Kondo Effect and topological phases in 1-D superconductors

RUTGERS

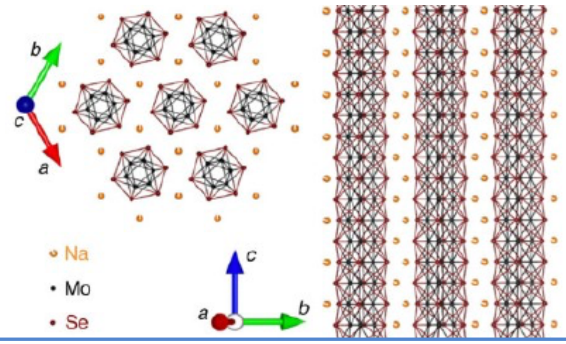
Natan Andrei
Rutgers university

Parmesh Pasnoori
Rutgers University

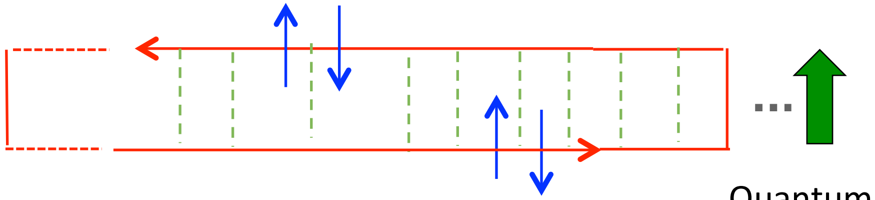
Patrick Azaria
Sorbonne Universite'

Colin Rylands
SISSA

1 D superconductor $Na_{2-\delta}Mo_6Se_6$.



Petrovic et al '16

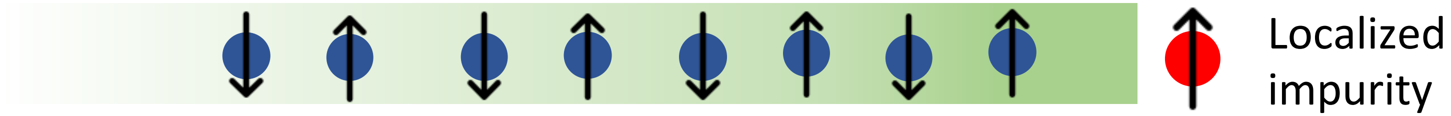


Quantum spin 1/2

1-D superconductor coupled to a Kondo spin

Atomtronics@Benasque2022

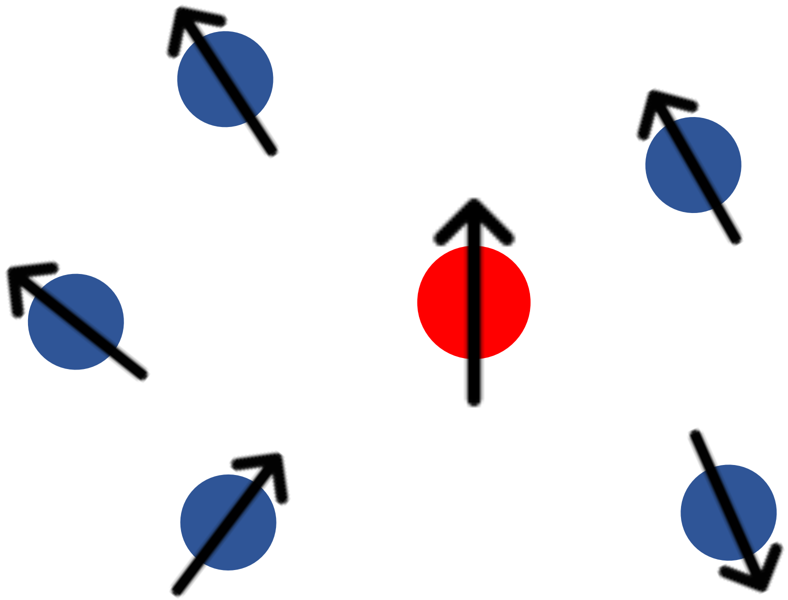
Overview



- Introduction
- Model – Kondo impurity at one edge
- Results: Superconductivity vs. Kondo: The phase diagram
- Some details on calculations
- Model – two Kondo impurities - one at each edge, emergence of boundary SUSY
- Conclusions & Outlook

Kondo effect in a metal

- Magnetic impurity in a sea of gapless fermions: $H_{\text{metal}} + H_{\text{imp}}$



$$H_{\text{metal}} = \sum_k \epsilon_k \psi^\dagger_{ka} \psi_{ka}$$

Metal: Fermi sea

$$H_{\text{imp}} = J \vec{S}_{\text{imp}} \cdot \vec{s}(0)$$

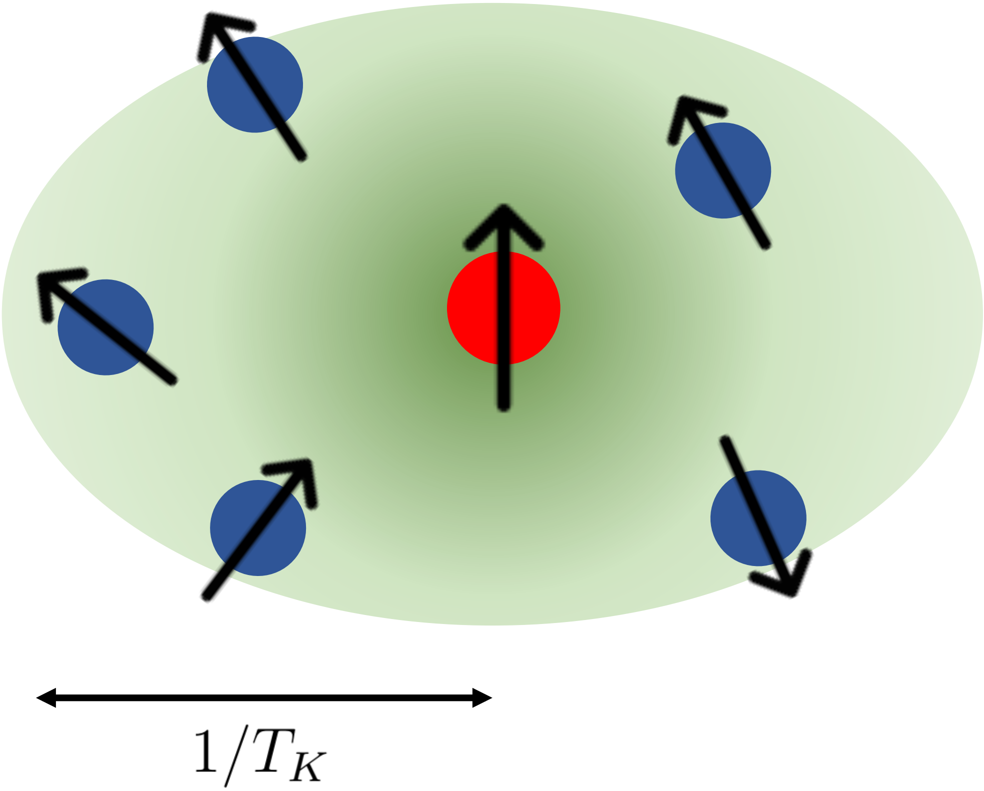
Kondo impurity coupled to the Fermi sea

$$\vec{s}(0) = \psi^\dagger_a(0) \vec{\sigma}_{ab} \psi_b(0)$$

Electrons spin density

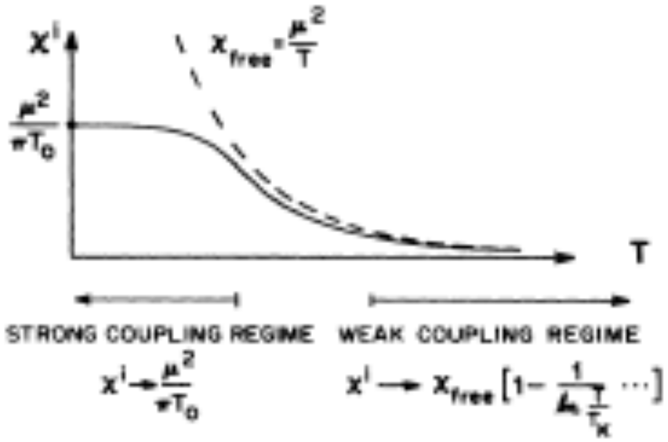
Kondo effect in a metal

Magnetic impurity in a sea of gapless fermions:



- For $J > 0$ spin ½ impurity is screened
  - *Singlet Ground State*
  - *Many-body screening*

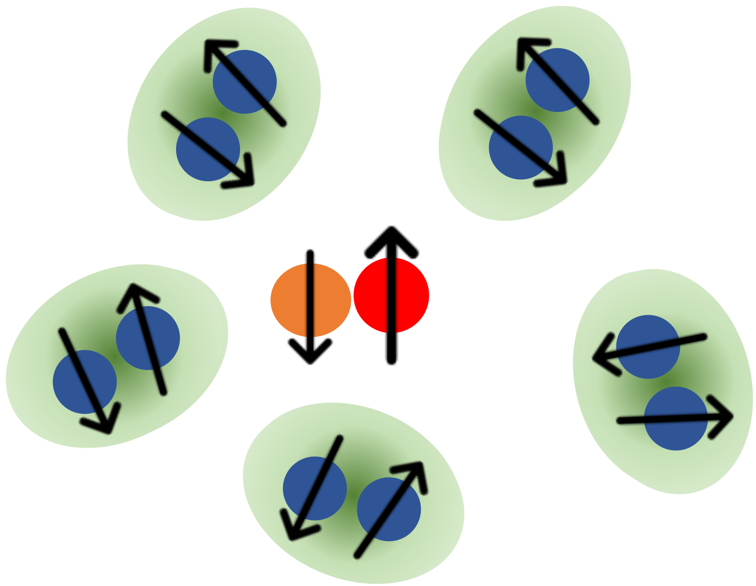
- Magnetic susceptibility and resistivity increase as $T \rightarrow 0$



- Non-perturbative scale $T_K \approx De^{-\pi/2J}$

- Impurity D.O.S $\rho_{\mathrm{imp}}(E) = \dfrac{T_K}{\pi(E^2 + T_K^2)}$

Kondo effect in a BCS superconductor



- Replace the metal with a superconductor

$$H_{\mathrm{metal}} + H_{\mathrm{pair}} + H_{\mathrm{imp}}$$

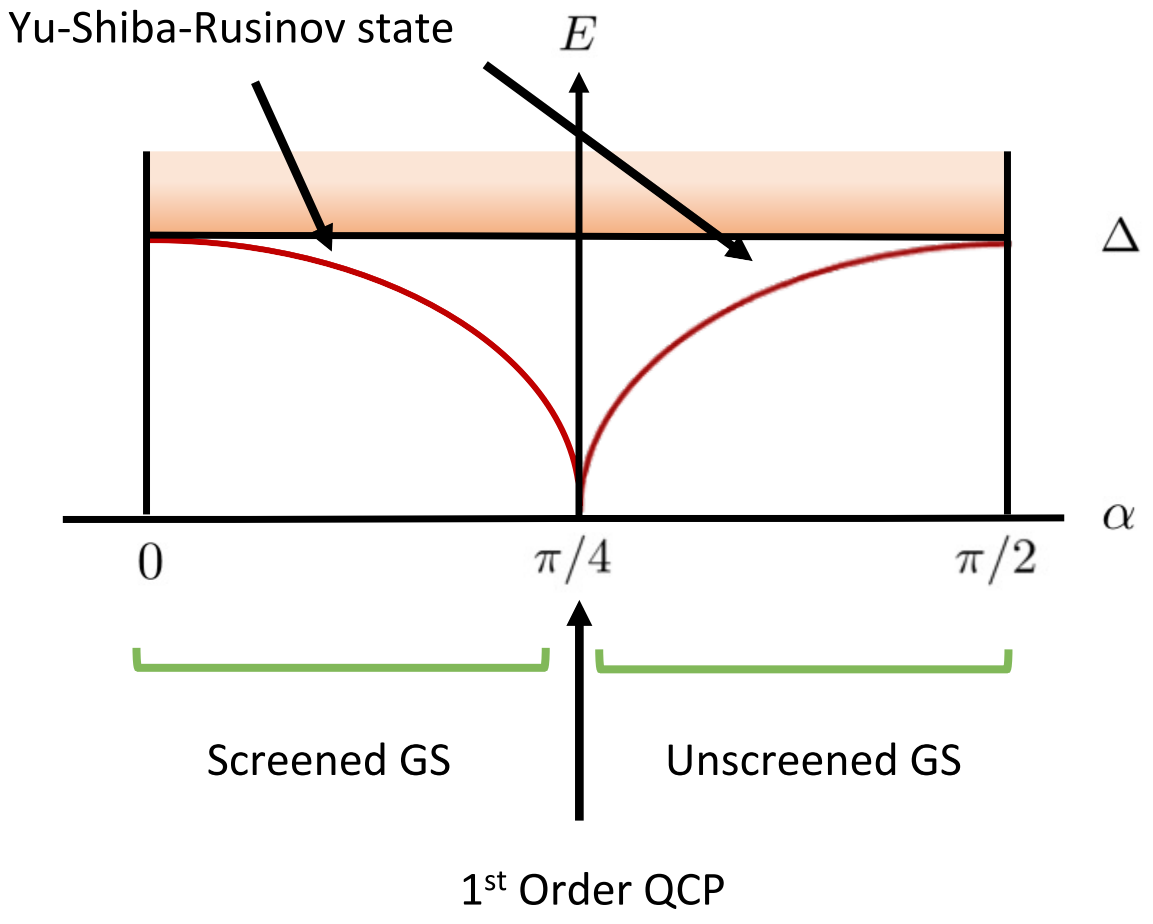
$$H_{\mathrm{pair}} = \Delta \int d\vec{x} \Psi_{\uparrow}^{\dagger}(\vec{x}) \Psi_{\downarrow}^{\dagger}(\vec{x}) + \mathrm{h.c}$$

- Competition between screening and Cooper pair formation

Yu (1965), Shiba (1968), Rusinov (1969)

- YSR approach: Mean Field theory (BCS) + classical spin

Classical spin in a BCS superconductor (YSR)



Yu (1965), Shiba (1968), Rusinov (1969)

- BCS superconductor, classical spin

$$H_{\mathrm{imp}} = J S^z_{\mathrm{imp}} s^z(0)$$

- Screening is done by a localized bound state

- BS energy is function of phase

$$\epsilon = -\Delta \sin(2\alpha(J))$$

with $$2\alpha = \pi/2 - JS$$

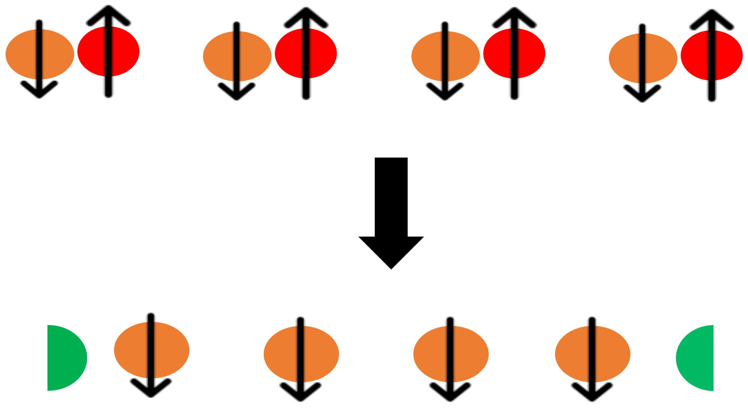
- Mediates a 1st order QPT

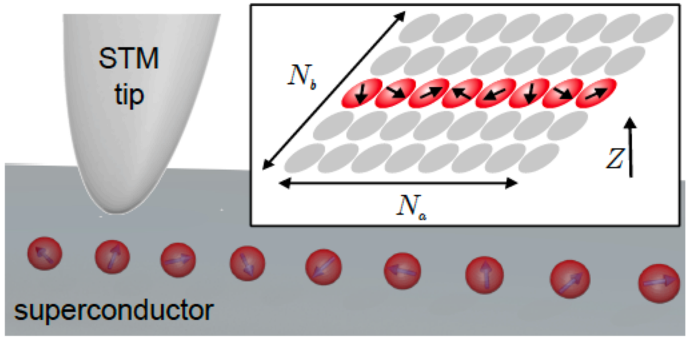
- Impurity quantum fluctuations - dress phase shift

$$\alpha(J) \to \tilde{\alpha}(\Delta/T_K)$$

Sakai *et.al.* J.Pys.Soc.Jpn. (1993)

Some applications: YSR states & topology





- Helical chain of impurities on a SC

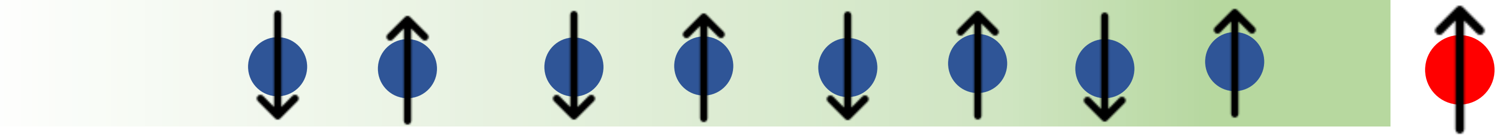
- YSR states hybridize, form a band below SC gap

- Realizes a topological SC, Majorana edge modes

Nadj-Perge, *et.al.* PRB (2013), Pientka, *et.al.* PRB (2013), Nadj-Perge, *et.al.* Science (2014), In BCS superconductors fluctuations are neglected

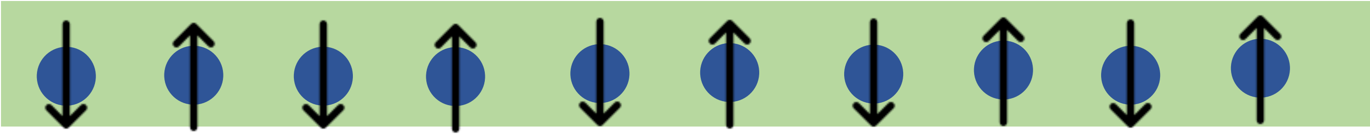
Q: What about SC's with strong fluctuations?

Ex: 1-d charge conserving superconductor coupled to a Kondo impurity



$$H = H_0 + H_{\text{int}} + H_{\text{imp}}$$

The bulk: 1d charge conserving superconductor



PBC:

N. Andrei, J. Lowenstein (1979)

G. Japaridze, A. Nersesyan, P. Wiegmann (1982)

OBC:

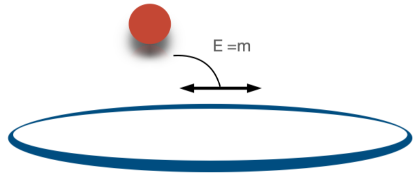
P. R. Pasnoori, N. Andrei, and P. Azaria, Phys. Rev. B 102, 214511 (2020)

P. R. Pasnoori, N. Andrei, and P. Azaria, Phys. Rev. B 104, 134519 (2021)

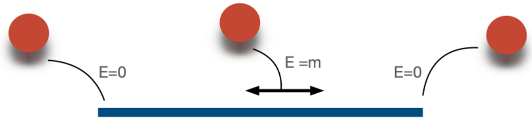
$$\mathcal{H} = \int_{-L/2}^{L/2} H_0 + H_{\rm int}$$

$$H_0 = -iv\left(\psi^\dagger_{Ra}\partial_x\psi_{Ra} - \psi^\dagger_{La}\partial_x\psi_{La}\right)$$

$$H_{\rm int} = \psi^\dagger_{Ra}\psi_{Rb}\left[g_\parallel\sigma^z_{ab}\sigma^z_{cd} + g_\perp(\sigma^x_{ab}\sigma^x_{cd} + \sigma^y_{ab}\sigma^y_{cd})\right]\psi_{Lc}\psi_{Ld}$$

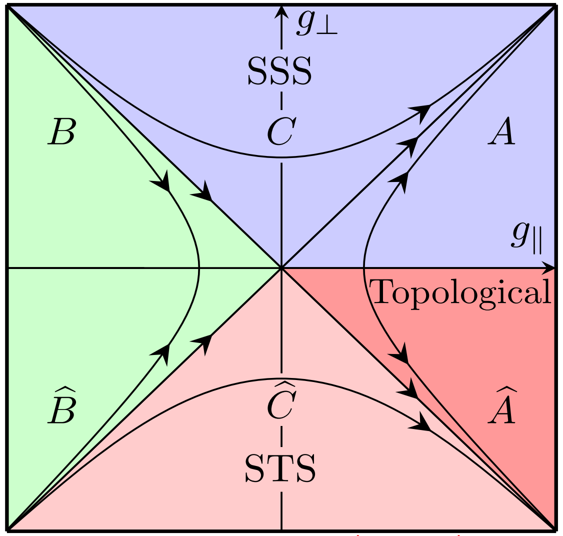


Periodic boundary conditions



Open boundary conditions

$$\mathcal{O}_{\rm SSS}(x) = \psi^\dagger_L\sigma^y\psi^\dagger_R$$



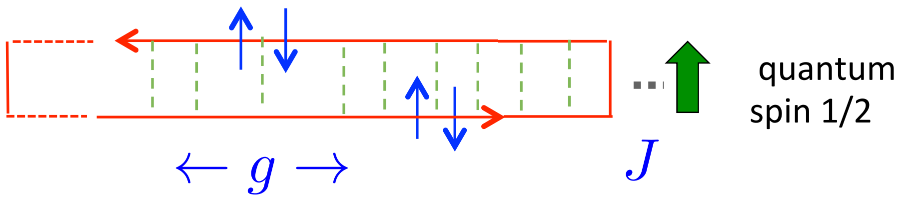
$$\mathcal{O}_{\rm STS}(x) = \psi^\dagger_L\sigma^x\psi^\dagger_R$$

- Strong e-e correlations
- Attractive spin exchange interaction for $g_\parallel > 0$
- Spin-charge separation
- Gapped spinons
- Gapless holons
- QLR SC order parameter :

$$\langle \mathcal{O}_{\rm SC}(x)\mathcal{O}_{\rm SC}(0)\rangle \sim \frac{1}{|x|^{1/2}}$$

SSS: $g_\perp > 0$, STS: $g_\perp < 0$

Bulk superconductor coupled to a quantum impurity



P. R. Pasnoori, N. Andrei, C. Rylands, and P. Azaria, (2021)

$$H = H_0 + H_{\text{int}} + \boxed{H_{\text{imp}}}$$

$$H_{\text{int}} = -g \int_{-\frac{L}{2}}^{0} dx\ \psi^\dagger_{Ra} \psi^\dagger_{Lc} \left(\vec{\sigma}_{ab} \cdot \vec{\sigma}_{cd}\right) \psi_{Rb} \psi_{Ld}$$

$$H_{\text{imp}} = -J \left(\vec{\sigma}_{ab} \cdot \vec{S}_{cd}\right) \psi^\dagger_{La}(0) \psi_{Rb}(0)$$

Open boundary conditions:

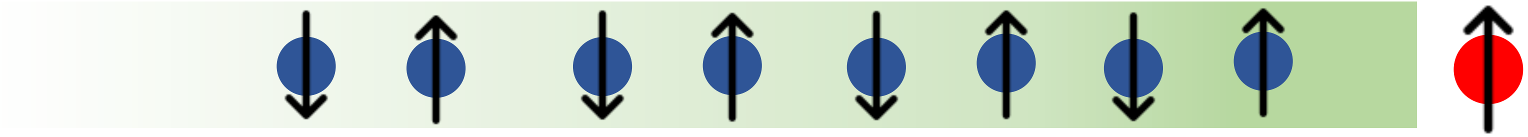
$$\psi^\dagger_{Ra}(0) = -\psi^\dagger_{La}(0) \qquad \psi^\dagger_{Ra}(-L/2) = -\psi^\dagger_{La}(-L/2)$$

We take the limit $g_\parallel = g_\perp = g > 0$ where the bulk is in the SSS phase.

Kondo coupling J is taken to be positive which corresponds to antiferromagnetic interaction.

Model is integrable for arbitrary Values of g and J

Competition :Kondo vs. SC



$$H = H_0 + H_{\rm int} + H_{\rm imp}$$

- Competition between nonperturbative effects and scales

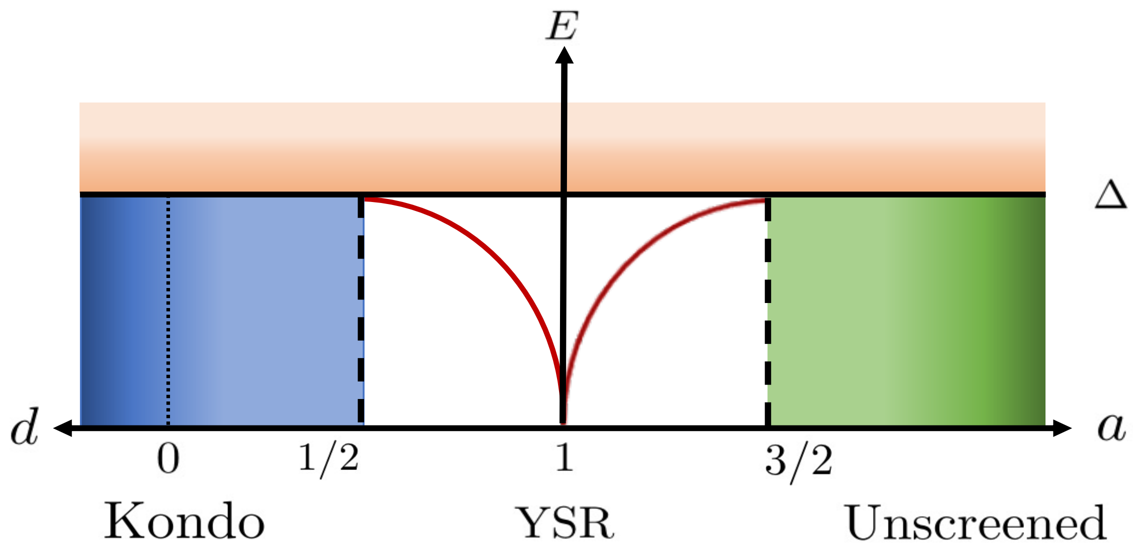
$$T_K \gg \Delta \quad T_K \approx \Delta \quad T_K \ll \Delta$$

  - Bulk fluctuations affect T_K

- Need nonperturbative techniques e.g. bosonization, semi-classical
- Model is integrable for isotropic couplings (maybe with anisotropy)

$$J_\perp = J_{||} \quad g_\perp = g_{||}$$

Model is integrable for any g and J



- Three regimes depending on relative values of g and J
  - Kondo
  - YSR
  - Unscreened

a (or d) RG invariant

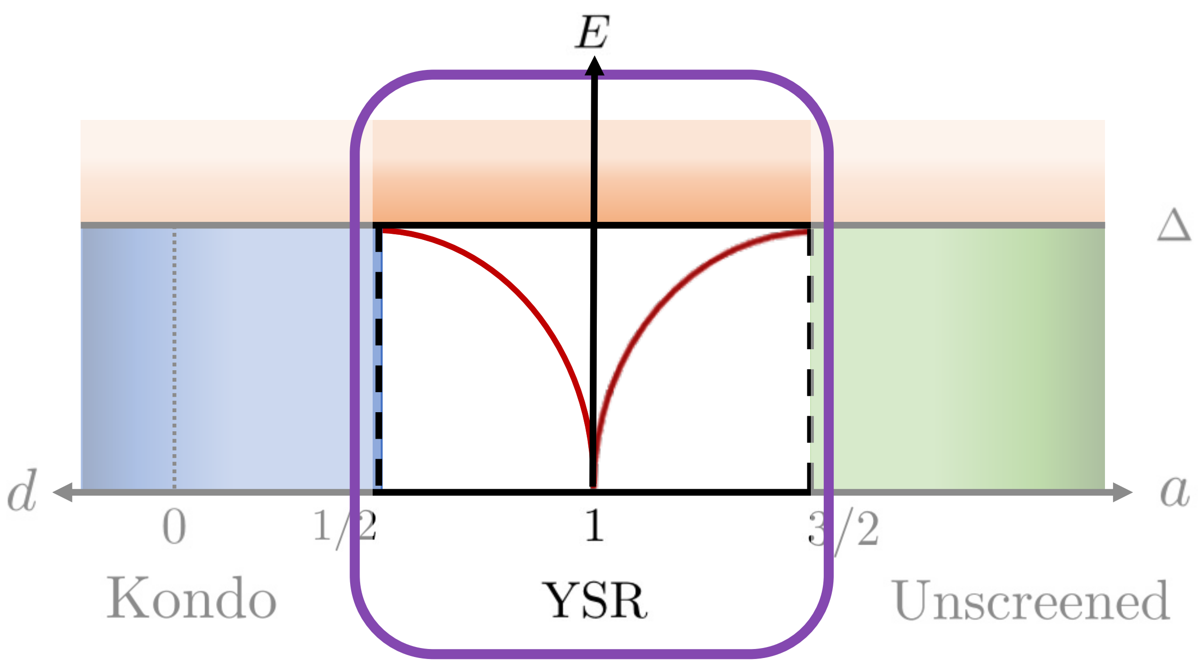
$J \gg 2g$ $\quad$ $J \sim 2g$ $\quad$ $J \ll 2g$

$$d = \sqrt{b^2 - \frac{b(1-3J^2/4)}{J} - 1} \qquad d \in \mathbb{R} \quad d = ia$$

Bulk parameter

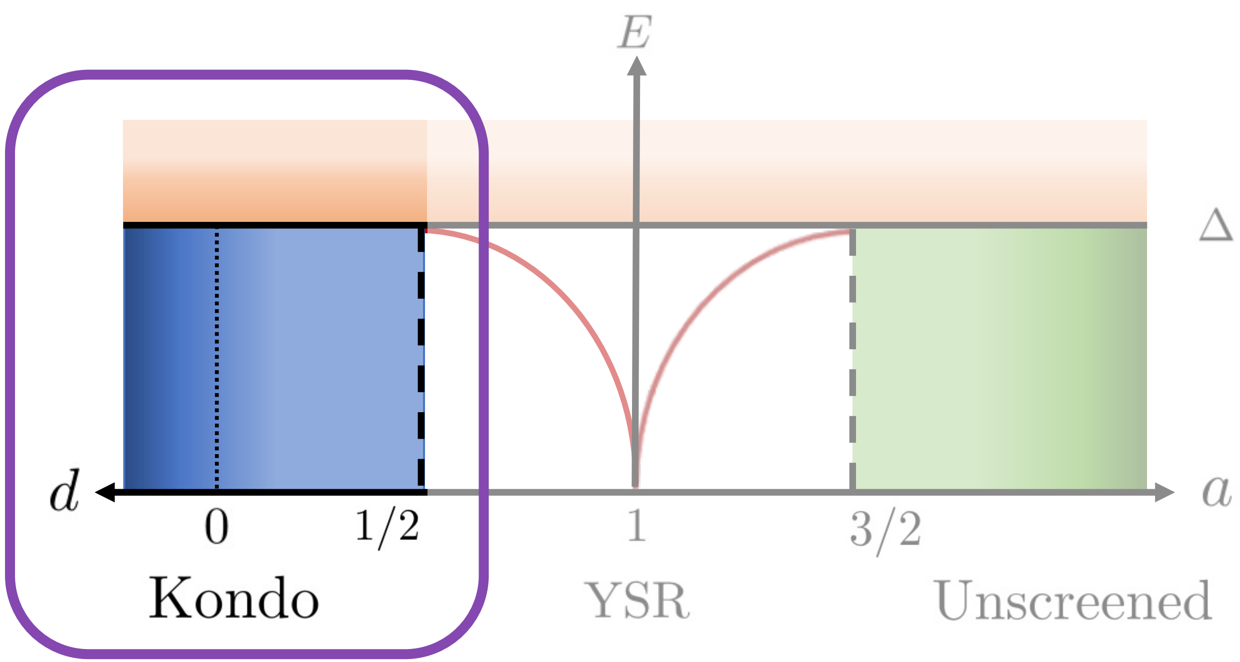
$$b = (1 - g^2/4)/2g$$

Results: Kondo effect in a 1-d superconductor



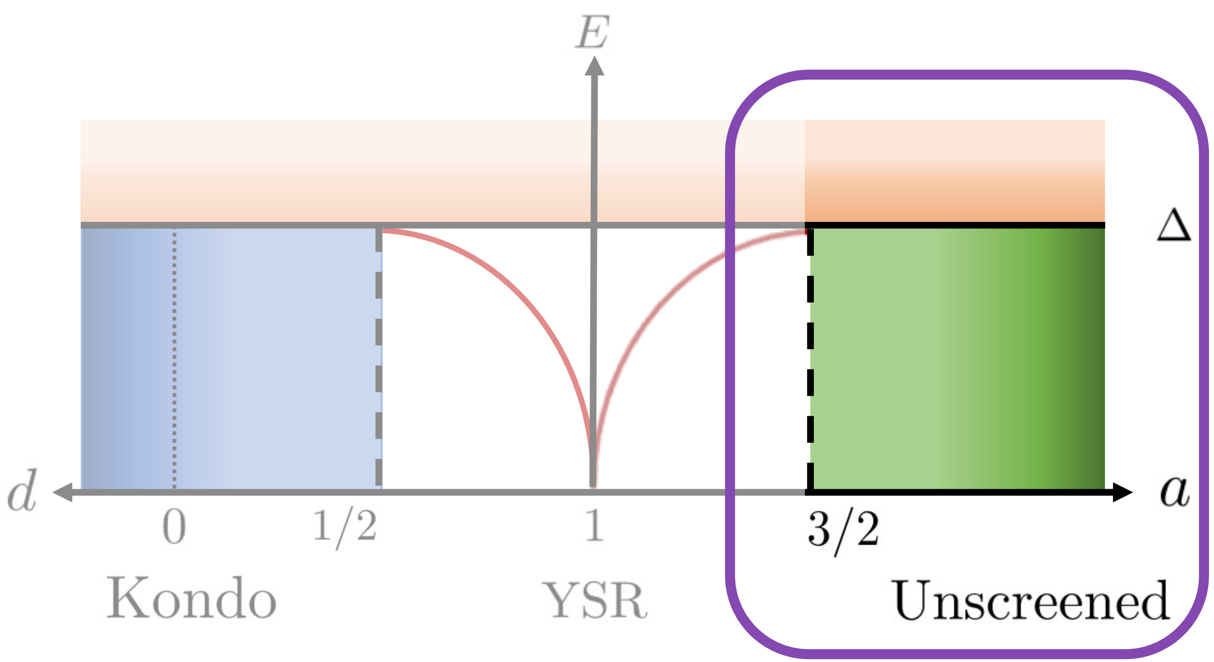
- YSR regime
  - YSR intragap bound state
  - Mediates a first order QPT
  - Single particle screening, no T_K
- Regime has semi-classical description

Results: Kondo effect in a 1-d superconductor



- Kondo Regime
  - No YSR bound states
  - Renormalized Kondo effect, singlet ground state
  - Crossover from many-body to few body screening as *a* is varied

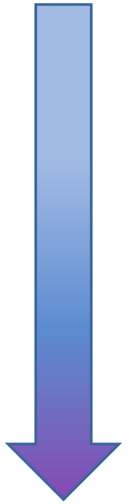
Results: Kondo effect in a 1-d superconductor



- Unscreened regime
  - No YSR states
  - Doublet ground state
  - Residual correlations between bulk & impurity

From Classical to Quantum description

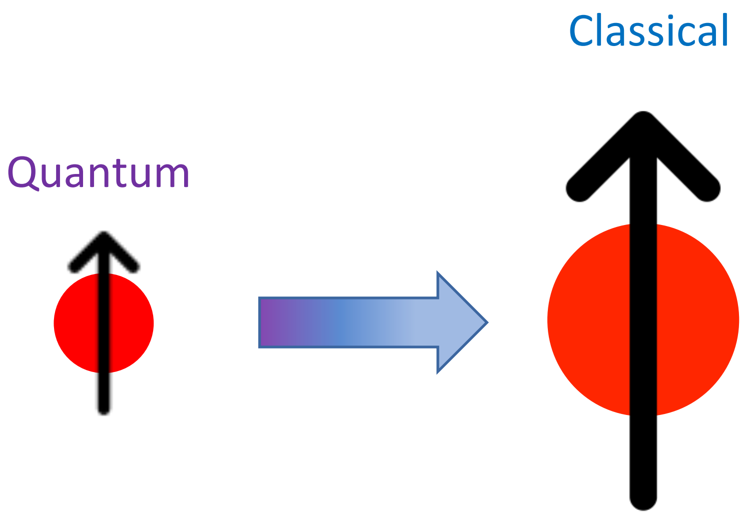
Classical



Quantum

- Classical impurity & semi-classical bulk
- Classical impurity & quantum bulk
- Quantum impurity & quantum bulk

Classical impurity



- Take the limit $S \to \infty$ $\;JS$ constant

$$H_{\rm imp} = -JS\,\left(\psi_L^\dagger \sigma^z \psi_R\right)(0)$$

- Bosonize the SC

$$H_0 + H_{\rm int} = \underbrace{\int dx\, \frac{1}{2}[(\partial_x\phi)^2 + (\partial_x\theta)^2] - \frac{m_0^2}{\beta^2}\cos(\beta\phi)}_{\text{Spin part}} + \underbrace{H_{\rm LL}}_{\text{charge part}}$$

$$\psi_{L(R),a} = \frac{e^{-i\sqrt{\pi}(\theta_a \pm \phi_a)}}{\sqrt{2\pi\delta}}$$

$$\frac{m_0^2}{\beta^2} \sim \frac{g_\perp}{(\pi\delta)^2}$$

- Chiral rotation removes impurity but changes b.c.

$$\phi(-L/2) = \frac{2\pi}{\beta}n \quad \phi(0) = \frac{2JS}{\beta}$$

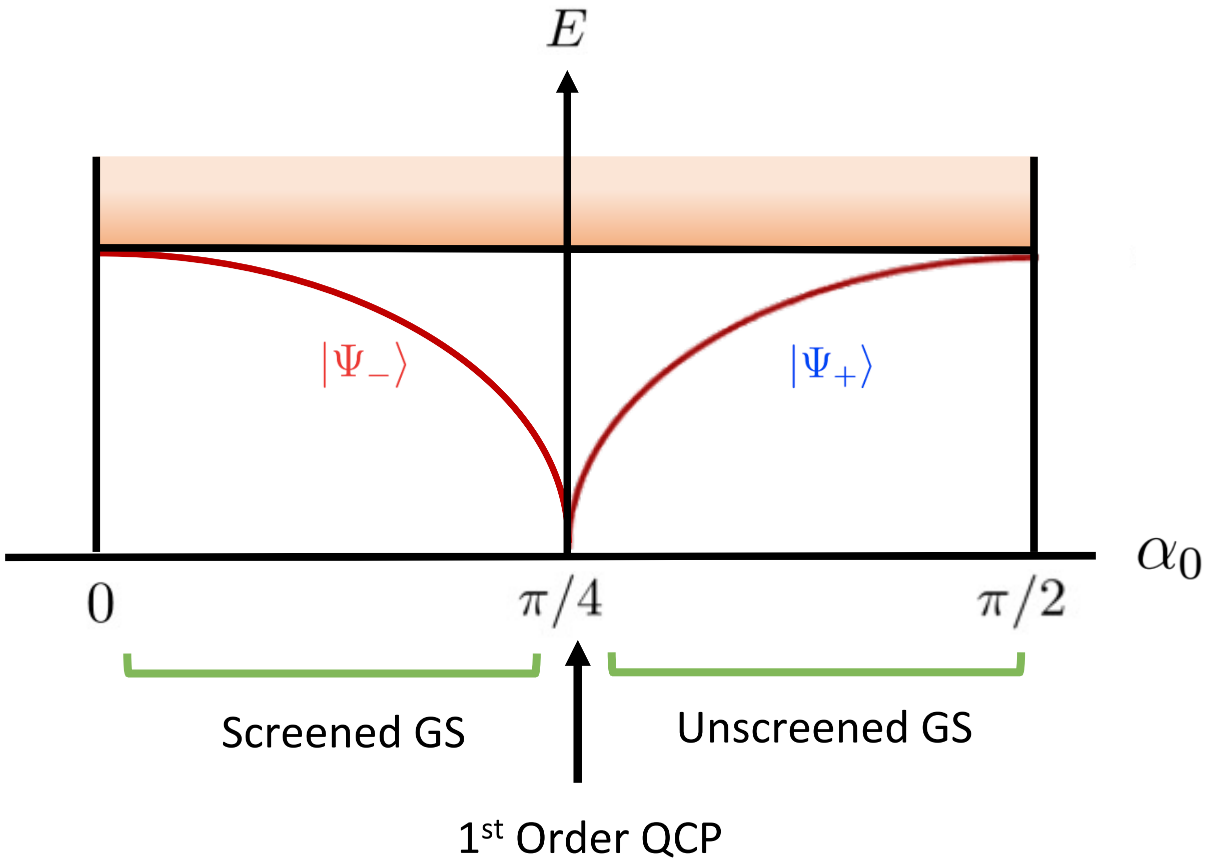
Semiclassical bulk (classical impurity)

- Take the limit $\beta \to 0$

$$\int dx\, \frac{1}{2}[(\partial_x\phi)^2 + (\partial_x\theta)^2] - \frac{m_0^2}{\beta^2}\cos(\beta\phi)$$

$$\phi(-L/2) = \frac{2\pi}{\beta}n \quad \phi(0) = \frac{2JS}{\beta}$$

SG model: Spectrum has kinks and anti-kinks



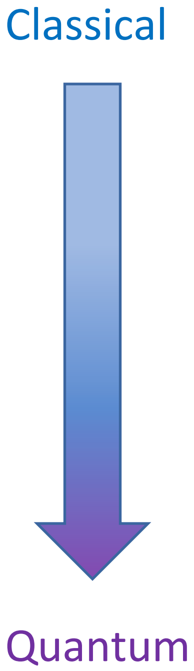
- Ground state/YSR state:

$|\Psi_+\rangle$ kink $\quad$ $|\Psi_-\rangle$ anti-kink

$$E_+ = \frac{4m_0}{\beta^2}(1 - \sin(2\alpha_0)) \quad E_- = \frac{4m_0}{\beta^2}(1 + \sin(2\alpha_0))$$

$$2\alpha_0 = \pi/2 - JS$$

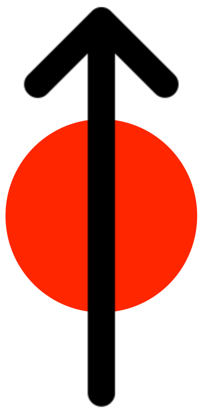
- Same results as for BCS SC: YSR state everywhere



- Classical impurity & semiclassical bulk
- Classical impurity & quantum bulk
- Quantum impurity & quantum bulk

Quantum bulk (classical impurity)

Classical



- Take the "opposite" limit $\beta^2 = 4\pi$, restore bulk fluctuations

$$H = \int dx\, \frac{1}{2}[(\partial_x\phi)^2 + (\partial_x\theta)^2] - \frac{m_0^2}{\beta^2}\cos(\beta\phi)$$

$$-\int_{-L/2}^{0} dx\, \Psi^\dagger(x)(i\sigma^z\partial_x + m\sigma^y)\Psi(x)$$

refermionize

$$\Psi_{L(R)} = \frac{e^{\mp i\sqrt{4\pi}\phi_{L(R)}}}{\sqrt{2\pi\delta}}$$

- Get Dirac equation with twisted b.c.

$$\Psi_R(-L/2) = \Psi_L(-L/2) \qquad \Psi_R(0) = -e^{i4\alpha_0}\Psi_L(0)$$

Luther & Emery, PRL (1974)

Quantum bulk (classical impurity)

- Dirac wave function with twisted b.c.

$$\chi_\lambda(x) = A_\lambda \begin{pmatrix} -ie^{-\pi\lambda/2} \\ e^{\pi\lambda/2} \end{pmatrix} e^{im\sinh(\pi\lambda)x} + A_{-\lambda} \begin{pmatrix} -ie^{\pi\lambda/2} \\ e^{-\pi\lambda/2} \end{pmatrix} e^{-im\sinh(\pi\lambda)x}$$

  - Energy $\epsilon(\lambda) = -m\cosh(\pi\lambda)$, λ rapidity parameter

- Boundary conditions lead to equations

$$e^{im\sinh(\pi\lambda)L} = \frac{\cosh\frac{\pi}{2}(\lambda - i(\frac{1}{2}-a))\cosh\frac{\pi}{2}(\lambda-\frac{i}{2})}{\cosh\frac{\pi}{2}(\lambda + i(\frac{1}{2}-a))\cosh\frac{\pi}{2}(\lambda+\frac{i}{2})}$$

  - Ground state: fill all negative energy modes
  - Excitations: $\lambda \to \lambda + i$ gapped

Boundary phase shift

$$A_\lambda/A_{-\lambda} = -\frac{\cosh\frac{\pi}{2}(\lambda - i(\frac{1}{2}-a))}{\cosh\frac{\pi}{2}(\lambda + i(\frac{1}{2}-a))}$$

$$a = 2\alpha_0/\pi$$

Pole/zero:

$$\lambda_{\rm bs} = \pm i(a - 1/2)$$

Quantum bulk (classical impurity)

- Pole indicates bound state - YSR

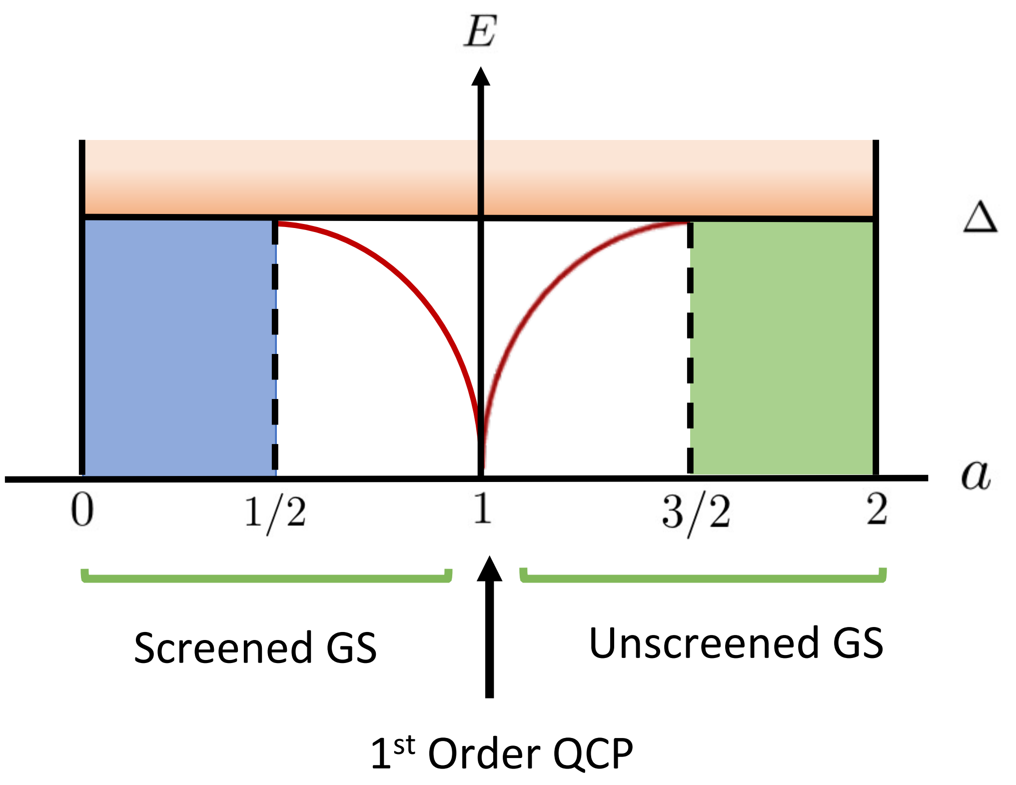
$$\chi_{\mathrm{b.s}}(x) = \xi_a^{-1/2} \begin{pmatrix} -e^{-i\pi a/2} \\ e^{i\pi a/2} \end{pmatrix} e^{x/\xi_a}$$

$$\xi_a = \left(-m\cos\left(\pi a\right)\right)^{-1}$$

- Only normalizable for $\frac{1}{2} < a < \frac{3}{2}$
- Energy is below gap

$$\epsilon = -m\sin\left(\pi a\right)$$

  - Include in GS when negative



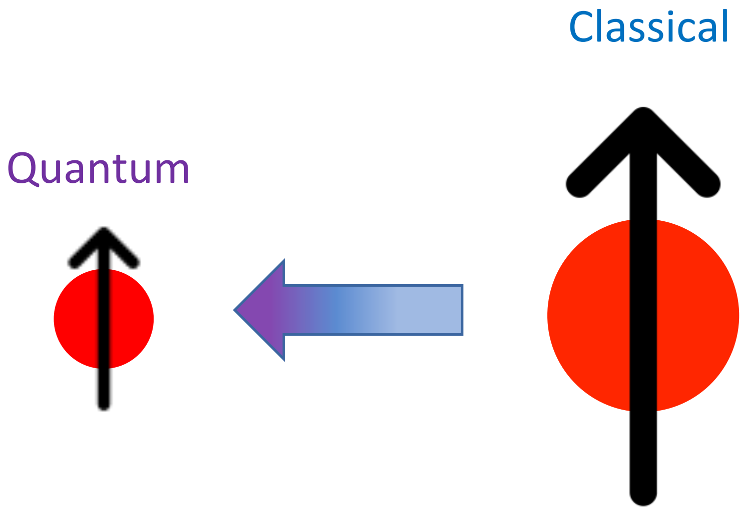
YSR region shrinks

Classical

Quantum

- Classical impurity & semiclassical bulk
- Classical impurity & quantum bulk
- Quantum impurity & quantum bulk

The Quantum model



- The model

$$H = \int_{-\frac{L}{2}}^{0} dx\, i(\psi^\dagger_{La}\partial_x\psi_{La} - \psi^\dagger_{Ra}\partial_x\psi_{Ra}) - 2g\psi^\dagger_{Ra}\psi^\dagger_{Lc}\left[\vec{\sigma}_{ab}\cdot\vec{\sigma}_{cd}\right]\psi_{Rb}\psi_{Ld}$$
$$-J\left[\vec{S}_{\alpha\beta}\cdot\vec{\sigma}_{ab}\right]\psi^\dagger_{La}(0)\psi_{Rb}(0)$$

$J_\perp = J_\parallel = J$

$g_\perp = g_\parallel = g$

- Model is integrable via nested Bethe Ansatz for arbitrary g & J

Pasnoori, Rylands & Andrei, PRR (2020)

Quantum model: Bethe states

- The Bethe states – divide configuration space to to (N+1) regions

$$|\{k_j\}\rangle = \sum_{\substack{\{a_j\},\{\sigma_j\} \\ Q}} \int \mathrm{d}\vec{x}\, F^{\{\sigma\}}_{\{a\}}(\vec{x}) \prod_{j=1}^{N} \psi^\dagger_{\sigma_j,a_j}(x_j)\,|0\rangle \otimes |a_0\rangle$$

with the wave function in region Q: $F^{\{\sigma\}}_{\{a\}}(\vec{x}) = \theta(x_Q) A^{\{\sigma\}}_{\{a\}}[Q] e^{i\sum_j^N \sigma_j k_j x_j}$

- Imposing open b.c. - the amplitude $A^{\{\sigma\}}_{\{a\}}[Q]$ satisfies:

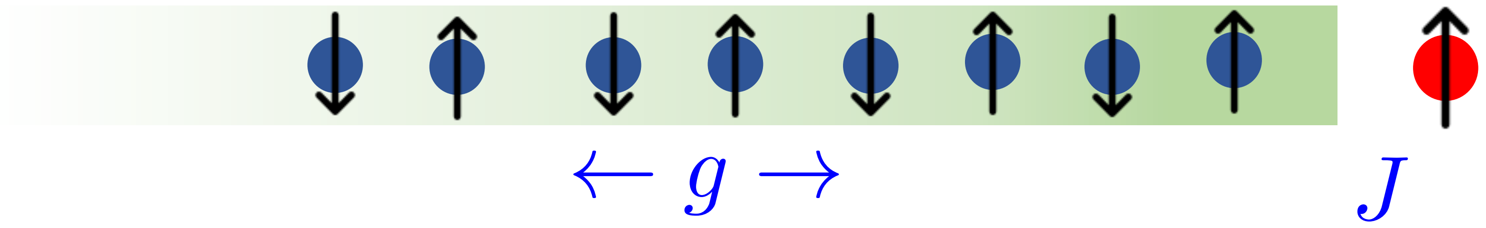
$$e^{-ik_jL} A_{\{a\}}[\mathbb{1}] = \mathrm{Tr}\left[\mathcal{T}(0)\right]^{\{b\}}_{\{a\}} A_{\{b\}}[\mathbb{1}]$$

- The monodromy matrix (from one edge to another)

$$\mathcal{T}(0) = R_{1\tau}(\lambda+b)...R_{N\tau}(\lambda+b)R_{0\tau}(\lambda+d)R_{0\tau}(\lambda-d)R_{N\tau}(\lambda-b)...R_{1\tau}(\lambda-b)$$

- $R_{ij}(\lambda)$ are the particle-particle scattering matrices

Quantum model: Bethe equations
- open boundary conditions



- Model integrable for arbitrary *g* and *J*
- *g* and *J* flow under RG
- *d* is RG invariant

$$\boxed{d = \sqrt{b^2 - \frac{b(1 - 3J^2/4)}{J} - 1}}$$

$$b = (1 - g^2/4)/2g$$

bulk parameter

The BA equations:

$$e^{-ik_jL} = \prod_{\alpha=1}^{M} \left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)\left(\frac{\lambda_\alpha - b - i/2}{\lambda_\alpha - b + i/2}\right)$$

$$\left(\frac{\lambda_\alpha - b + i/2}{\lambda_\alpha - b - i/2}\right)^N \left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)^N \left(\frac{\lambda_\alpha - d + i/2}{\lambda_\alpha - d - i/2}\right)\left(\frac{\lambda_\alpha + d + i/2}{\lambda_\alpha + d - i/2}\right)$$

$$= \prod_{\alpha\neq\beta}^{M} \left(\frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}\right)\left(\frac{\lambda_\alpha + \lambda_\beta + i}{\lambda_\alpha + \lambda_\beta - i}\right)$$

The energy:

$$E = \sum_{j=1}^{N} k_j$$

Spin charge separation
- Gapless holons
- gapful spinons

Quantum model: Bethe equations – the bulk terms

$$e^{-ik_jL} = \prod_{\alpha=1}^{M} \boxed{\left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)\left(\frac{\lambda_\alpha - b - i/2}{\lambda_\alpha - b + i/2}\right)}$$

$$\boxed{\left(\frac{\lambda_\alpha - b + i/2}{\lambda_\alpha - b - i/2}\right)^N \left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)^N} \left(\frac{\lambda_\alpha - d + i/2}{\lambda_\alpha - d - i/2}\right)\left(\frac{\lambda_\alpha + d + i/2}{\lambda_\alpha + d - i/2}\right)$$

bulk

$$= \prod_{\alpha \neq \beta}^{M} \left(\frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}\right)\left(\frac{\lambda_\alpha + \lambda_\beta + i}{\lambda_\alpha + \lambda_\beta - i}\right)$$

- RG invariants and scales

The bulk parameter

$$\boxed{b = (1 - g^2/4)/2g}$$

Bulk scale

$\Delta = De^{-\pi b}$ Bulk scale, the gap

$D = N/L$ - cutoff

Quantum model: Bethe equations –the impurity terms

$$e^{-ik_jL} = \prod_{\alpha=1}^{M} \left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)\left(\frac{\lambda_\alpha - b - i/2}{\lambda_\alpha - b + i/2}\right)$$

$$\left(\frac{\lambda_\alpha - b + i/2}{\lambda_\alpha - b - i/2}\right)^N \left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)^N \boxed{\left(\frac{\lambda_\alpha - d + i/2}{\lambda_\alpha - d - i/2}\right)\left(\frac{\lambda_\alpha + d + i/2}{\lambda_\alpha + d - i/2}\right)}$$

$$= \prod_{\alpha\neq\beta}^{M} \left(\frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}\right)\left(\frac{\lambda_\alpha + \lambda_\beta + i}{\lambda_\alpha + \lambda_\beta - i}\right)$$

- RG invariants and scales

$$\boxed{d = \sqrt{b^2 - \frac{b(1-3J^2/4)}{J} - 1}}$$

Note: the RG invariant can be real (denoted *a*) ore imaginary (denoted *d*)

Impurity scale

$\Delta e^{\pi d}$ or $\Delta e^{\pi a}$

$d \in \mathbb{R}$ $\qquad$ $d = ia$

Quantum model: Bethe equations

$$e^{-ik_jL} = \prod_{\alpha=1}^{M} \left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)\left(\frac{\lambda_\alpha - b - i/2}{\lambda_\alpha - b + i/2}\right)$$

$$\left(\frac{\lambda_\alpha - b + i/2}{\lambda_\alpha - b - i/2}\right)^N \left(\frac{\lambda_\alpha + b + i/2}{\lambda_\alpha + b - i/2}\right)^N \left(\frac{\lambda_\alpha - d + i/2}{\lambda_\alpha - d - i/2}\right)\left(\frac{\lambda_\alpha + d + i/2}{\lambda_\alpha + d - i/2}\right)$$
$$= \prod_{\alpha\neq\beta}^{M} \left(\frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}\right)\left(\frac{\lambda_\alpha + \lambda_\beta + i}{\lambda_\alpha + \lambda_\beta - i}\right)$$

- **Bulk** solutions are strings (complex pair) – gapped Spinons

$$\epsilon(\lambda) = \Delta \cosh(\pi\lambda)$$ (for all values of RG inv)

- **Boundary** spectrum: Three regimes depending on

$\boxed{d \in \mathbb{R} \text{ or } a < \frac{1}{2}}$ $\boxed{\frac{1}{2} < a < \frac{3}{2}}$ $\boxed{a > \frac{3}{2}}$

Quantum model: Solution in thermodynamic limit

- Ground state has all $\lambda \in \mathbb{R}$ distributed according to BAE

$$\sum_{\sigma=\pm} \underbrace{N\varphi_1(\lambda+\sigma b)}_{\text{bulk}} + \underbrace{\varphi_1(\lambda+\sigma d)}_{\text{impurity}} + \underbrace{\varphi_1(\lambda)+\varphi_2(\lambda)}_{\text{boundary}} = \rho(\lambda) + \int_{-\infty}^{\infty} \mathrm{d}\mu\, \varphi_2(\lambda-\mu)\rho(\mu)$$

$$\varphi_n(x) = \frac{1}{\pi}\frac{n/2}{(n/2)^2+x^2}$$

The density: $$\rho(\lambda) = \rho_{\text{bulk}}(\lambda) + \rho_{\text{imp}}(\lambda) + \rho_{\text{bdry}}(\lambda)$$

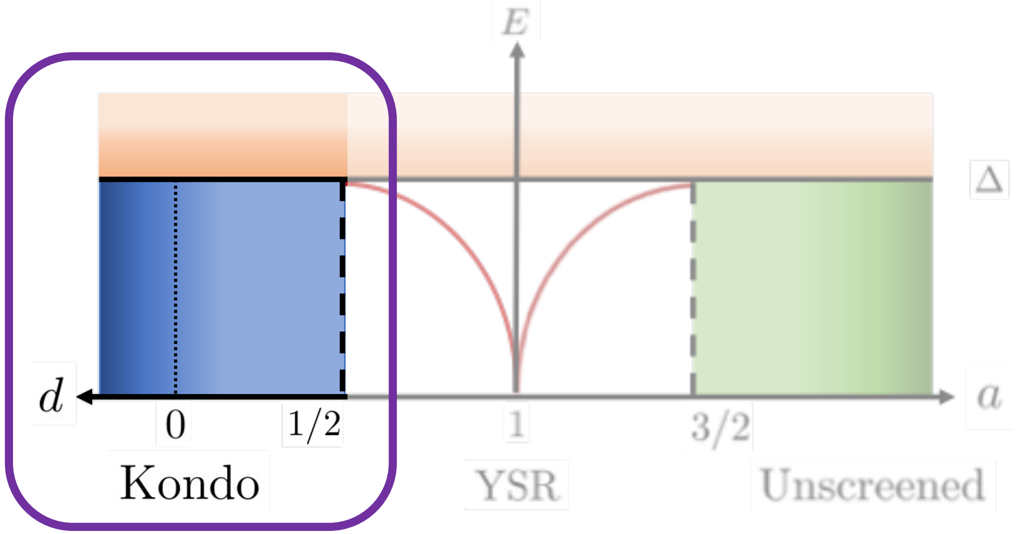
- Solve for $\rho(\lambda)$ by Fourier Transform

Quantum model: Kondo Regime $d \in \mathbb{R} \quad a < \frac{1}{2}$

- The total spin $S_z = N - 2\int d\lambda \rho(\lambda) = 0 \quad \longrightarrow$ many body singlet
- Density of states $\rho(E) = \left|\frac{\rho(\lambda)}{p'(\lambda)}\right|$

$$\rho_{\text{bulk}}(E) = \frac{L}{2}\frac{E}{\pi\sqrt{E^2 - \Delta^2}}$$

$$\frac{L}{2}\frac{\rho_{\text{imp}}(E)}{\rho_{\text{bulk}}(E)} = \frac{\Delta \cosh \pi d}{(E^2 + \Delta^2 \sinh^2 \pi d)}$$



- Defines a renormalized Kondo scale

$$T_K = \Delta\sqrt{1 + \cosh^2 \pi d} \underset{J \gg g}{\longrightarrow} De^{-\pi/2J}$$

- Smooth crossover from many-body to single particle screening

$\Delta = T_K$ when $a = 1/2$

Quantum model: YSR regime $\frac{1}{2} < a < \frac{3}{2}$

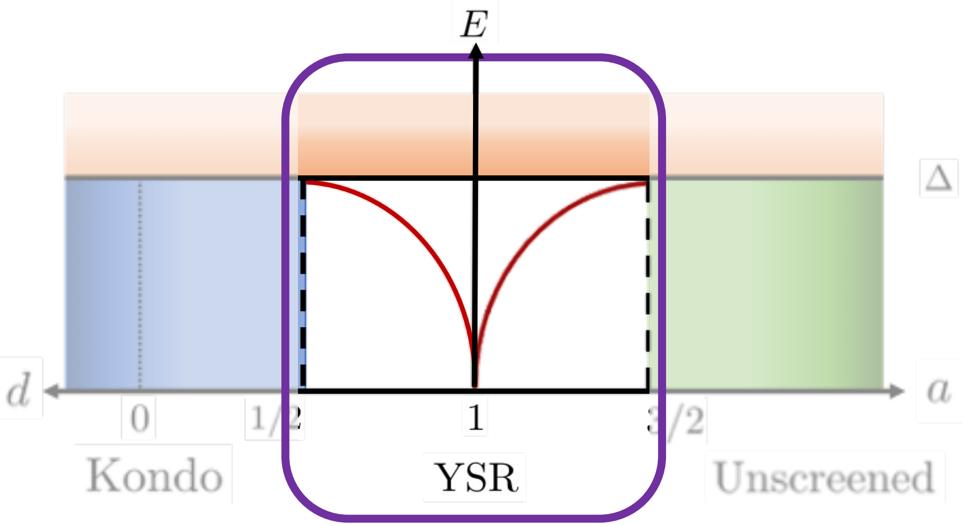
There exists three states with spin $S^z = 0$, $S^z = \pm 1/2$ corresponding to impurity being screened and unscreened respectively.

The impurity is screened by a bound state which is described by the close boundary string

$$\lambda_{\text{bs}} = \pm i(a - 1/2)$$

Energy of the bound state is below the gap:

$$\epsilon = -\Delta \sin(\pi a)$$



First order QPT at $a = 1$

$a < 1$

- Ground state has λ_{bs} $S^z = 0$
  - impurity screened
- YSR state has no λ_{bs} $S^z = \pm 1/2$

$a > 1$

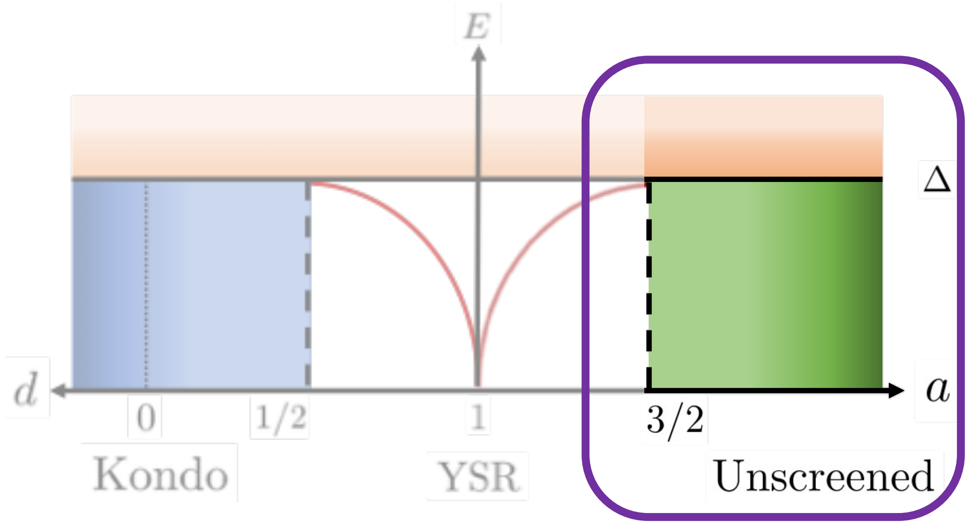
- Ground state has no λ_{bs} $S^z = \pm 1/2$ impurity not screened
- YSR state has λ_{bs} $S^z = 0$

Quantum model: Unscreened regime

- For $a > \frac{3}{2}$ have "wide" boundary string

$$\lambda_{\rm bs} = \pm i(a - 1/2)$$

- Only a solution in presence of excitations
  - Not a bound state, flips spin of impurity



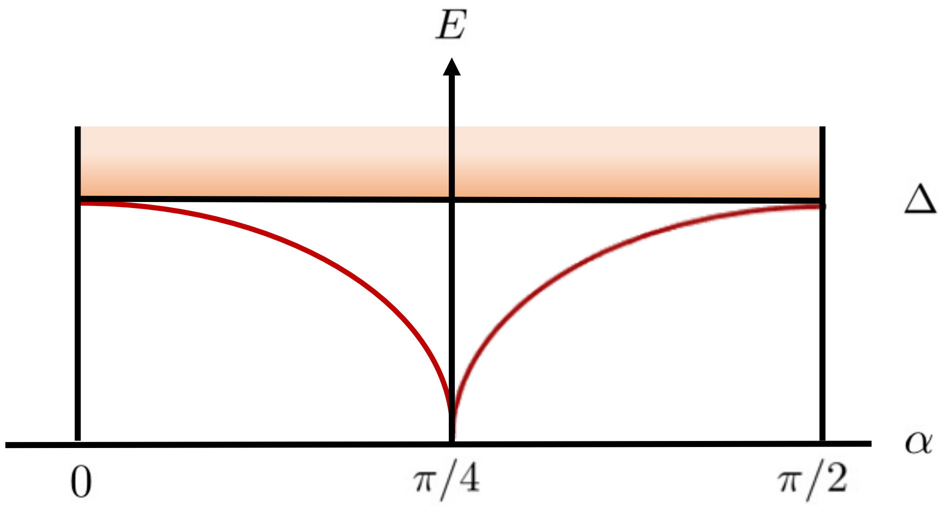
- Ground state is doubly degenerate

$$|{\rm GS}\rangle \equiv \{|+1/2\rangle, |-1/2\rangle\}$$

- No YSR states, still some residual scale $\Delta e^{\pi a}$

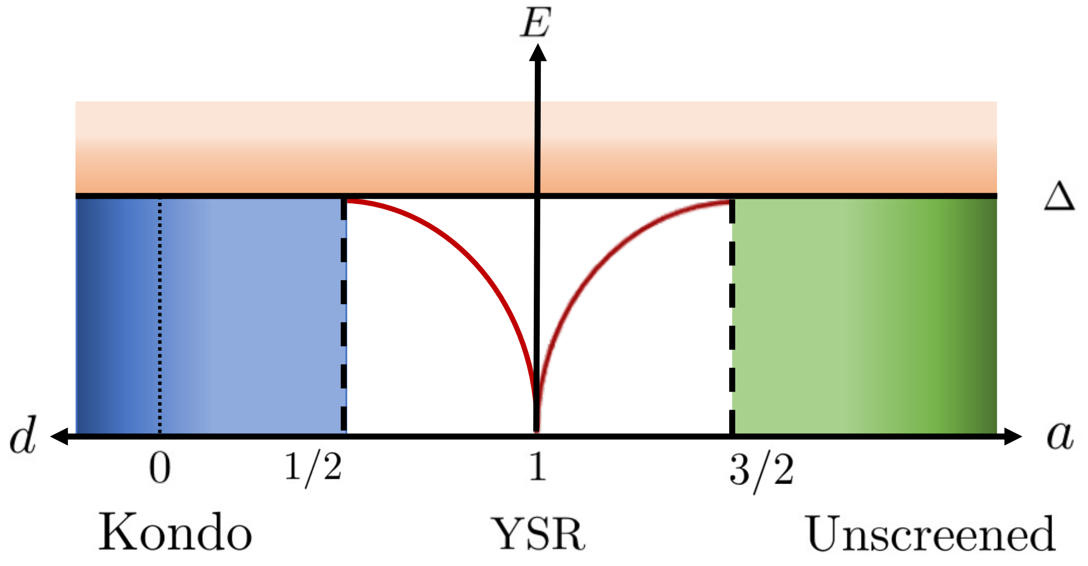
In Sum: Kondo effect in a 1-d superconductor

Semi-classical vs. Quantum



screened unscreened

Quantum fluctuations



Semi-classical description

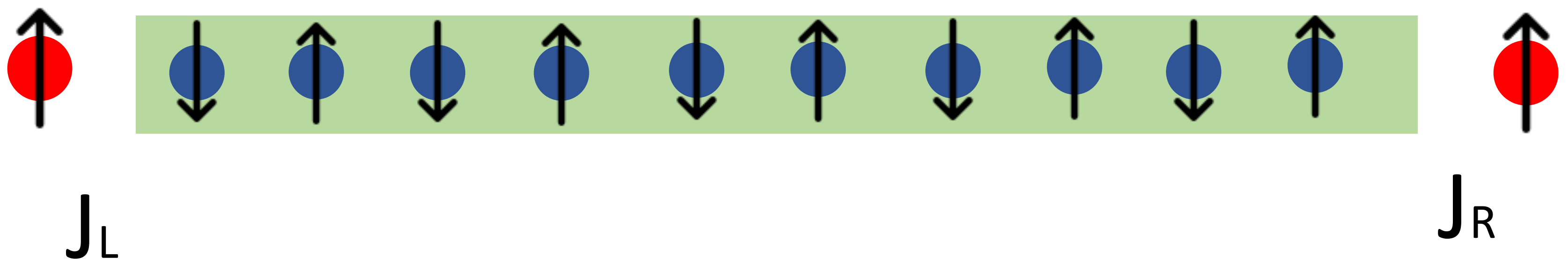
- Two phases
  - Screened
  - Unscreened

Quantum description

- Three regimes
  - Kondo
  - YSR
  - Unscreened

Main feature: Quantum fluctuations destroy YSR over most of the phase diagram

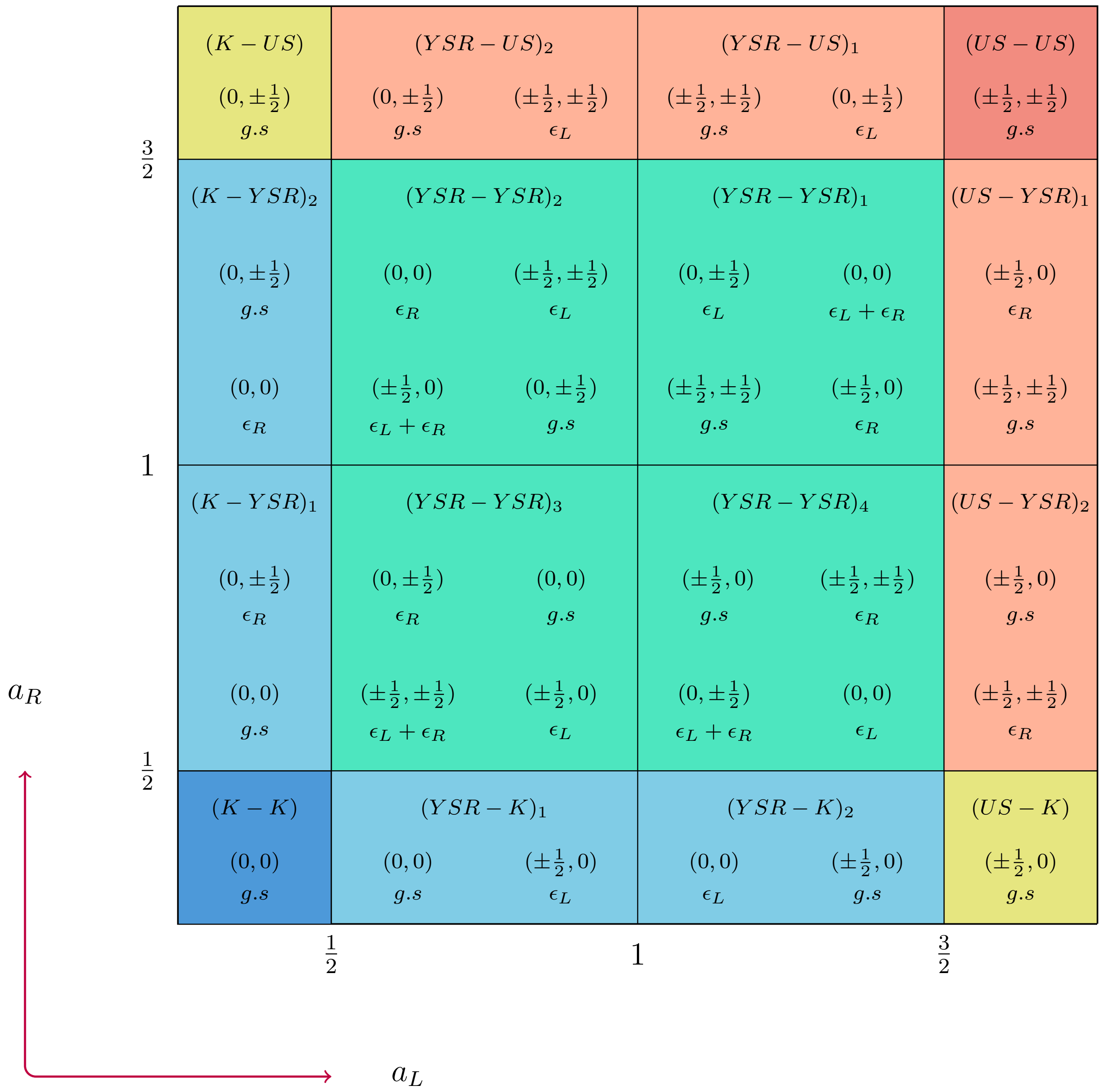
Two impurities



- Model integrable for arbitrary values of bulk and impurity coupling strengths: three free parameters.
- Two RG invariants corresponding to two impurities - a_L, a_R

Two impurity model phase diagram:

Several regimes depending on the coupling strengths of the two impurities



- Symmetry Protected Topological phase: Entanglement of the edges and emergence of boundary SUSY
- Reduction to Majorana modes by applying strong magnetic fields at the edges
- At the point $a_R = a_L = 1$, states form irreps of $spl(2,1) \otimes spl(2,1)$ SUSY algebra

YSR-YSR regime (green squares) - work in progress

- For $\frac{1}{2} < a < \frac{3}{2}$ have extra "close" boundary string solution $\lambda_{\mathrm{bs}} = \pm i(a - 1/2)$

$a < 1$		
Ground state has	λ_{bs}	$S^z = 0$
YSR state has no	λ_{bs}	$S^z = \pm 1/2$

$a > 1$		
Ground state has no	λ_{bs}	$S^z = \pm 1/2$
YSR state has	λ_{bs}	$S^z = 0$

Total of nine states exist: $(\pm\frac{1}{2}, \pm\frac{1}{2})\,, \quad (\pm\frac{1}{2}, 0)\,, \quad (0, \pm\frac{1}{2})\,, \quad (0,0)$

- Introduce fermionic operators $c^\dagger_{Lm}, c^\dagger_{Rm}$ creating bound states at the left and right edges.

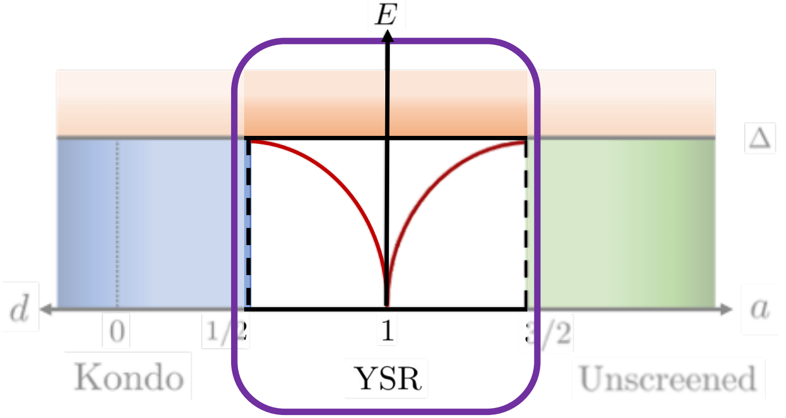
- This allows us to define the operators:

$$X_{\alpha,0\uparrow} = (1 - n_{\alpha,\downarrow})c_{\alpha,\uparrow}, \quad X_{\alpha,0\downarrow} = (1 - n_{\alpha,\uparrow})c_{\alpha,\downarrow}, \quad \alpha = L, R$$

$$S^+_\alpha = S^x_\alpha + iS^y_\alpha = X_{\alpha,\uparrow\downarrow}, \quad S^-_\alpha = S^x_\alpha - iS^y_\alpha = X_{\alpha,\downarrow\uparrow}$$

$$S^z = \frac{1}{2}\left(X^{\uparrow\uparrow} - X^{\downarrow\downarrow}\right), \quad S^0_\alpha = \frac{1}{2}\left(1 + X^{00}\right)$$

with $$S^a_\alpha = \sum_{m,n} c^\dagger_{\alpha m}\sigma^a_{mn}c_{\alpha n}, \quad a = x, y, z, 0 \quad \alpha = L, R$$



YSR-YSR regime

Nine states in the YSR-YSR regime:

$(\pm\frac{1}{2},\pm\frac{1}{2})\,, \quad (\pm\frac{1}{2},0)\,, \quad (0,\pm\frac{1}{2})\,, \quad (0,0)$

At each edge we have the following states:

$|1,0,0\rangle$: impurity is screened

$|\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}\rangle$: impurity is unscreened

Irreducible representations:
Total fermionic parity is even or odd corresponding to total baryon number being integer or half odd integer

Both impurities screened: $|2,0,0\rangle = -|1,0,0\rangle \otimes |1,0,0\rangle$

Both impurities unscreened: $|1,1,\pm1\rangle = |\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}\rangle$

$$|1,1,0\rangle, |1,0,0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle \pm |\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle\right)$$

One impurity is screened and the other is unscreened:

$$|\frac{3}{2},\frac{1}{2},\pm\frac{1}{2}\rangle_{\pm} = \frac{1}{\sqrt{2}}\left(|1,0,0\rangle \otimes |\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}\rangle \pm |\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}\rangle \otimes |1,0,0\rangle\right)$$

Irreducible representation: $[b,q]$. b is the baryon number and q is the spin. It may contain multiplets

$|b,q,q_3\rangle, \quad |b+\frac{1}{2},q-\frac{1}{2},q_3\rangle \quad |b-\frac{1}{2},q-\frac{1}{2},q_3\rangle, \quad |b,q-1,q_3\rangle$

$q_3 = -q\ldots q$ is the spin S^z

Odd operators: $V_{\pm}, W_{\pm}$ increase and decrease the baryon number by $\frac{1}{2}$ resp. $\pm$ operators increase and decrease spin S^z by $\frac{1}{2}$.

Even operators: $Q_{\pm}, Q_3$ are the usual spin operators.

Irrep of $[\frac{1}{2},\frac{1}{2}] \otimes [\frac{1}{2},\frac{1}{2}]$ is $[1,1] \oplus [\frac{3}{2},\frac{1}{2}]$

$[\frac{1}{2},\frac{1}{2}]$ contains the multiplets $|1,0,0\rangle, |\frac{1}{2},\frac{1}{2},m\rangle$

$[1,1]$ contains the multiplets $|1,1,m\rangle, |\frac{3}{2},\frac{1}{2},m\rangle$

$[\frac{3}{2},\frac{1}{2}]$ contains the multiplets $|\frac{3}{2},\frac{1}{2},m\rangle, |2,0,0\rangle, |1,0,0\rangle$

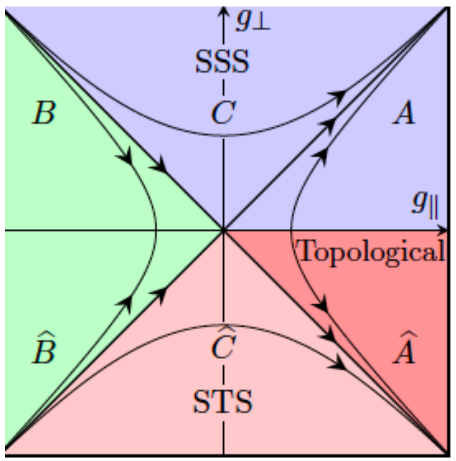
The two irreps correspond to space parity +1, -1 resp.

Reduce edge symmetry:

- Apply a magnetic field to the impurities at the edge

- Obtain SC (in the SSS - Spin Singlet SC phase) with magnetic fields at the edges

$$\mathcal{H}_{int} = -\psi_R^\dagger \psi_L^\dagger \left[g_\perp (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) + g_\parallel \sigma^z \otimes \sigma^z \right] \psi_L \psi_R$$



- Fractionalized $S^z = 1/4$ edge modes will appear

  - The SSS SC with twisted BC at the edge $B \sim \sigma^z$

$$\Psi_{Ra}(+L/2) = -B_{ab}\Psi_{Lb}(+L/2)$$
$$\Psi_{Ra}(-L/2) = -\Psi_{Lb}(-L/2)$$

is dual to the SC in the STS (Spin Triplet SC) phase

$$\widehat{\psi}_L = \psi_L, \; \widehat{\psi}_R = i\sigma^z \psi_R \quad H(\psi, g_\parallel, g_\perp) = H(\widehat{\psi}, g_\parallel, -g_\perp)$$

SSS –Hamiltonian *STS - Hamiltonian*

- The STS Hamiltonian is topological, with fractionalized spin edge modes and Majorana Algebra *Keselman, Berg '15 –'18,* Pasnoori, Andrei, Azaria '20

Breaking the symmetries at the edges

The Hilbert space breaks into four towers. Each labelled by local fermionic parity quantum numbers

$$\mathcal{P}_{\mathcal{L},\mathcal{R}} \equiv \sigma^z_{\mathcal{L},\mathcal{R}} = 2a^\dagger_{\mathcal{L},\mathcal{R}} a_{\mathcal{L},\mathcal{R}} - 1.$$

States	$\lvert -\frac{1}{2}\rangle$	$\lvert 0\rangle_{\epsilon'_{\mathcal{L}}}$	$\lvert 0\rangle_{\epsilon'_{\mathcal{R}}}$	$\lvert -\frac{1}{2}\rangle$
$(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{R}})$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$	$(-1,-1)$
$\mathcal{P} = -\mathcal{P}_{\mathcal{L}}\mathcal{P}_{\mathcal{R}}$	-1	$+1$	$+1$	-1

Local fractional spin quantum numbers exist: $S^z_{\mathcal{L},\mathcal{R}} = \frac{1}{4}\sigma^z_{\mathcal{L},\mathcal{R}}$

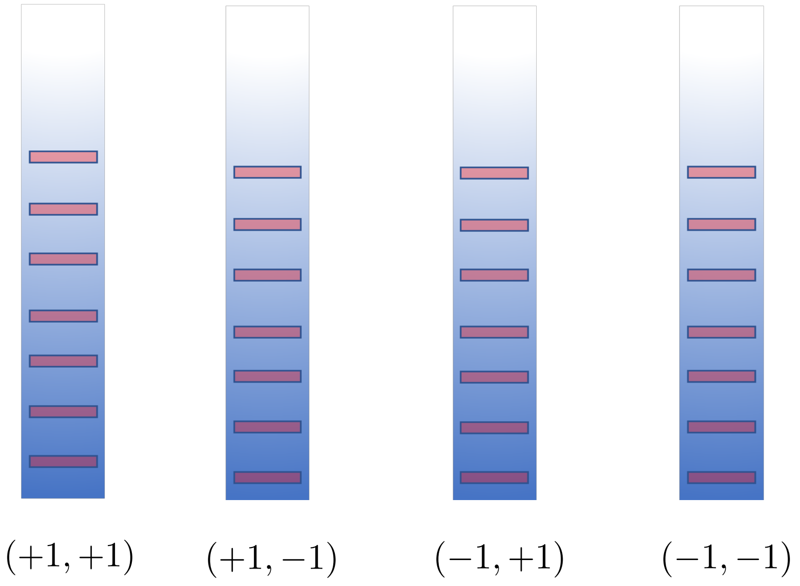
$$\begin{aligned}
\lvert \pm \frac{1}{2}\rangle &= \lvert \pm 1/4\rangle_{\mathcal{L}} \otimes \lvert \pm 1/4\rangle_{\mathcal{R}}, \\
\lvert 0\rangle_{\epsilon'_{\mathcal{R}}} &= \lvert -1/4\rangle_{\mathcal{L}} \otimes \lvert +1/4\rangle_{\mathcal{R}}, \\
\lvert 0\rangle_{\epsilon'_{\mathcal{L}}} &= \lvert +1/4\rangle_{\mathcal{L}} \otimes \lvert -1/4\rangle_{\mathcal{R}}.
\end{aligned}$$

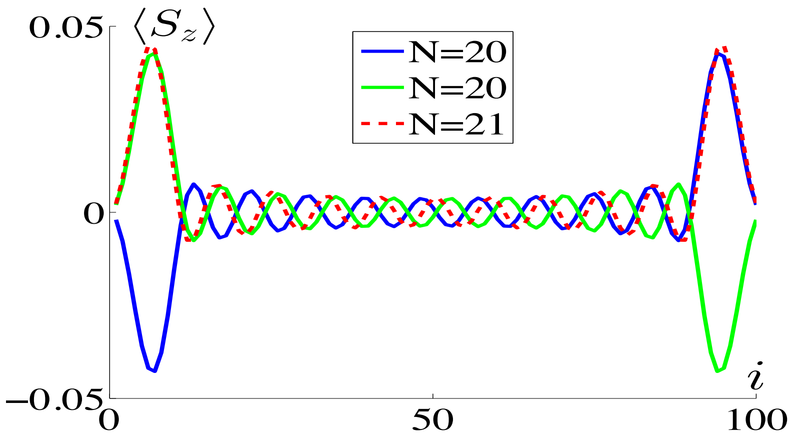
Fractionalization of the $\mathbb{Z}_2$ symmetry group and Majorana ZEM:

$$\mathbb{Z}_2 = \mathbb{Z}_{2,\mathcal{L}} \otimes \mathbb{Z}_{2,\mathcal{R}} \qquad \mathbb{Z}_{2,\mathcal{L},\mathcal{R}} = \{1, \sigma^x_{\mathcal{L},\mathcal{R}}\}$$

$$\sigma^y_{\mathcal{L},\mathcal{R}} = -i(a^\dagger_{\mathcal{L},\mathcal{R}} - a_{\mathcal{L},\mathcal{R}}) \quad \sigma^x_{\mathcal{L},\mathcal{R}} = (a^\dagger_{\mathcal{L},\mathcal{R}} + a_{\mathcal{L},\mathcal{R}})$$

$$\{\sigma^\mu_r, \sigma^\nu_{r'}\} = 2\delta_{rr'}\delta^{\mu\nu}, \quad (\mu,\nu) = (x,y), \quad (r,r') = (\mathcal{L},\mathcal{R})$$





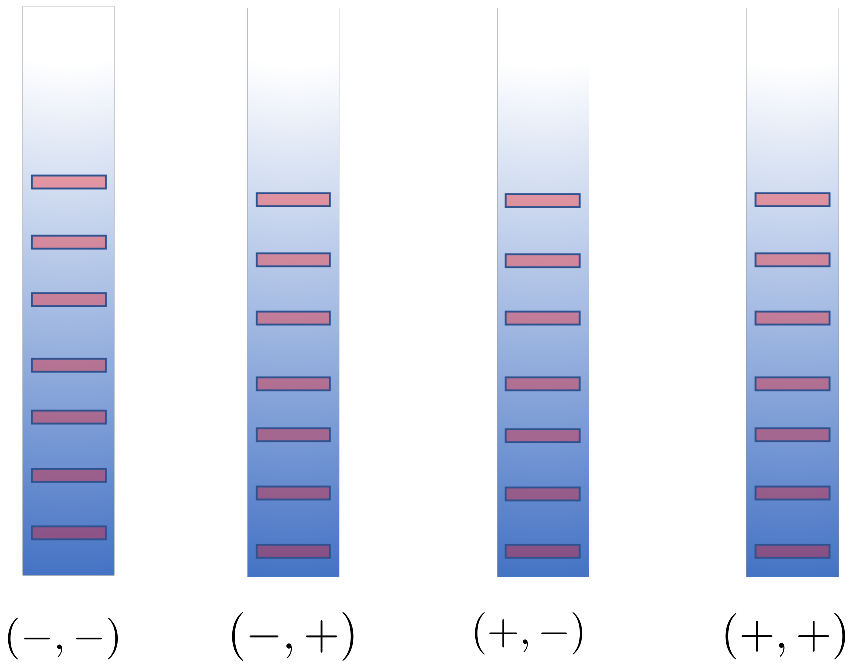
A. Keselman and E. Berg, Phys. Rev. B 91, 235309 (2015).

Majorana ZEM

- Hilbert space breaks into four towers labelled by local fermionic parity quantum numbers:

$$(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{R}}) = (\pm 1, \pm 1) \qquad \mathcal{P}_{\mathcal{L},\mathcal{R}} \equiv \sigma^z_{\mathcal{L},\mathcal{R}} = 2a^\dagger_{\mathcal{L},\mathcal{R}} a_{\mathcal{L},\mathcal{R}} - 1.$$

$$\mathcal{P} = (-1)^N = -\mathcal{P}_{\mathcal{L}} \mathcal{P}_{\mathcal{R}}$$



- Conservation of total fermionic parity and existence of local fermionic parities leads to entanglement of the edges

$$|+,\pm\rangle = \sqrt{1/2}\,(\,|+\rangle_L \otimes |-\rangle_R \;\pm\; |+\rangle_L \otimes |-\rangle_R\,)$$

$$|-,\pm\rangle = \sqrt{1/2}\,(\,|+\rangle_L \otimes |+\rangle_R \;\pm\; |-\rangle_L \otimes |-\rangle_R\,)$$

- Majorana operators change the local fermionic parities and hence map states corresponding to different towers :

$$\sigma^x_{\mathcal{L},\mathcal{R}} = (a^\dagger_{\mathcal{L},\mathcal{R}} + a_{\mathcal{L},\mathcal{R}}) \qquad \lambda_L = \sigma^z_L, \quad \lambda_R = -i\sigma^y_L \sigma^x_R$$

$$\sigma^y_{\mathcal{L},\mathcal{R}} = -i(a^\dagger_{\mathcal{L},\mathcal{R}} - a_{\mathcal{L},\mathcal{R}}) \quad \eta_L = \sigma^z_L \sigma^x_R, \quad \eta_R = \sigma^y_R$$

$$\{\sigma^\mu_r, \sigma^\nu_{r'}\} = 2\delta_{rr'}\delta^{\mu\nu}, \quad (\mu,\nu) = (x,y), \quad (r,r') = (\mathcal{L},\mathcal{R})$$

$$[\lambda_r, \eta_{r'}] = 0, \quad \{\lambda_r, \lambda_{r'}\} = \{\eta_r, \eta_{r'}\} = 2\delta_{rr'}$$

- Reduce the magnetic field, the impurity spin fluctuates – interplay Majorana and Kondo spin?

Reduce Symmetry:

- Apply a magnetic field to the impurities at the edges

- Break the SU(2) symmetry at the edges by applying strong magnetic fields to the magnetic impurities

- Effective boundary Hamiltonian : using $a^\dagger_{\mathcal{L},\mathcal{R}} \sim \psi^\dagger(x \simeq \pm L/2)$

$$\begin{aligned} h_B - E_{0,0} &= m_{\mathcal{L}} a^\dagger_{\mathcal{L}} a_{\mathcal{L}} + m_{\mathcal{R}} a^\dagger_{\mathcal{R}} a_{\mathcal{R}} \\ S^z &= \frac{1}{2}(a^\dagger_{\mathcal{L}} a_{\mathcal{L}} + a^\dagger_{\mathcal{R}} a_{\mathcal{R}} - 1). \end{aligned}$$

- Bound state operators commute with the Hamiltonian $[a_{\mathcal{L},\mathcal{R}}, H] = 0$

Two impurity model

- Quantum fluctuations at the edges lead to SPT phase: entanglement between the two edges
- Preserving SU(2) symmetry at the edges leads to enhancement of the symmetry at the boundaries: SUSY
- Breaking the SU(2) symmetry by applying strong magnetic fields reduces SUSY algebra to Clifford algebra: Majorana modes

Conclusions & Outlook

- YSR states are destroyed by quantum fluctuations
- Facilitate 1^{st} order QPT – semiclassical description

- Nature of the Kondo cloud
- Unscreened dynamical scale
- Massive Thirring – shrinking of YSR region
- Multiple impurities, Quench dynamics, other SCs

- Quantum fluctuations at the edges lead to SPT phase: entanglement between the two edges

- Preserving SU(2) symmetry at the edges leads to enhancement of the symmetry at the boundaries: SUSY

- Breaking the SU(2) symmetry by applying strong magnetic fields reduces SUSY algebra to Clifford algebra: Majorana modes