A Non-local non-linear convection-diffusion problem on metric trees

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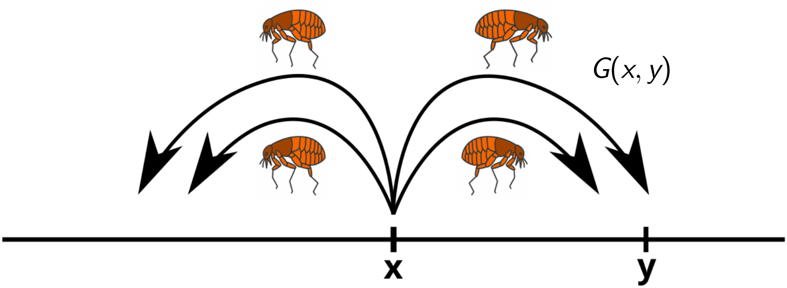
Partially supported by national Grant 0794/2020
"Spectral Methods in Hyperbolic Geometry" PN-III-P4-ID-PCE-2020-0794

Structure of the talk

1. Intuition
   - Non-local movement on the real line
   - Shrinking the movement
2. The structure of the graph we work on
3. Transport equation on tree
   - Non-local and local transport on tree
   - Convergence result for the transport problem
4. Convection-diffusion problem on tree
   - Non-local and local convection-diffusion on tree
   - Convergence result for the convection-diffusion problem

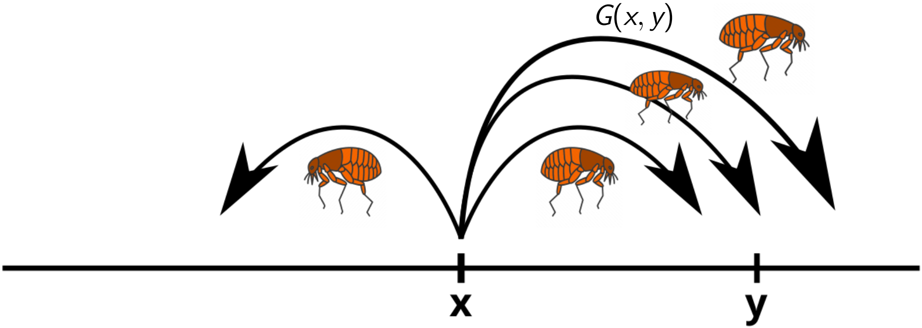
Fleas jumping on a mattress

Assuming the mattress is $\mathbb{R}$.



$G(x, y)$ accounts for the probability that the fleas will jump from x to y.

If $G(x, y) = J(d(x, y)) \Rightarrow$ equal movement in every direction
$\Rightarrow$ diffusion effect

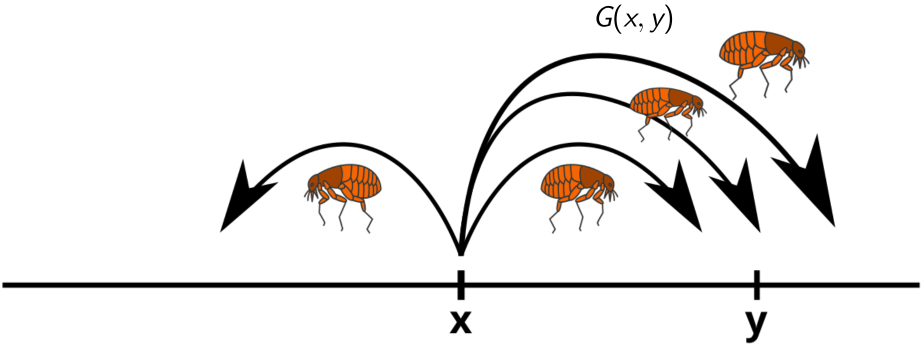


If, for example, $G(x, y) > G(y, x)$ for $x < y$, we obtain a drift to the right

$\Rightarrow$ transport behaviour

How many fleas run away from x?

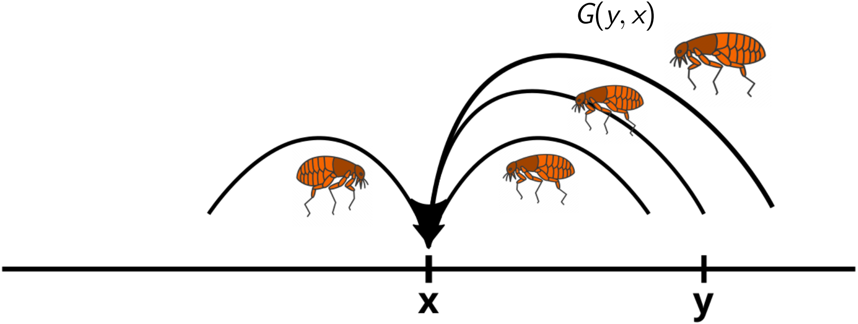
$u(x) =$ number of fleas at point x.



$$\int_{\mathbb{R}} G(x,y)u(x)dy$$ fleas running away from x

How many fleas are coming to x?

$u(y) =$ number of fleas at point y.

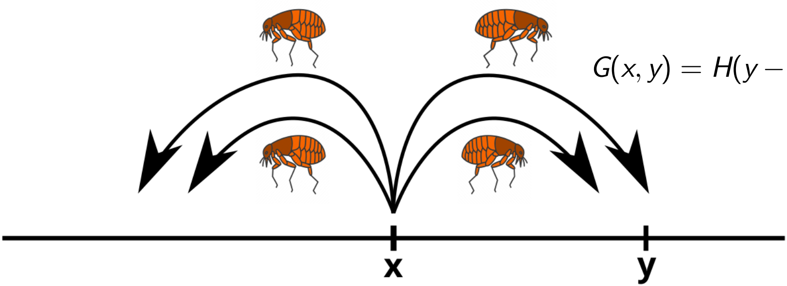


$$\int_{\mathbb{R}} G(y,x)u(y)dy \quad \text{fleas comming in } x$$

The evolution equation

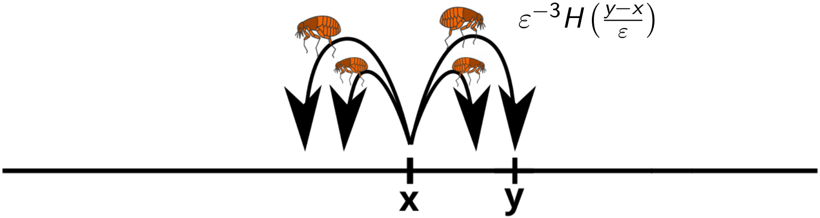
$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{R}} G(y,x)u(t,y)dy - \int_{\mathbb{R}} G(x,y)u(t,x)dy, & x \in \mathbb{R}, t \geq 0 \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \quad \text{(NLTR)}$$

Shrinking the movement – diffusion



$$G(x, y) = H(y - x)$$

Shrinking the movement – diffusion



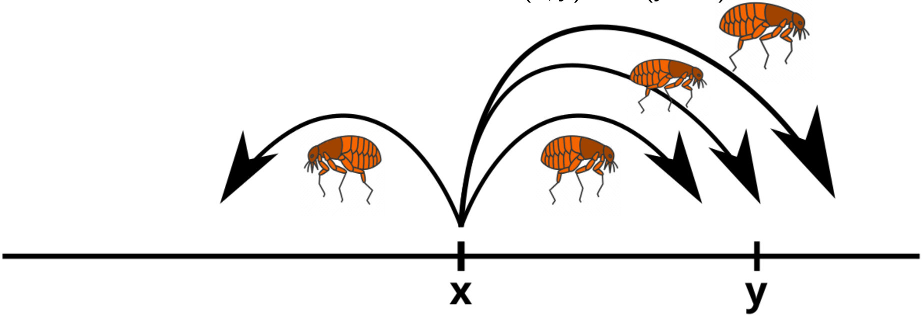
In the limit: local diffusion behaviour.

$$\partial_t u(t,x) = A \partial_{xx} u(t,x)$$

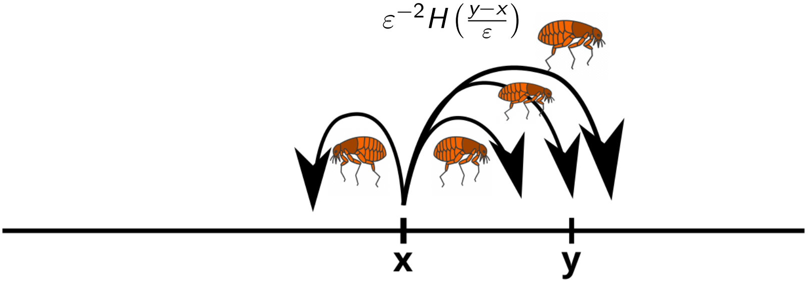
$$A = \frac{1}{2} \int_0^{\infty} H(x) \cdot x^2 dx$$

Shrinking the movement – transport

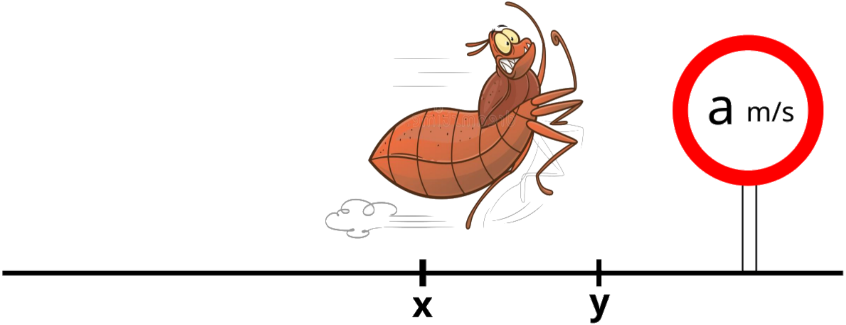
$$G(x, y) = H(y - x)$$



Shrinking the movement – transport



The transport equation



In the limit: drift effect = transport equation:

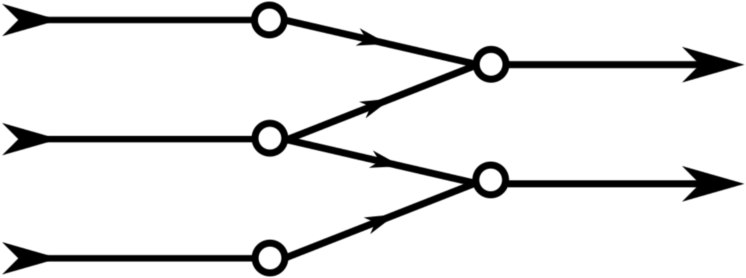
$$\partial_t u(t,x) = -a \cdot \partial_x u(t,x)$$

$$a = \int_{\mathbb{R}} H(x) \cdot x \, dx$$

The graph

$\Gamma = (V, E)$ connected oriented tree.

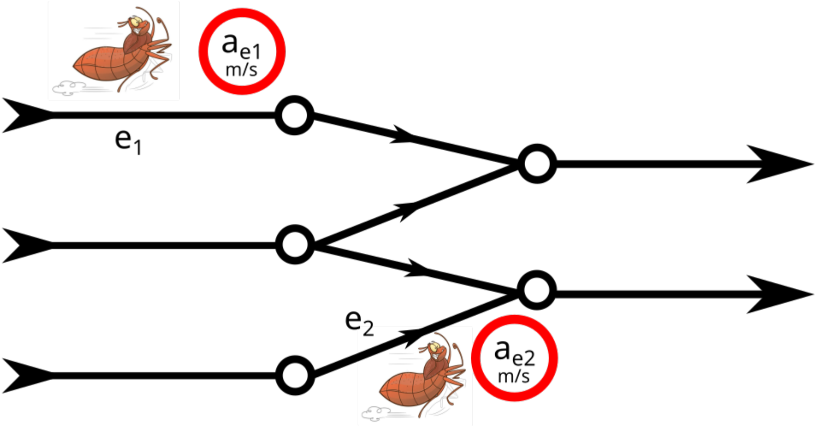
The infinite edges are either inputs or outputs – the only edges which are connected to single vertices.



Every infinite edge is parametrised $(-\infty, 0]$ or $[N+1, \infty)$.

Every finite edge is parametrised as $[i, i+1]$, with $i \in \{0, 1, \ldots, N\}$.

Fleas running on the edges



For every $e \in E$, we associate $a_e > 0$ the speed of the fleas on that edge.

A condition regarding the speeds

The network handles the flow perfectly

$$\sum_{e \in E^F(v)} a_e = \sum_{e \in E^I(v)} a_e \tag{Sa}$$

The sum of incoming speeds = The sum of outgoing speeds

Local transport equation on tree

$$\begin{cases} \partial_t u_e(t,x) = -a_e u_e(t,x), & x \in e \subset \Gamma, t \geq 0 \\ u(0,x) = u_0(x), & x \in \Gamma \\ \displaystyle\sum_{e \in E_v} \nu(v,e) \cdot a_e \cdot u_e(v) = 0, & \forall v \in V \\ u_{e_1}(v) = u_{e_2}(v), & \forall v \in V, \quad \forall e_1, e_2 \in E_v^I \end{cases} \tag{LT}$$

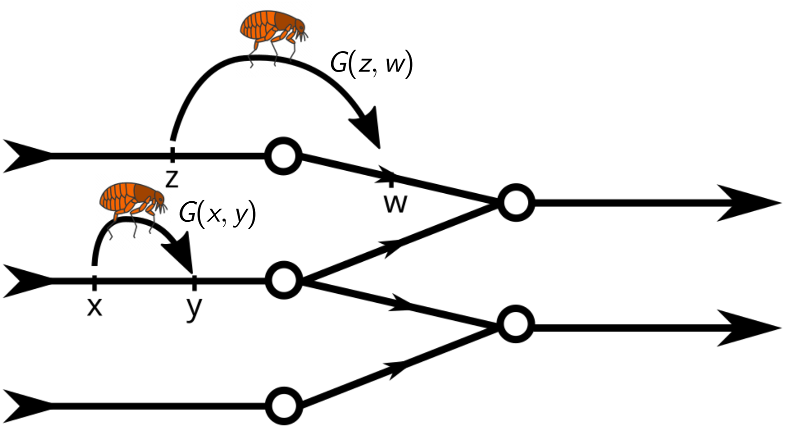
- $\nu(v,e) = \pm 1$ for edges e that enter/exit the vertex v

  The flow of flees is conserved in every vertex
- $E_v^I :=$ edges that emerge from the vertex v.

  The flees are evenly distributed when exiting a vertex

The problem (LT) is well-posed

Can this be achieved as a limit of non-local eq.?

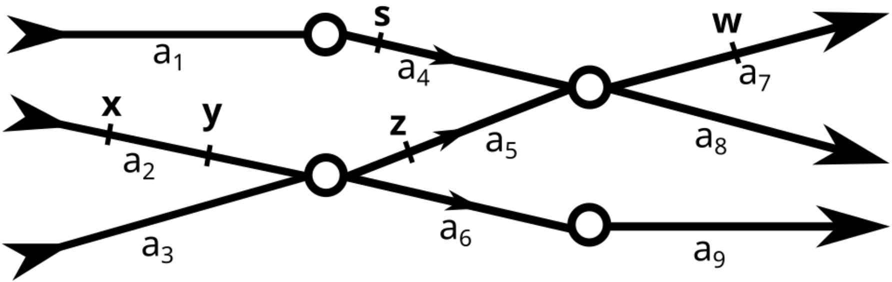


$G(x,y)$ accounts for the probability for fleas to jump from x to y.

Constructing such a G

$H : \mathbb{R} \to [0, \infty)$ integrable.

$$\int_{\mathbb{R}} H(x) \cdot x \, dx = 1$$

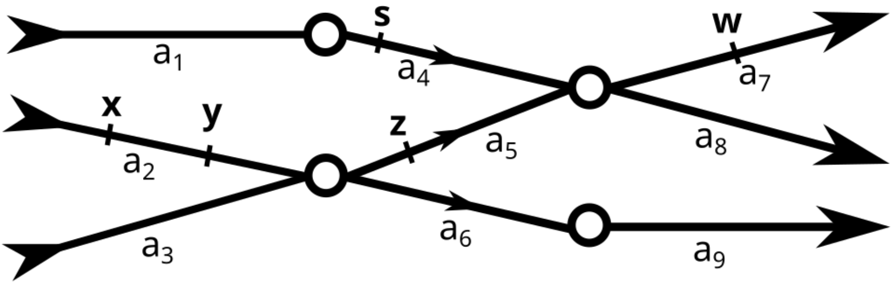


$$G(x,y) = a_2 H(y-x)$$

$$G(x,z) = a_2 \cdot \frac{a_5}{a_5+a_6} H(z-x)$$

$$G(x,w) = a_2 \cdot \frac{a_5}{a_5+a_6} \cdot \frac{a_7}{a_7+a_8} H(w-x)$$

$$G(x,s) = 0$$



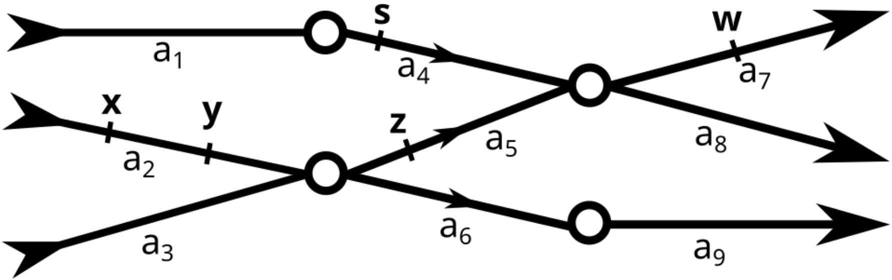
$$G(y,x) = a_2 H(x-y)$$

$$G(z,x) = a_2 \cdot \frac{a_5}{a_5+a_6} H(x-z)$$

$$G(w,x) = a_2 \cdot \frac{a_5}{a_5+a_6} \cdot \frac{a_7}{a_7+a_8} H(x-w)$$

$$G(s,x) = 0$$

Shrinking the kernel G



$$G_\varepsilon(y,x) = a_2 \cdot \varepsilon^{-2} H\left(\frac{x-y}{\varepsilon}\right)$$

$$G_\varepsilon(z,x) = a_2 \cdot \frac{a_5}{a_5+a_6} \varepsilon^{-2} H\left(\frac{x-z}{\varepsilon}\right)$$

Non-local transport problem

Shrunk non-local transport problem on Γ

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \left[\int_\Gamma G_\varepsilon(y,x)u^\varepsilon(t,y)dx - \int_\Gamma G_\varepsilon(x,y)u^\varepsilon(t,x)dy\right], & t \geq 0, x \in \Gamma \\ u^\varepsilon(0,x) = u_0(x), & x \in \Gamma \end{cases} \quad (\text{NLT}\varepsilon)$$

The problem is well-posed in L^2, assuming the integrability of H.

Convergence to the local problem – transport equation

Theorem

Assume that supp$(H) \in [0, \infty)$ *(this gives the direction of the movement left* $\to$ *right)*
Let $u_0 \in H^1(\Gamma)$ *satisfy the coupling conditions imposed for the local problem:*

$$\sum_{e \in E_v} \nu(v, e) \cdot a_e \cdot u_{0,e}(v) = 0, \qquad \forall v \in V$$

$$u_{0,e_1}(v) = u_{0,e_2}(v), \qquad \forall v \in V, \quad \forall e_1, e_2 \in E_v^I$$

Then, the sequence $(u^\varepsilon)_{\varepsilon > 0}$ *of solutions of* (NLTε) *converges weakly in every* $L^2([0, T] \times \Gamma)$, $T > 0$
to the solution u *of the local transport problem* (LT).

Idea of proof

Lemma

Let $H \in L^1(\mathbb{R})$, $\varphi \in H^1(\mathbb{R})$. Assume that:

$$\int_{\mathbb{R}} H(x) \cdot x \, dx < \infty$$

Then, for every $T > 0$,

$$\int_0^T \int_{-\infty}^{\infty} \left(\varepsilon^{-2} \int_{-\infty}^{\infty} H\left(\frac{y-x}{\varepsilon}\right) |\varphi(t,y) - \varphi(t,x) - (y-x) \cdot \partial_x \varphi(t,x)| \, dy \right)^2 dxdt$$

converges to 0 as $\varepsilon \to 0$.

The convection-diffusion problem

Shrunk non-local convection-diffusion problem on Γ

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \varepsilon^{-3} \int_\Gamma J\left(\frac{d(x,y)}{\varepsilon}\right)(u^\varepsilon(t,y) - u^\varepsilon(t,x))dy \\ \qquad + \int_\Gamma G_\varepsilon(y,x) f(u^\varepsilon(t,y))dy - \int_\Gamma G_\varepsilon(x,y) f(u^\varepsilon(t,x))dy, \ x \in \Gamma, t \geq 0 \\ u^\varepsilon(0,x) = u_0(x), \qquad x \in \Gamma \end{cases} \quad (\text{NLCD}\varepsilon)$$

J continuous in 0, $J(0) \neq 0$,

$$\int_0^\infty J(r) \cdot r^2 \, dr < \infty$$

$$f(r) = |r|^{m-1} r$$

Local convection-diffusion on Γ

$$\begin{cases} \partial_t u(t,x) = A\partial_{xx}u(t,x) - a_e\partial_x(f(u(t,x))), & x \in e \subset \Gamma, t \geq 0 \\ u_{e_1}(t,v) = u_{e_2}(t,v), & v \in V, \quad e_1, e_2 \in E_v \\ u(0,x) = u_0, & x \in \Gamma \end{cases} \tag{LCD}$$

The weak formulation of the problem above:

$$\mathrm{X} = \{u \in H^1(\Gamma) : u_{e_1}(v) = u_{e_2}(v), v \in V, e_1, e_2 \in E_v\}$$

$$\begin{cases} u \in L^2([0,T],\mathrm{X}) \cap C([0,T],L^2(\Gamma)), u_t \in L^2([0,T],\mathrm{X}') \\ \langle u_t(t),\psi\rangle_{\mathrm{X}',\mathrm{X}} + A\cdot(u_x(t),\psi)_{L^2(\Gamma)} - (a(\cdot)f(u(t)),\psi)_{L^2(\Gamma)} = 0, \\ \qquad \text{a.e. } t \in [0,T], \quad \forall \psi \in \mathrm{X} \\ u(0,x) = u_0(x) \end{cases} \tag{LCDW}$$

Existence and uniqueness – similar to [Cazacu et al., 2022].

Convergence non-local $\to$ local in the conv.-diff. setting

Theorem

Let $u_0 \in L^1(\Gamma) \cap L^\infty(\Gamma)$. The sequence $(u^\varepsilon)_{\varepsilon>0}$ of solutions of (NLCDε) converge weakly in every $L^2([0,T] \times \Gamma)$ to the unique weak solution of the local convection-diffusion problem (i.e. (LCDW)), with:

$$A = \frac{1}{2} \int_0^\infty J(r) \cdot r^2 \, dr$$

The proof is inspired by the paper [Ignat et al., 2020].

A compactness result used in the proof

Taken from [Ignat et al., 2015].

Theorem

$(a,b) \in \mathbb{R}$ open, possibly unbounded, $(u^\varepsilon)_{\varepsilon>0}$ bounded in $L^2([0,T]\times(a,b))$

$$\liminf_{\varepsilon\to 0} \varepsilon^{-3} \int_0^T \int_a^b \int_a^b J\left(\frac{d(x,y)}{\varepsilon}\right) |u^\varepsilon(y) - u^\varepsilon(x)|^2 dxdydt < \infty \qquad (1)$$

Then

1. *If $u^\varepsilon \rightharpoonup u$ in $L^2([0,T]\times(a,b))$, then:*
$$u \in L^2([0,T], H^1(a,b));$$
2. *If $D \subseteq (a,b)$ open, bounded and*
$$\|\partial_t u^\varepsilon\|_{L^2([0,T],H^{-1}(D))} \text{ uniformly bounded in } \varepsilon > 0$$
then $(u^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2([0,T]\times D)$ on a subsequence.

References

[Cazacu et al., 2022] Cazacu, C. M., Ignat, L. I., Pazoto, A. F., and Rossi, J. D. (2022).
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