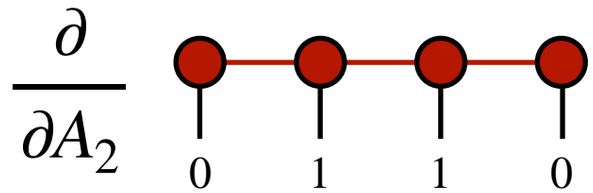


$$
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & & & & \vdots & & & & &
\end{array}
$$

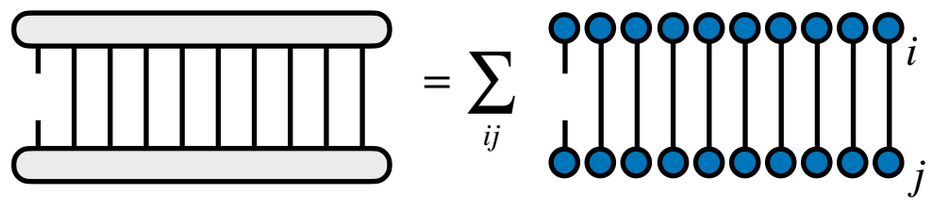
There are many ways one can devise to optimize MPS for machine learning objectives

Let's discuss two:

1. Alternating Optimization of Tensors



2. Density matrix algorithm

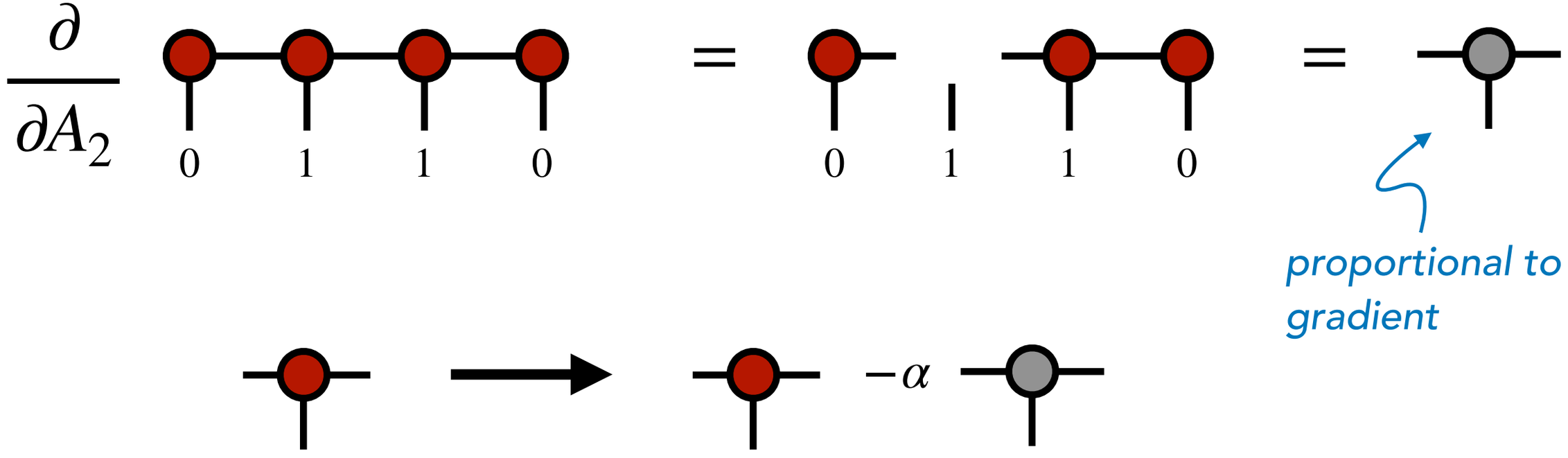


1. Alternating Optimization of MPS Tensors

Alternating Optimization of MPS Tensors

Optimize one MPS tensor at a time

Project data samples through MPS tensors to obtain gradient:

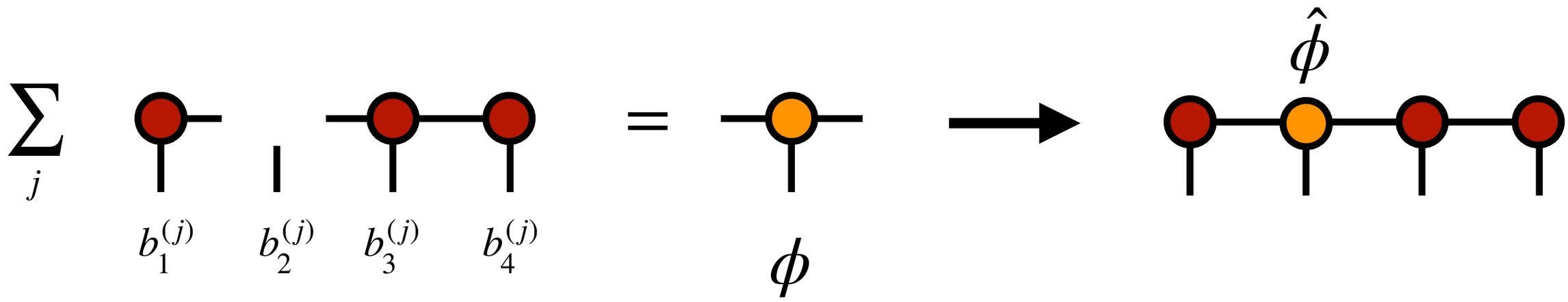


Scaling **linear** in training set size & **linear** in length of MPS

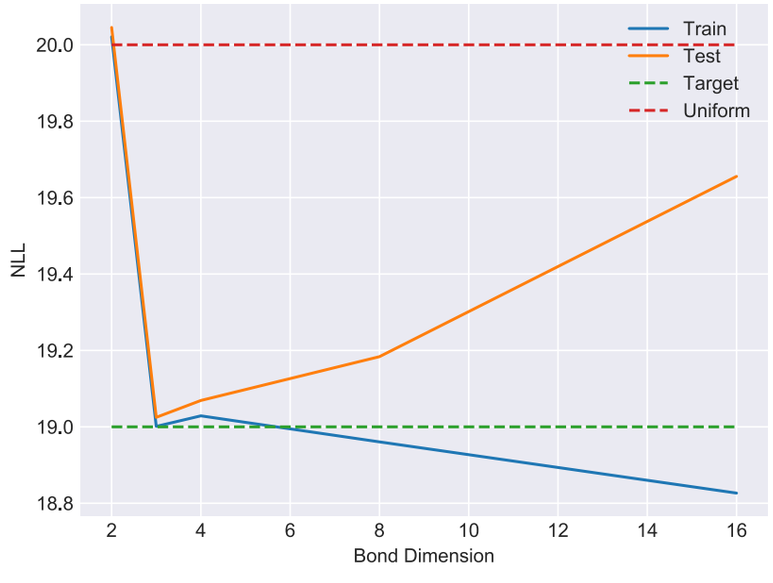
Alternating Optimization of MPS Tensors

Used by Stokes and Terilla to study even-parity data set

Locally optimal update for generative training:



Bond dimension related to generalization gap:



Stokes, Terilla, arxiv:1902.06888

Alternating Optimization of MPS Tensors

But alternating optimization has some drawbacks:

- can get stuck in local minimum
- requires good initialization to work
- results depend on how long algorithm is run

Alternatives?

2. Density Matrix Algorithm

Density Matrix Algorithm

This algorithm involves a **deterministic** strategy:

write 'perfect' answer then compress into MPS

Because every step of the algorithm involves linear algebra, we could product a theoretical prediction of generalization performance

Tai-Danae Bradley

John Terilla

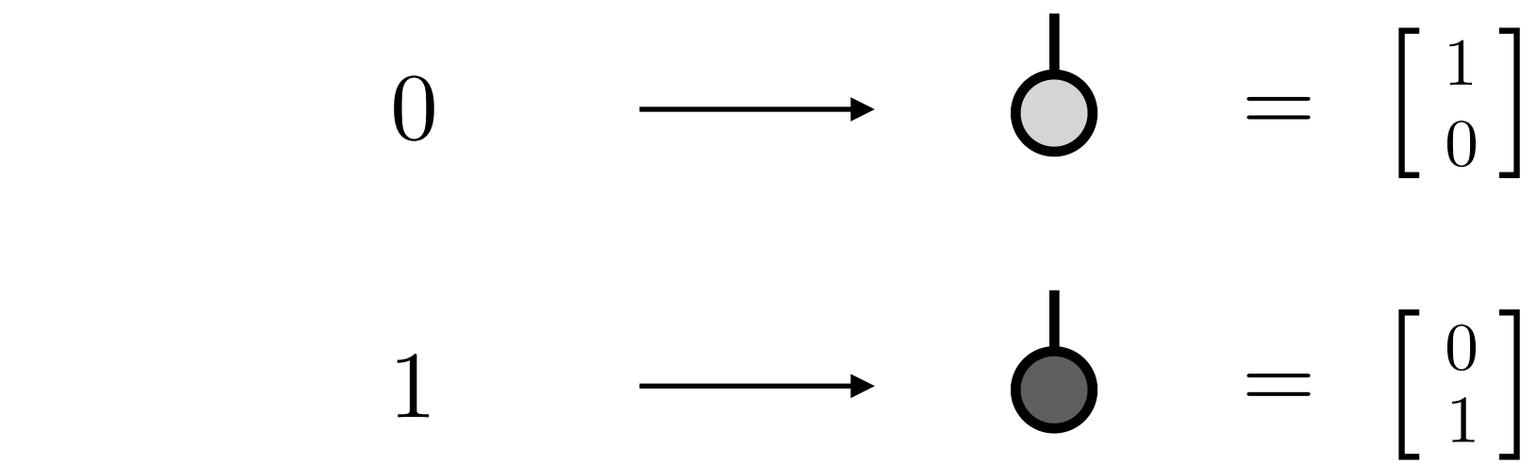
Bradley, Stoudenmire, Terilla, arxiv:1910.07425

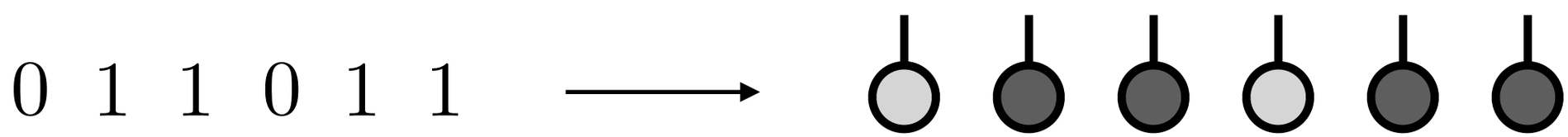
First let's review the learning algorithm

Then give a summary of the theoretical prediction

Density Matrix MPS Learning Algorithm

Map data to rank-one tensors





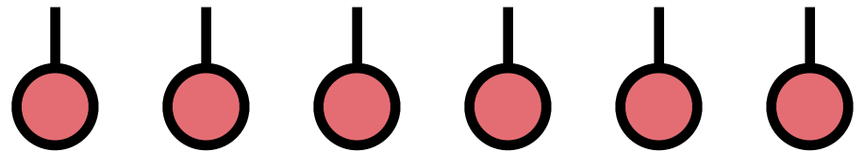
$\mathbf{x}_i$ $\Phi(\mathbf{x}_i)$

Density Matrix MPS Learning Algorithm

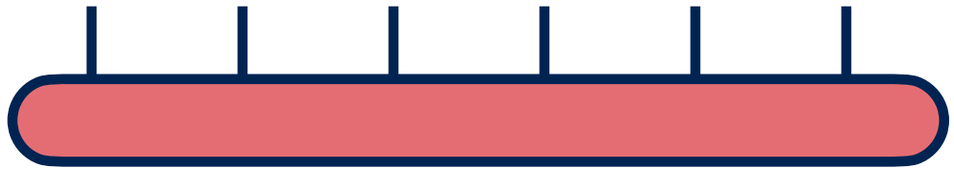
Define "target tensor" which perfectly overfits training data

$$N = |\mathcal{T}|$$

$$\psi^{s_1 s_2 s_3 s_4 s_5 s_6} = \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{T}}$$



$$=$$



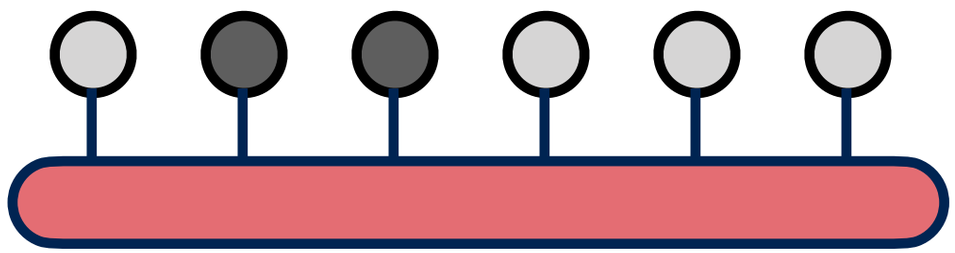
(Do not actually perform sum yet!)

See also: Stokes, Terilla, arxiv:1902.06888

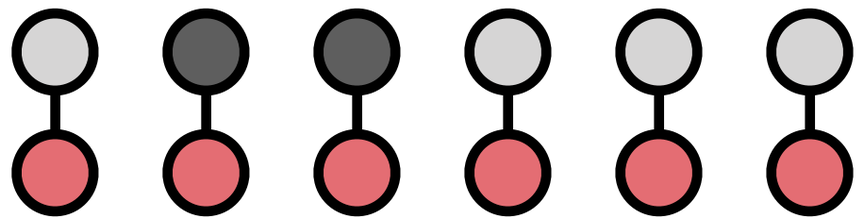
Density Matrix MPS Learning Algorithm

Perfectly overfits because data vectors orthonormal

$$p(\mathbf{x}_i) =$$



$$= \left| \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{T}} \right.$$



$$\left. \right|^2$$

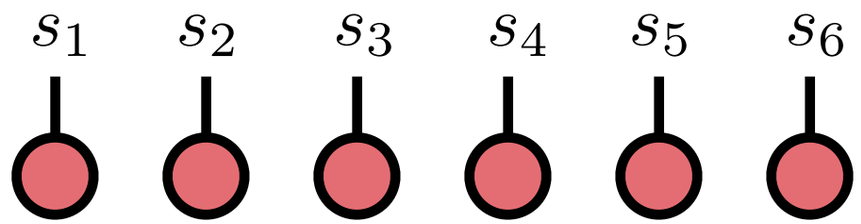
$$= \quad \frac{1}{N} \qquad \text{if} \qquad i \in \mathcal{T}$$

otherwise zero

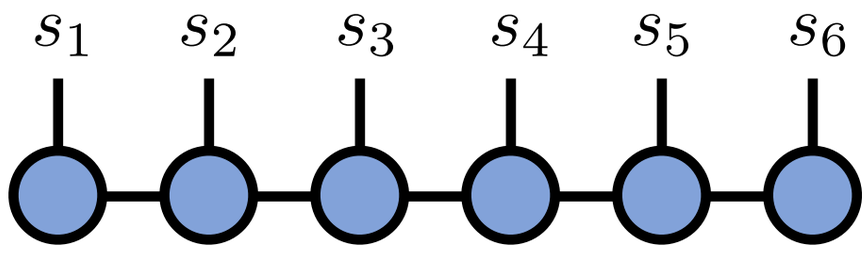
Density Matrix MPS Learning Algorithm

Compute closest MPS to this tensor
(restricting to bond dimension 2)

$$\psi^{s_1 s_2 s_3 s_4 s_5 s_6} = \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{T}}$$



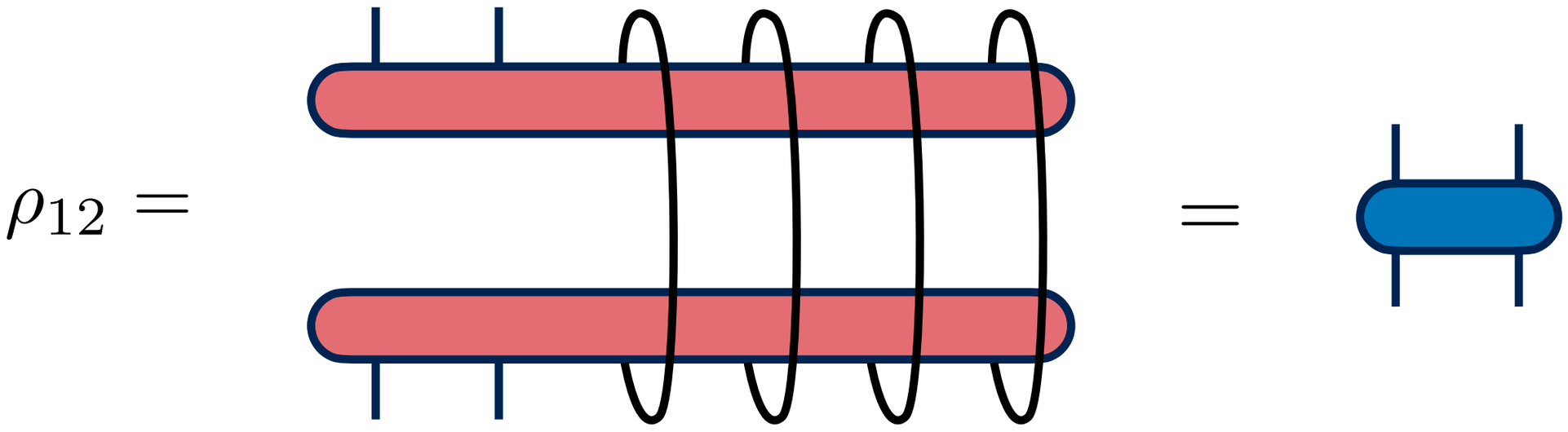
$\approx$



Density Matrix MPS Learning Algorithm

Algorithm is *density matrix algorithm* for compressing tensor to MPS form (closely related to TT-SVD and DMRG)

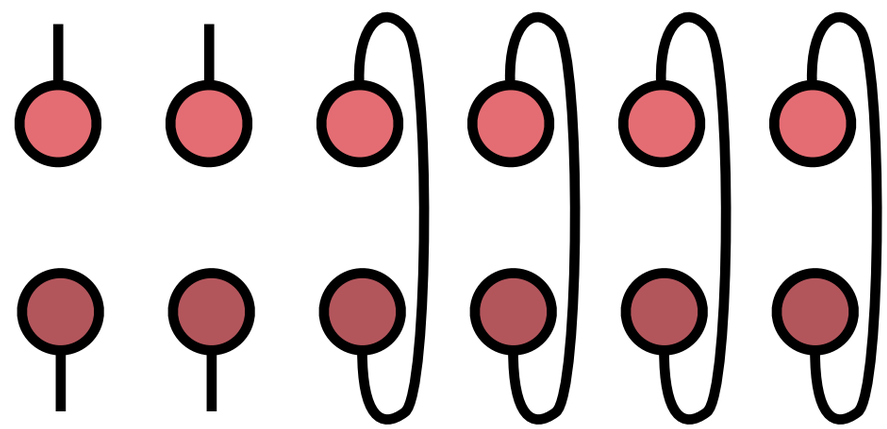
Compute Hermitian, positive semi-definite matrix



Density Matrix MPS Learning Algorithm

Can compute density matrix with
summed form of ψ inserted

$$\rho_{12} = \sum_{j,j' \in \mathcal{T}}$$



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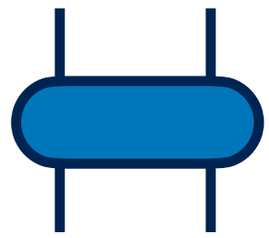
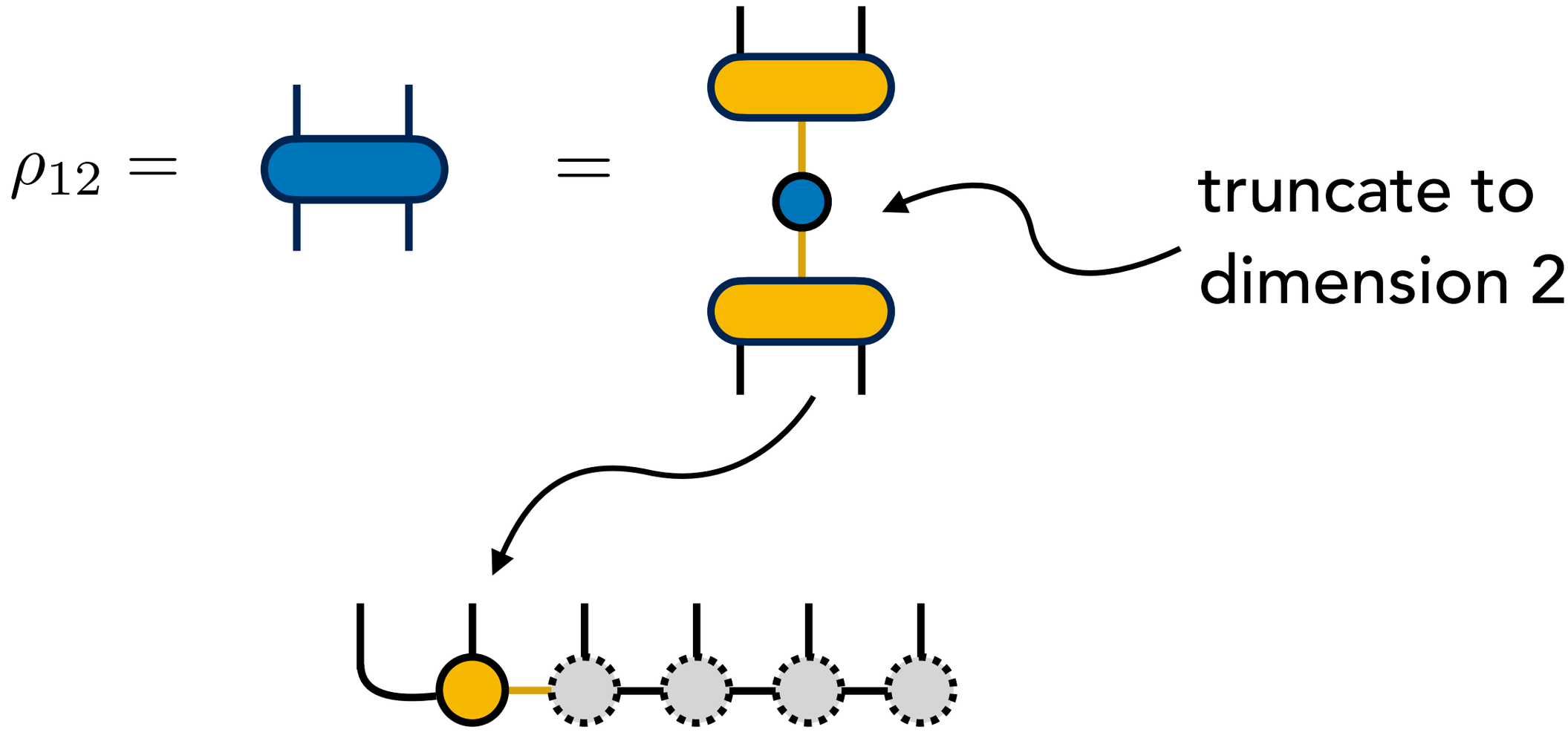


Density Matrix MPS Learning Algorithm

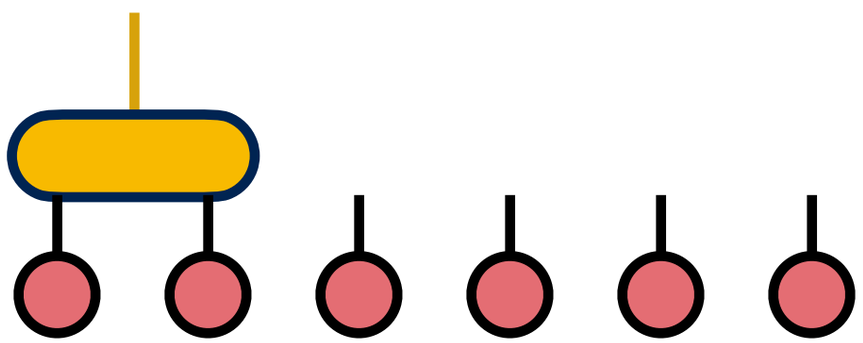
Diagonalizing ρ_{12} gives first & second MPS tensors



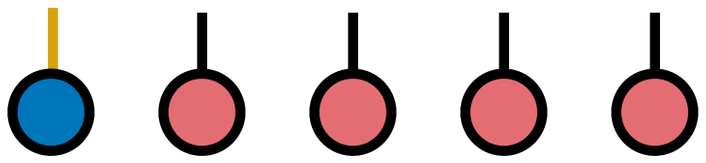
Density Matrix MPS Learning Algorithm

Use diagonalizing unitary to compress data

$$\psi^{s_1 s_2 s_3 s_4 s_5 s_6} = \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{T}}$$



$$= \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{T}}$$

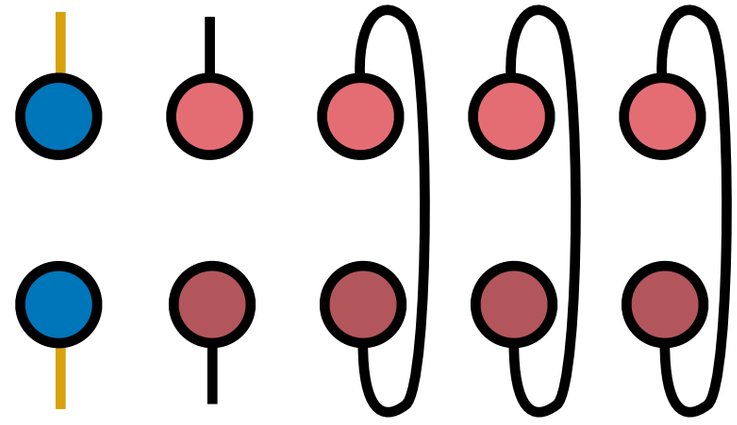


& recursively repeat algorithm to generate all MPS tensors

Density Matrix MPS Learning Algorithm

Computing next density matrix ρ_3

$$\rho_3 = \sum_{j,j' \in \mathcal{T}}$$



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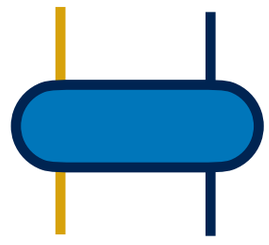
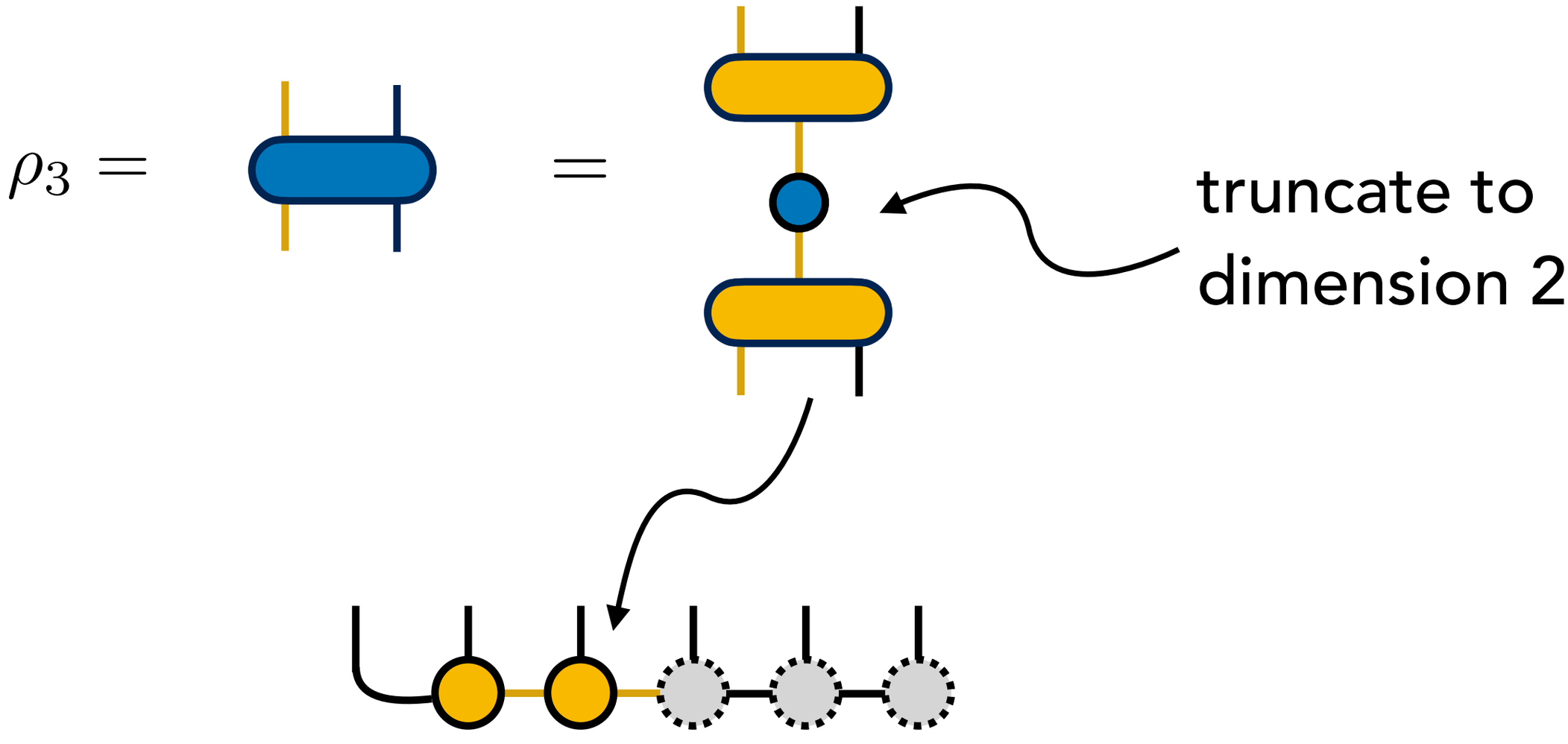


Density Matrix MPS Learning Algorithm

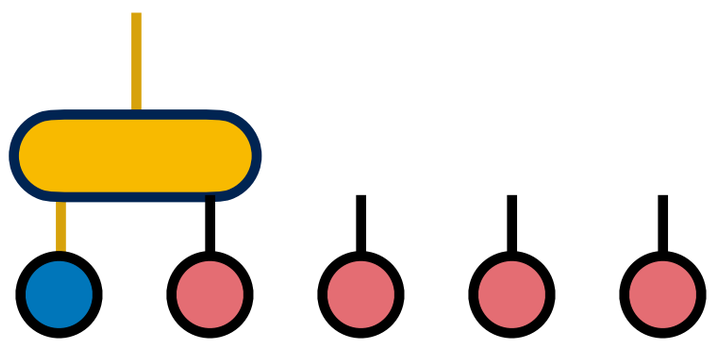
Diagonalizing ρ_3 gives third MPS tensor



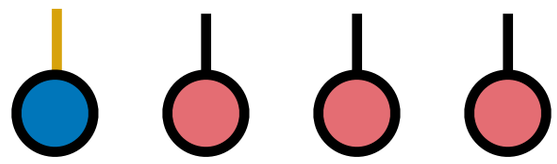
Density Matrix MPS Learning Algorithm

Use diagonalizing unitary to compress data

$$\psi^{s_1 s_2 s_3 s_4 s_5 s_6} = \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{T}}$$



$$= \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{T}}$$



& recursively repeat algorithm to generate all MPS tensors

Density Matrix MPS Learning Algorithm

Deterministic algorithm for generative modeling

Truncation of density matrix determines
bond dimension of MPS

Intentionally make 'error' from overfitted model
in order to generalize

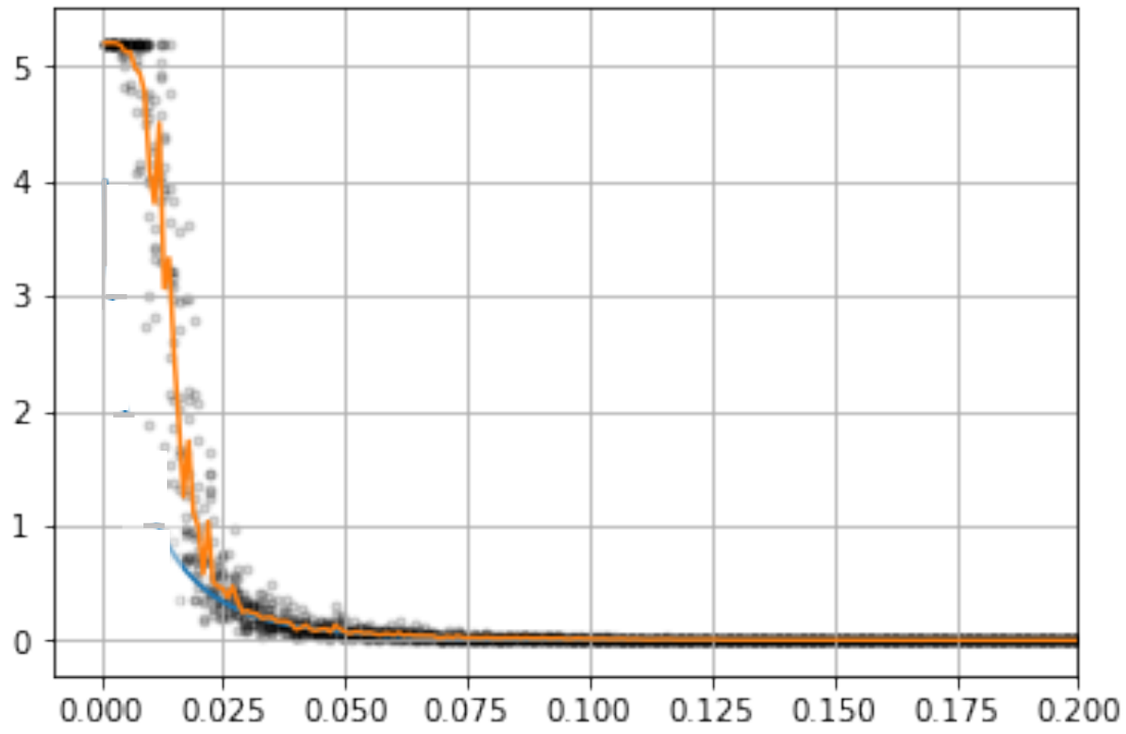
compression = learning

Density Matrix MPS Learning Algorithm

How well does it work?

L=16 results:

Bhattacharyya distance to true distribution



Fraction of data used as training set

Bhattacharyya distance:

$$D_B = -\ln\left(\sum_x \sqrt{p(x)q(x)}\right)$$

can evaluate exactly for parity data set

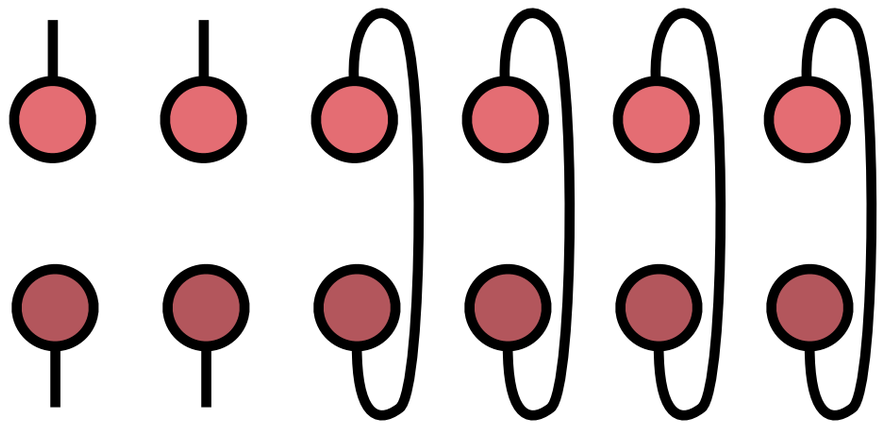
Sketch of the theoretical prediction

Tai-Danae Bradley

John Terilla

Observe that density matrices have a block-diagonal form (blocks for even & odd parity 'sectors')

$$\rho_{12} = \sum_{j,j' \in \mathcal{T}}$$



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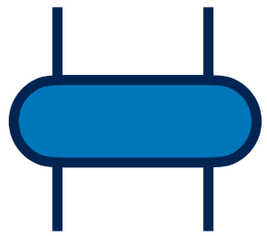


$$\rho_{12} = \begin{bmatrix} d_1 & s_e & & \\ s_e & d_2 & & \\ & & d_3 & s_o \\ & & s_o & d_4 \end{bmatrix}$$

Reason is trace over bits 3,4,...,N pairs identical suffixes (& identical parity), otherwise zero

Assume two most dominant eigenvectors come from the two blocks (one from each)

$$\rho_{12} = \begin{array}{c} \\ 00 \\ 11 \\ 01 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 11 & 01 & 10 \end{array} \\ \begin{bmatrix} d_1 & s_e & & \\ s_e & d_2 & & \\ & & d_3 & s_o \\ & & s_o & d_4 \end{bmatrix} \end{array}$$

$$|E_2\rangle = \cos\theta_2|00\rangle + \sin\theta_2|11\rangle$$
$$|O_2\rangle = \cos\phi_2|01\rangle + \sin\phi_2|10\rangle$$

(*Notation* $|\mathbf{v}\rangle$ *means a vector* $\mathbf{v}$)

Easily find expressions for angles in terms of entries of density matrix

$$\rho_{12} = \begin{array}{c} \\ 00 \\ 11 \\ 01 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 11 & 01 & 10 \end{array} \\ \begin{bmatrix} d_1 & s_e & & \\ s_e & d_2 & & \\ & & d_3 & s_o \\ & & s_o & d_4 \end{bmatrix} \end{array}$$

$$|E_2\rangle = \cos\theta_2|00\rangle + \sin\theta_2|11\rangle$$
$$|O_2\rangle = \cos\phi_2|01\rangle + \sin\phi_2|10\rangle$$

$$\theta_2 = \arctan\left(\frac{2s_e}{\sqrt{G_e^2 + 4s_e^2} + G_e}\right) \qquad \phi_2 = \arctan\left(\frac{2s_o}{\sqrt{G_o^2 + 4s_o^2} + G_o}\right)$$

$$G_e = |d_1 - d_2| \qquad G_o = |d_3 - d_4|$$

$$\rho_{12} = \begin{array}{c} \\ 00 \\ 11 \\ 01 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 11 & 01 & 10 \end{array} \\ \begin{bmatrix} d_1 & s_e & & \\ s_e & d_2 & & \\ & & d_3 & s_o \\ & & s_o & d_4 \end{bmatrix} \end{array}$$

$$\theta_2 = \arctan\left(\frac{2s_e}{\sqrt{G_e^2 + 4s_e^2} + G_e}\right) \qquad \phi_2 = \arctan\left(\frac{2s_o}{\sqrt{G_o^2 + 4s_o^2} + G_o}\right)$$

$$G_e = |d_1 - d_2| \qquad G_o = |d_3 - d_4|$$

Importantly, if gaps G_e, G_o zero, perfect learning occurs as long as s_e, s_o non-zero

Non-zero gaps are what drive errors in learning (coming from strings with unique suffixes)

$$\rho_{12} = \begin{array}{c} \\ 00 \\ 11 \\ 01 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 11 & 01 & 10 \end{array} \\ \begin{bmatrix} d_1 & s_e & & \\ s_e & d_2 & & \\ & & d_3 & s_o \\ & & s_o & d_4 \end{bmatrix} \end{array}$$

$$\theta_2 = \arctan\left(\frac{2s_e}{\sqrt{G_e^2 + 4s_e^2} + G_e}\right) \qquad \phi_2 = \arctan\left(\frac{2s_o}{\sqrt{G_o^2 + 4s_o^2} + G_o}\right)$$

$$G_e = |d_1 - d_2| \qquad G_o = |d_3 - d_4|$$

Quantities d_1, d_2, d_3, d_4 are frequency of observing a certain prefix (00,11,01,10). So "gaps" are from imbalance in symmetry-related training examples.

Use combinatorics of even-parity data set (taking a finite fraction as training set)

Determine facts such as:

$s_e =$ number bitstrings with even prefix (00,11) having same suffix

$$\mathbb{E}[s_e] = \frac{f \cdot |\mathcal{T}|}{4}$$

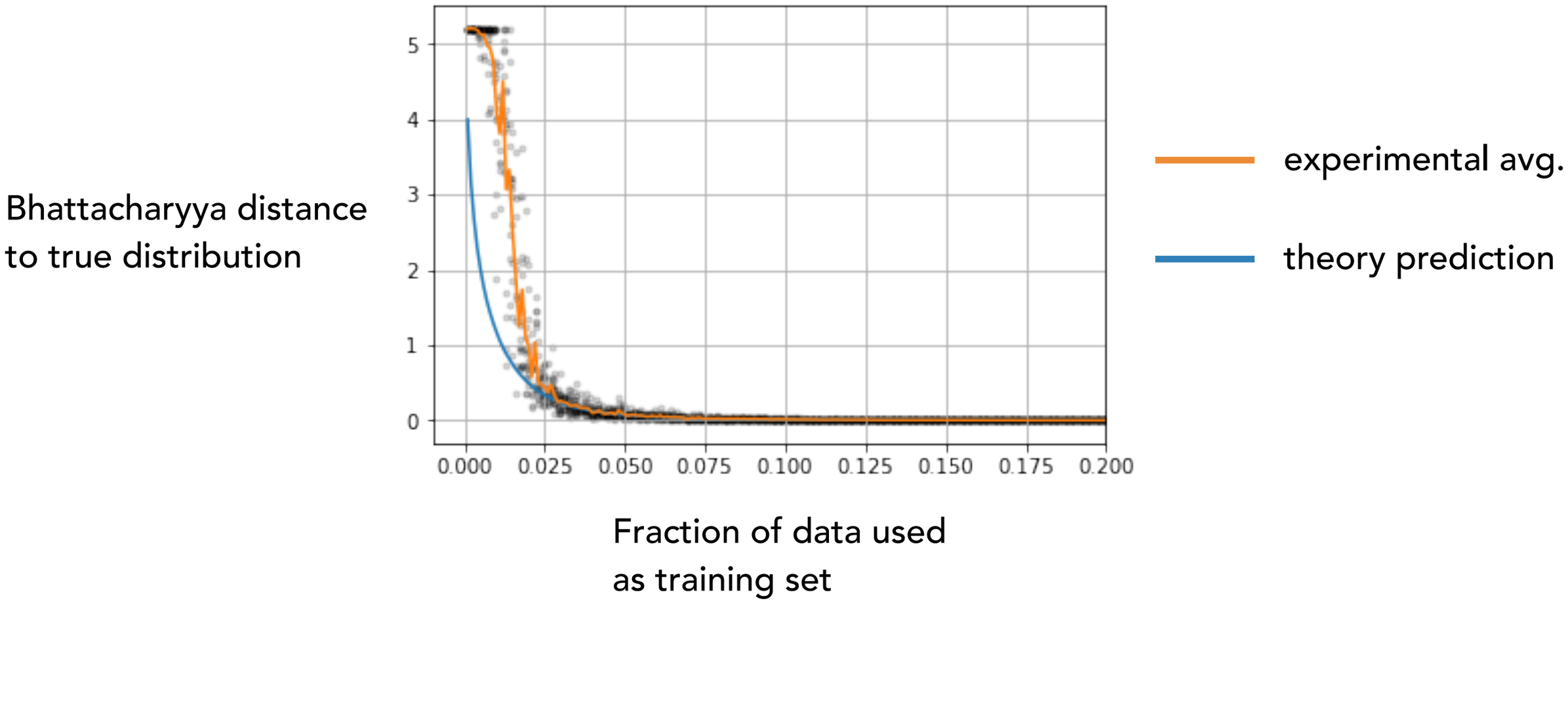
$$\mathbb{E}[G_e] = \sum_{d_1=0}^{r} |2d_1 - r| \frac{\binom{n}{d_1}\binom{n}{r-d_1}}{\binom{2n}{r}}$$

$r = |\mathcal{T}|/2$

$n = 2^{L-3}$

From combinatorial formulas, can estimate
typical MPS tensors as function of training set fraction

Encouraging agreement with experimental results (L=16)



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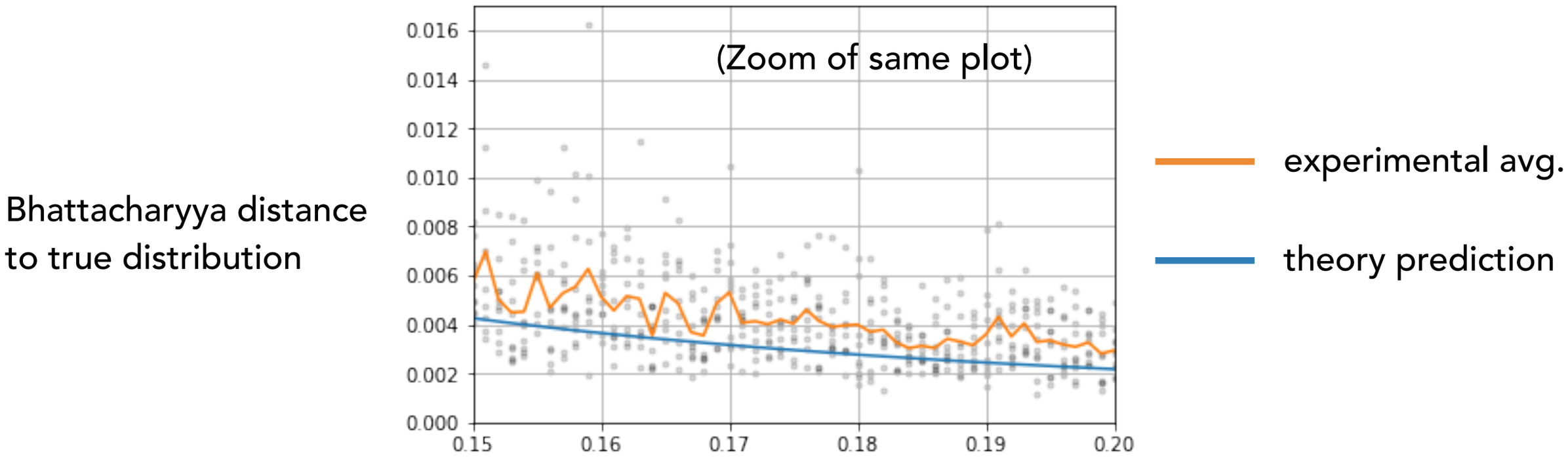
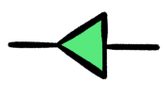
Encouraging agreement with experimental results (L=16)

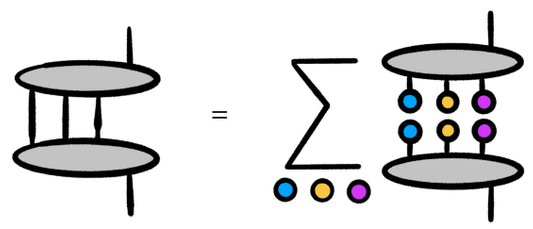


For an expanded discussion, see Tai-Danae's recent thesis *"At the Interface of Algebra and Statistics"* *arxiv:2004.05631*



An **isometric embedding** U is a linear map from a space V to a space W of larger dimension that preserves the lengths of vectors. Such a map satisfies $U^\dagger U = \mathrm{id}_V$ but $UU^\dagger \neq \mathrm{id}_W$. In words,

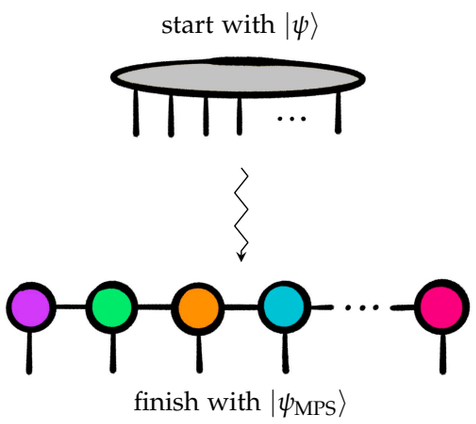
Tai-Danae Bradley
@math3ma (on Twitter)

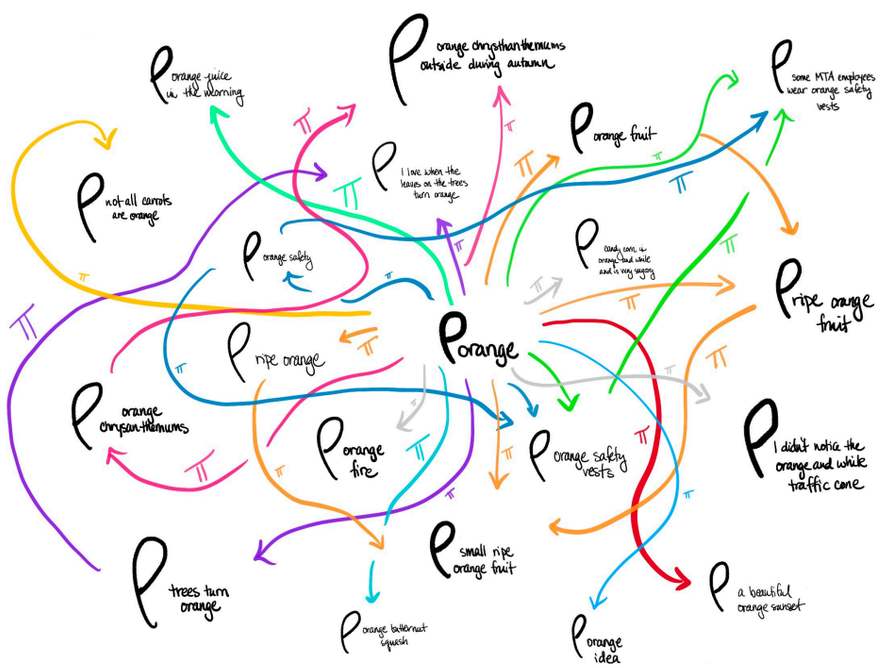


In symbols,

$$\rho_Y = \sum_{(a,b,c)\in A\times B\times C} M|abc\rangle\langle abc|M^\dagger$$

where we are using the fact that $A_i|\psi\rangle = M|x_i\rangle$ for each i, as remarked in the main text.





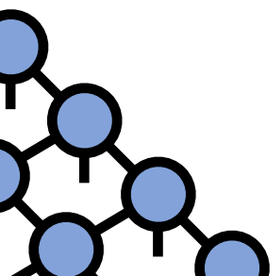
$$\rho_3 = \quad = \quad =$$

Future Directions to Pursue

- Develop theoretical prediction into rigorous, probabilistic bounds

- Apply this algorithm & theory for other synthetic data

- Find ways to estimate, just from summary stats of data, how likely learning is to succeed for what complexity of model

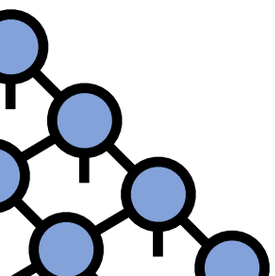


Summary

- Tensor networks can be used as machine learning models with interesting training algorithms, offering theoretical insights

- Can apply to variety of tasks with good results, sometimes state-of-the-art

- Importantly, opportunity to investigate deeper questions about learning from data



Recommended resource for learning about machine learning:

Physics Reports
Volume 810, 30 May 2019, Pages 1-124

A high-bias, low-variance introduction to Machine Learning for physicists

Pankaj Mehta [a] ✉, Marin Bukov [b] ✉, Ching-Hao Wang [a], Alexandre G.R. Day [a], Clint Richardson [a], Charles K. Fisher [c], David J. Schwab [d]

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https://doi.org/10.1016/j.physrep.2019.03.001

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