Nonlocal transport equations in inviscid flows and aggregation models: singularity formation

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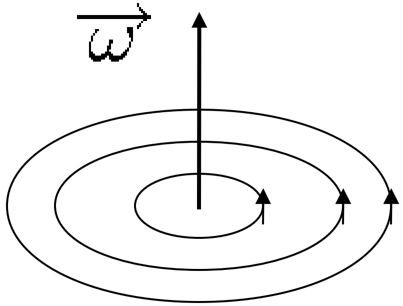
1.- Euler and Navier-Stokes equations. Models.
2.- Smoluchowski equation.
3.- Singularity formation.
4.- Selfsimilar solutions.



Euler equations (inviscid fluid, 1755) :

$$\begin{aligned} \frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v} \cdot \overrightarrow{\nabla})\overrightarrow{v} &= -\overrightarrow{\nabla} p \quad \text{on } \Omega \\ \overrightarrow{\nabla} \cdot \overrightarrow{v} &= 0 \quad \text{on } \Omega \end{aligned}$$

Vorticity :

$$\overrightarrow{\omega} = \nabla \times \overrightarrow{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}$$

Vorticity equation :

$$\begin{cases} \dfrac{\partial \overrightarrow{\omega}}{\partial t} + (\overrightarrow{v} \cdot \overrightarrow{\nabla})\overrightarrow{\omega} = (\overrightarrow{\omega} \cdot \overrightarrow{\nabla})\overrightarrow{v} \\ \overrightarrow{\omega}(\overrightarrow{x}, 0) = \overrightarrow{\omega}_0(\overrightarrow{x}) \end{cases}$$

Local existence (Kato 1972)

Biort-Savart's law:

$$\vec{v}(\vec{x},t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3} \times \vec{\omega}(\vec{y})d\vec{y}$$

$$\equiv \int_{\mathbb{R}^3} K(\vec{x}-\vec{y})\vec{\omega}(\vec{y})d\vec{y}$$

Then:

$$(\vec{\omega}\cdot\nabla)\,\vec{v}(\vec{x},t) = \left[\int_{\mathbb{R}^3} \nabla_{\vec{x}} K(\vec{x}-\vec{y})\vec{\omega}(\vec{y})d\vec{y}\right]\vec{\omega}$$

Euler eqn:

$$\frac{D\vec{\omega}}{Dt} = \left[\int_{\mathbb{R}^3} \nabla_{\vec{x}} K(\vec{x}-\vec{y})\vec{\omega}(\vec{y})d\vec{y}\right]\vec{\omega}$$

Singular integral operator acting on vorticity

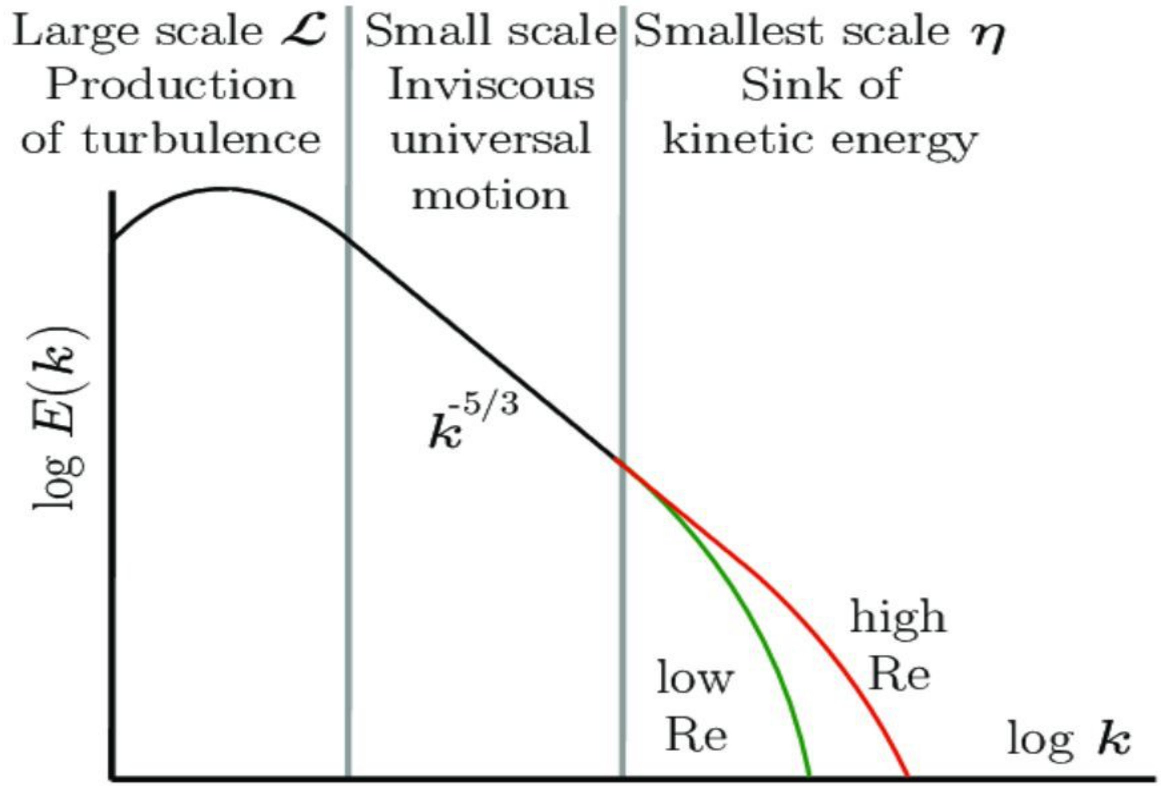
Navier-Stokes equations (viscous fluid, 1822) :

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} &= -\vec{\nabla} p + \mu \Delta \vec{v} \quad \text{on } \Omega \\ \vec{\nabla} \cdot \vec{v} &= 0 \quad \text{on } \Omega \end{aligned}$$

Vorticity equation :

$$\left\{ \begin{array}{c} \frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} = (\vec{\omega} \cdot \vec{\nabla})\vec{v} + \mu \Delta \vec{\omega} \\ \vec{\omega}(\vec{x}, 0) = \vec{\omega}_0(\vec{x}) \end{array} \right.$$

Local existence: Leray 1934



As an immediate application of these formulas, we can rederive the prediction of Onsager [24] that Hölder singularities $h \leq 1/3$ are required in the velocity field in order for energy dissipation to persist in the limit $\nu \to 0$.

The quasigeostrophic equation

Constantin, Majda, Tabak 1994

θ : Temperature field.

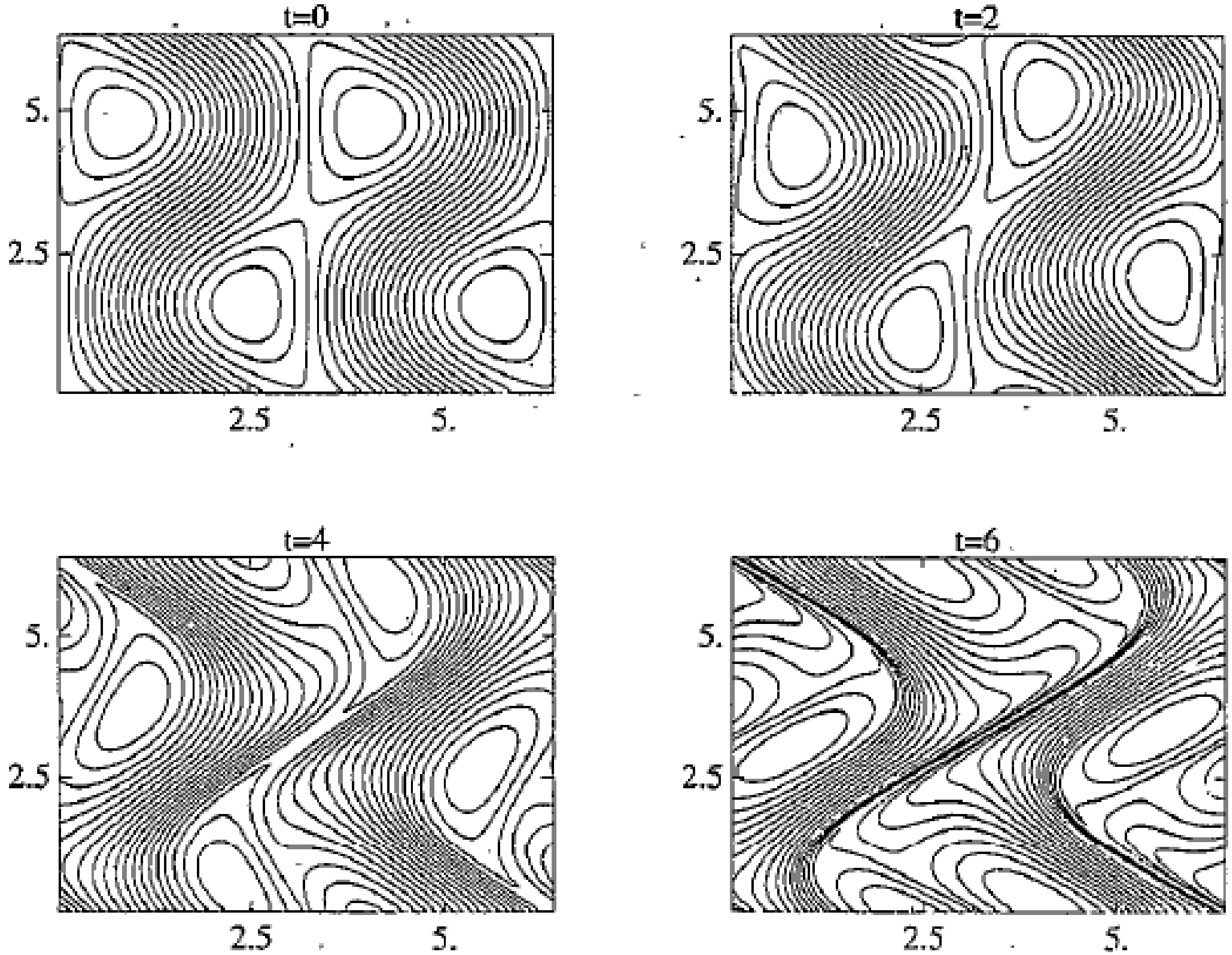
$$\begin{aligned} \frac{\partial \theta}{\partial t} + \overrightarrow{v} \cdot \nabla \theta &= 0 \\ \nabla \cdot \overrightarrow{v} &= 0 \end{aligned}$$

$$\overrightarrow{v} = (v_1, v_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right)$$

$$(-\Delta)^{\frac{1}{2}} \psi = -\theta$$

$$(-\Delta)^{\frac{1}{2}} \psi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \, |\mathbf{k}| \, \widehat{\psi}(\mathbf{k}) d\mathbf{k}$$

Level lines of θ



Is there a finite time singularity in the derivatives of θ ?

Formation of sharp fronts

$$u = -\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta = -R^{\perp}\theta,$$

$$R^{\perp}\theta = (-R_2\theta, R_1\theta)$$

$$R_j(\theta)(x,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \frac{(x_j - y_j)\theta(y,t)}{|x-y|^3} dy.$$

Q-G equation:

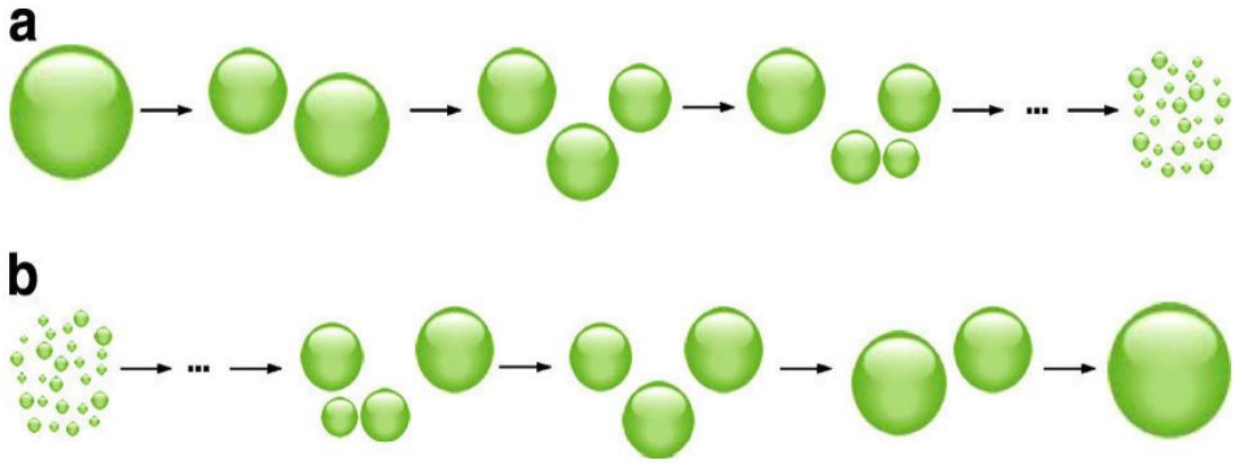
$$\begin{aligned} \theta_t - (R^{\perp}\theta)\cdot\nabla\theta &= 0 \\ \theta(x,0) &= \theta_0(x) \end{aligned}$$

Singularities?
Unknown

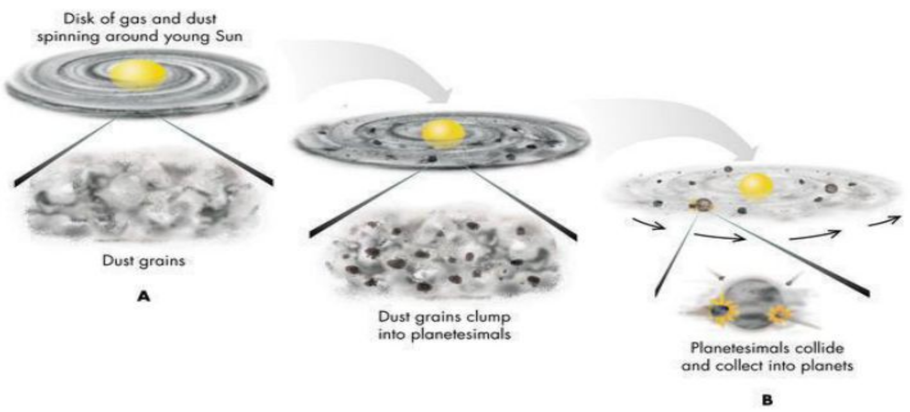
1D analog:

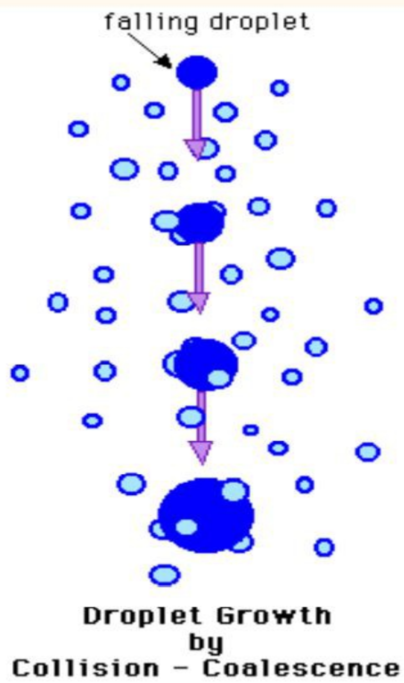
$$\begin{aligned} \theta_t - (H\theta)\theta_x &= 0 \\ \theta(x,0) &= \theta_0(x) \end{aligned}$$

Coagulation



Applications to Coagulation-fragmentation





Smoluchowski equation (1916):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{c}\,(x,t) = \boldsymbol{Q}_{\mathrm{C}}\,(\mathrm{c})$$

$$\boldsymbol{Q}_{\mathrm{C}}\,(\mathrm{c}) := \frac{1}{2}\int_0^x K\,(x-y,y)\,\mathrm{c}\,(x-y,t)\,\mathrm{c}\,(y,t)\;\mathrm{d}y \;-\mathrm{c}\,(x,t)\int_0^\infty K\,(x,y)\,\mathrm{c}\,(y,t)\;\mathrm{d}y$$

The mass depending and symmetric operator $K(x, y)$ is known as the coagulation kernel and describes the rate at which particles of the given size coagulate.

Unfortunately, most of the physically interesting models correspond to equations that are not exactly solvable.

Discrete version:

$$\frac{\partial n(x_i,t)}{\partial t} = \frac{1}{2}\sum_{j=1}^{i-1} K(x_i - x_j, x_j) n(x_i - x_j, t) n(x_j, t) - \sum_{j=1}^{\infty} K(x_i, x_j) n(x_i, t) n(x_j, t).$$

- It has been possible to discover analytical solutions exclusively for a limited number of kernels; the principal cases include: $K_c(x, y) = 1$, $K_+ (x, y) = x + y$, $K_\times (x, y) = xy$.
- For more general kernels, studies have essentially relied on numerical methods.
- Existence, positivity and uniqueness result for all time for kernels growing at most **linearly**.

INTERESTS:

- **singularities** in finite time occur for some Kernels: very strong coagulation creates infinitely dense clusters.
- **self-similar solutions**.

Laplace transform: $$\varphi(\eta,t)=\int_0^{\infty}e^{-\eta x}c(x,t)dx$$

Regularized Laplace transform: $$\Phi(\eta,t)=-\int_0^{\infty}(e^{-\eta x}-1)xc(x,t)dx$$

$$\begin{aligned}\partial_t\phi(\eta,t)&=-(\phi(\eta,t))^2, && \lambda=0;\\ \partial_t\phi(\eta,t)-\phi(\eta,t)\partial_\eta\phi(\eta,t)&=-\phi(\eta,t), && \lambda=1;\\ \partial_t\Phi(\eta,t)-\Phi(\eta,t)\partial_\eta\Phi(\eta,t)&=0, && \lambda=2.\end{aligned}$$

$$K(x,y)=(xy)^{1-\varepsilon}$$

Fractional Burgers equation: $$\partial_t\Phi(\eta,t)-D^{-\varepsilon}\Phi(\eta,t)\partial_\eta(D^{-\varepsilon}\Phi(\eta,t))=0$$

$$D^{-\varepsilon}\Phi\equiv\frac{1}{\Gamma(1-\gamma)}\int_0^{\infty}\frac{\Phi(\eta+\zeta)-\Phi(\zeta)}{\zeta^{\gamma}}d\zeta$$

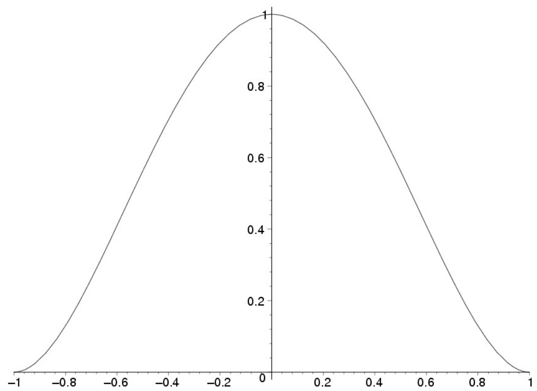
$$\varepsilon=1-\gamma$$

Singularities

Take

$$\theta_t - (H\theta)\,\theta_x = 0$$

A Córdoba, D Córdoba, MAF. Annals of mathematics, 1377-1389, 2005.



$Supp(\theta_0(x)) \subset [-L, L]$, $\theta_0(x)$ symmetric

$$Supp(\theta(x,t)) \subset [-L, L]$$
$$\max_x \theta(x,t) = \theta(0,t)$$

Theorem. $\theta(x,t)$ becomes singular in finite time.

Proof.

$$\frac{d}{dt}\left(\int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx\right) = -\int_0^L \frac{\theta_x(H\theta)}{x^{1+\delta}}dx = -\int_0^\infty \frac{(1-\theta)_x H(1-\theta)}{x^{1+\delta}}dx\ .$$

Let

Plancherel´s identity for Mellin transforms

$$-\int_0^\infty \frac{f_x(x)(Hf)(x)}{x^{1+\delta}}dx = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{A(\lambda)}B(\lambda)d\lambda \equiv I$$

where

$$A(\lambda) = -(i\lambda - \frac{1}{2} - \frac{\delta}{2})F(\lambda)$$

$$B(\lambda) = \frac{1+\cos\left((-i\lambda + \frac{3}{2} + \frac{\delta}{2})\pi\right)}{\sin\left((-i\lambda + \frac{3}{2} + \frac{\delta}{2})\pi\right)}F(\lambda)$$

$$F(\lambda) \equiv \int_0^\infty x^{i\lambda - \frac{3}{2} - \frac{\delta}{2}} f(x)dx \, .$$

so that $I = \int_{-\infty}^{\infty} M(\lambda)\,|F(\lambda)|^2\,d\lambda$ where

$$\frac{1}{C}(1+|\lambda|) \le Re\,\{M(\lambda)\} \le C(1+|\lambda|)$$

Then $$I \ge \frac{1}{2\pi C}\int_{-\infty}^{\infty} |F(\lambda)|^2\,d\lambda = \frac{1}{C}\int_0^\infty \frac{1}{x^{2+\delta}} f^2(x)dx$$

$$\ge C_\delta \int_0^\infty \frac{1}{x^{2+\delta}} f^2(x)dx$$

We prove that I is positive and bounded from below by a certain Weighed norm of f.

Since

$$\int_0^L \frac{(1-\theta)}{x^{1+\delta}} dx \le \left(\int_0^L \frac{(1-\theta)^2}{x^{2+\delta}} dx \right)^{\frac{1}{2}} \left(\int_0^L \frac{1}{x^{\delta}} dx \right)^{\frac{1}{2}}$$

$$\le \left(\frac{L^{1-\delta}}{1-\delta} \right)^{\frac{1}{2}} \left(\int_0^\infty \frac{(1-\theta)^2}{x^{2+\delta}} dx \right)^{\frac{1}{2}}$$

one can write

$$\frac{d}{dt} \int_0^L \frac{(1-\theta)}{x^{1+\delta}} dx \ge C_{L,\delta} \left(\int_0^L \frac{(1-\theta)}{x^{1+\delta}} dx \right)^2$$

The inequality

$$\int_0^L \frac{(1-\theta)}{x^{1+\delta}} dx \le \sup_x \frac{1-\theta}{x} \int_0^L \frac{dx}{x^{\delta}} \le \frac{L^{1-\delta}}{1-\delta} \sup_x |\theta_x|$$

implies $\sup_x |\theta_x|$ blows up.

$$\theta_0(x) = \begin{cases} (1-x^2)^2, & if -1 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

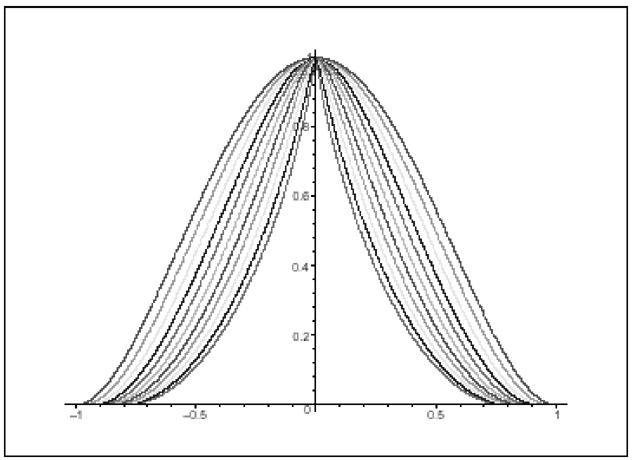


Figure 3: $\theta(x,t)$

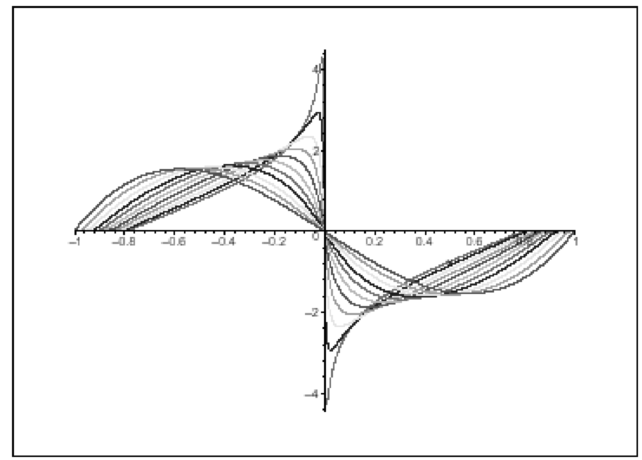


Figure 4: $\theta_x(x,t)$

$$u_t - (D^a u)\left(D^b u\right)_x = 0$$

Selfsimilar solution:

$$\begin{aligned} u(x,t) &= (t_0 - t)^\alpha f\left(\frac{x}{(t_0-t)^\beta}\right) \\ f(\eta) &\sim c_\pm |\eta|^{\frac{\alpha}{\beta}} \text{ as } \eta \to \infty \end{aligned}$$

so that

$$u(x,t) \sim c_\pm |x|^{\frac{\alpha}{\beta}} \text{ for } x = O(1)$$

By dimensional analysis

$$\alpha - 1 = 2\alpha - (a+b+1)\beta$$

that is

$$\alpha = (a+b+1)\beta - 1$$

and $f(\eta)$ solves

$$-[(a+b+1)\beta - 1]\, f + \beta \eta f_\eta - (D^a f)\left(D^b f\right)_x = 0$$

$$f(\eta) \sim c_\pm |\eta|^{(a+b+1)-\frac{1}{\beta}} \text{ as } \eta \to \infty$$

$$u(x,t_0) = c_\pm |x|^{(a+b+1)-\frac{1}{\beta}}$$

Nonlinear integrodifferential ODE with eigenvalue β to be determined by the condition at infinity (selfsimilarity of the second kind)

Nonlocal transport equation: $\theta_t = D^{(\gamma)}(\theta)\theta_x,$

MAF, Eggers, to appear in Nonlinearity 2019

(Toy model for Euler eqn.)

$$D^{(0)}(\theta) = H(\theta).$$

Existence of singularities: A. Córdoba, D. Córdoba, MAF, Ann. of Math. 2005

$$\theta = (t_0 - t)^{\alpha}\Theta\left(\frac{x}{(t_0-t)^{\beta}}\right)$$

$$\alpha - 1 = 2\alpha - (1+\gamma)\beta$$

$$\beta = \frac{1+\alpha}{1+\gamma}$$

$$\theta \sim |x|^{\frac{\alpha(1+\gamma)}{1+\alpha}}, \text{ as } |x| \to 0$$

$$\gamma = 0 \to |x|^{0.541...}$$

$$\gamma = 1 \to |x|^{\frac{4}{3}}$$

$$\alpha \approx 1.181\ldots \text{ for } \gamma = 0,$$

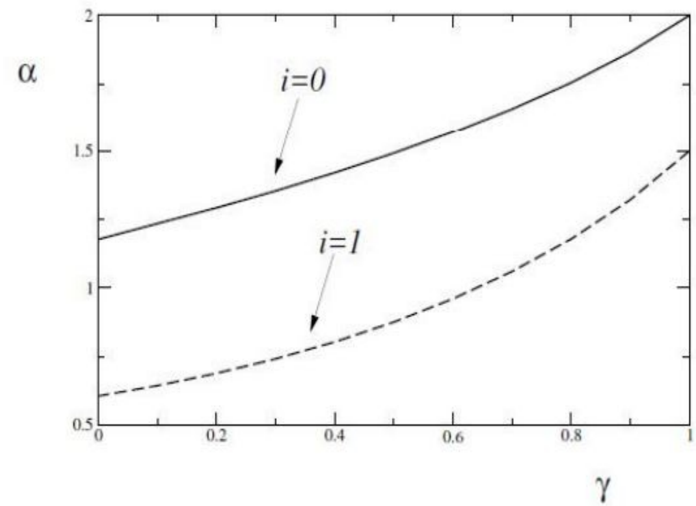


Figure 2. The exponent α as a function of γ. The ground state ($i = 0$) is shown as the solid line, the first unstable branch ($i = 1$) is the dashed line. For $\gamma = 1$, $\alpha = 2$ in the ground state, and $\alpha = 3/2$ in the first unstable state.

Selfsimilarity of the second Kind (Barenblatt)

Smoluchowski equation

J. Eggers, MAF 2022

$$\theta_t - \left(D^{\gamma-1}\theta\right)\left(D^{\gamma-1}\theta\right)_\eta = 0$$

Selfsimilar solution:

$$-\nu\Phi(\eta) + \eta\Phi'(\eta) = \frac{1}{2}\frac{dF^2}{d\eta},$$

$$F \equiv \frac{1}{\Gamma(1-\gamma)}\int_0^\infty \frac{\Phi(\eta+\zeta)-\Phi(\zeta)}{\zeta^\gamma}d\zeta$$

$$\theta(\eta, t_0) = c\,|\eta|^\nu$$

$$c(x, t_0) \sim \frac{1}{x^{2+\nu}},\ \text{as } x \to \infty$$

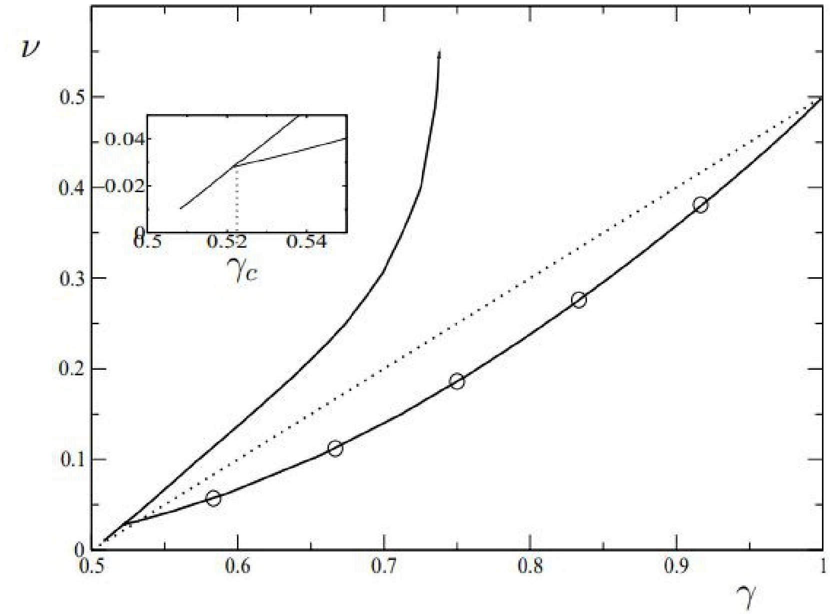


FIG. 1. The exponent ν for γ between $1/2$ and 1. One branch (the lower branch) starts from $\gamma = 1$ and ends by intersecting another branch (the upper branch) at $\gamma_c \approx 0.5214$. The scaling $\nu = \gamma - 1/2$ [14] is shown as the dotted line. The inset shows a detail of the bifurcation, where the two branches meet at γ_c. The circles mark the numerical calculations of Lee [14].

Numerical method

- The function Φ is discretized on a grid $\eta_i, i = 1 \ldots k$, where $\eta_1 = 0$ and $\Phi_i = \Phi(\eta_i)$.
- Local refinement near the origin, based on the width of the peak of Φ', while the grid spacing increases geometrically for large η.
- Integral up to η_k, we use a formula equivalent to the trapezoidal rule, but taking into account the singularity at the origin explicitly; The value of $\Phi(\eta+\zeta)$ is found by interpolating from the fixed grid.

This reduces the lower branch problem to a system of nonlinear equations defined on the grid η_i, where the k variables are the index ν, the amplitude, and the values $\Phi_i, i = 2 \ldots k-1$ of the profile.

The system of equations is solved using Newton's method, using $\eta_k = 10^{10}$ up to $\gamma \approx 0.53$; for smaller γ, the solution is extended to $\eta_k \approx 10^{70}$. To obtain a good initial guess, we start from the "ground state" solution for $\gamma = 1$ as an initial condition, and continue the solution branch in small steps of γ. The Newton iteration eventually fails to converge for $\gamma \approx 0.5217$, indicating a bifurcation toward the upper branch. For higher order branches we once again start from the exact solution for $\gamma = 1$, but choosing $j > 1$. In this case $\Phi'(0) = j/(1+j)$, which we use as the normalization for all values of γ.

Conclusions

- Classical models in fluid mechanics lead to nonlocal transport equations. Also the classical models for coagulation.
- Close relation between kinetic type models and nonlocal transport equations via integral transforms.
- Appearence of singularities where smooth initial data turn into Hölder continuous functions of nontrivial order.
- Crucial role of selfsimilar solutions.