Space-time domain decomposition

of optimal control problems for linear hyperbolic systems

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Introduction

Problem formulation

Let $y(t,x) \in \mathbf{R}^d$, $t \in I_k$, $x \in [0,L]$, denote the state and let

$$\Lambda = diag\,(\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_d) \in \mathbf{R}^{d\times d}$$

with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0 > \lambda_{m+1} \geq \cdots \geq \lambda_d.$$

We use the block-matrix abbreviation

$$\Lambda := diag\left(\Lambda^+, \Lambda^-\right),$$

with $\Lambda^+ := diag(\lambda_1, \ldots, \lambda_m)$ and $\Lambda^- := diag(\lambda_{m+1}, \ldots, \lambda_d)$.

Boundary conditions

We denote the first m components of the state by y^+ and the remaining $d-m$ components by y^- such that $y = (y^+, y^-)$. We consider here separated boundary conditions. Define the block matrix K

$$K := \begin{bmatrix} K^{00} & K^{01} \\ K^{10} & K^{11} \end{bmatrix},$$

$$K^{00} = 0 \in \mathbf{R}^{m\times m}, \quad K^{01} \in \mathbf{R}^{m\times d-m}, K^{10} = 0 \in \mathbf{R}^{d-m\times m}, \quad K^{11} \in \mathbf{R}^{d-m\times d-m}.$$

Let B_d signify the input operator for distributed controls u.

Hyperbolic system (P)

$$
\begin{aligned}
\partial_t y + \Lambda \partial_x y &= B_d u_d, && (t,x) \in [0,T] \times (0,L),\\
\begin{pmatrix} y^+(t,0) \\ y^-(t,L) \end{pmatrix} &= K \begin{pmatrix} y^+(t,L) \\ y^-(t,0) \end{pmatrix} && t \in [0,T],\\
y(0,x) &= y_0(x), && x \in (0,L),
\end{aligned}
$$

The optimal control problem

$$J(u_d, y) := \frac{\kappa}{2} \int_0^T \int_0^1 \|y - y_d\|^2 dt + \frac{\nu}{2} \int_0^T \int_0^1 \|u_d\|^2 dt$$

The considered control problem is thus given by

$$\min_{u,y} \quad J(u, y) \quad \text{s.t.} \quad (u, y) \text{ satisfies } \text{ P } .$$

Adjoint boundary conditions and optimality condition

The boundary matrix $\tilde{K}$ is given by

$$\tilde{K} := diag((\Lambda^+)^{-1}, |(\Lambda^-)^{-1}|)K^\top diag(\Lambda^+, |\Lambda^-|),$$

By taking the directional derivative of $\mathcal{L}(u, y, p)$ w.r.t. u_d in the direction $\tilde{u_d}$

$$u_d(t, x) = \frac{1}{\nu} B_d^\top p(t, x), \qquad (t, x) \in (0, T) \times (0, L).$$

Optimality system

We obtain the following optimality conditions governing the adjoint variable p:

$$
\begin{aligned}
\partial_t p + \Lambda \partial_x p &= \kappa(y - y_d), && (t,x) \in (0,T) \times (0,L), \\
\begin{pmatrix} p^+(t,L) \\ p^-(t,0) \end{pmatrix} &= \tilde{K} \begin{pmatrix} p^+(t,0) \\ p^-(t,L) \end{pmatrix}, && t \in (0,T), \\
p(T,x) &= 0, && x \in (0,L).
\end{aligned}
$$

Derivation of the second order problem

We take the adjoint equation, first multiply by Λ and differentiate wrt x and secondly differentiate wrt to t:

$$\Lambda\partial_{xt}p + \Lambda^2\partial_{xx} = \kappa(\frac{1}{\nu}B_dB_d^Tp - \partial_t y - \Lambda\partial_x y_d)$$
$$\partial_{tt}p + \Lambda\partial_{xt}p = \kappa(\partial_t y - \partial_t y_d).$$

The second equation yields

$$\kappa\partial_t y = \partial_{tt}p + \Lambda\partial_{xt}p + \kappa\partial_t y_d$$

which used in the first equation of to obtain

$$\partial_{tt}p + 2\Lambda\partial_{tx}p + \Lambda^2\partial_{xx}p - \frac{\kappa}{\nu}B_dB_d^Tp = \kappa(\Lambda\partial_x y_d + \partial_t y_d) =: -\kappa f. \qquad \text{(SOE)}$$

Standard div-grad formulation

We introduce the block matrix

$$\mathcal{A} := \begin{pmatrix} I & \Lambda \\ \Lambda & \Lambda^2 \end{pmatrix},$$

which is symmetric and positive semi-definite.
Then (SOE) turns into a degenerate Poisson equation in divergence form

$$-\mathrm{div}\mathcal{A}\nabla p + \frac{\kappa}{\nu} B_d B_d^T p = \kappa f.$$

Boundary conditions

Notice that $\mathcal{A} \in \mathbb{R}^{2d\times 2d}$, such that each block of $\mathcal{A} =: (a_{ij})_{i,j=1}^2$ is in $\mathbb{R}^{d\times d}$. We denote the trace of p by $\gamma(p)(x), x \in \partial\Omega := (0,T)\times(0,L)$. Then, understanding that the indices $i,j = 1,2$ relate to the direction t,x, respectively, we have

$$\partial_{\nu_{\mathcal{A}}} p(t,x) := \sum_{i,j=1}^{2} a_{ij}\gamma(\partial_j p(t,x))\nu_i(t,x), \; (t,x) \in \partial\Omega$$

Using this definition, we can write the co-normal derivative explicitly as follows

$$\begin{aligned}
\partial_{\nu_{\mathcal{A}}} p(t,0) &= -\Lambda(\partial_t p(t,0) + \Lambda\partial_x p(t,0)), && t \in (0,T)\\
\partial_{\nu_{\mathcal{A}}} p(t,L) &= \Lambda(\partial_t p(t,L) + \Lambda\partial_x p(t,L)), && t \in (0,T)\\
\partial_{\nu_{\mathcal{A}}} p(0,x) &= -(\partial_t p(0,x) + \Lambda\partial_x p(0,x)), && x \in (0,L)\\
\partial_{\nu_{\mathcal{A}}} p(T,x) &= \partial_t p(T,x) + \Lambda\partial_x p(T,x), && x \in (0,L)
\end{aligned}$$

Final system

We obtain a system of semi-elliptic boundary value problems in Ω:

$$
\begin{aligned}
-\mathrm{div}\mathcal{A}\nabla p + \frac{\kappa}{\nu} B_d B_d^T p &= -\kappa\left(\partial_t y_d + \Lambda \partial_x y_d\right) =: \kappa f & (t,x) &\in \Omega \\
\mathcal{B}_{11}\partial_{\nu_\mathcal{A}} p(t,x) &= -\kappa \mathcal{B}_{11} y_d(t,x),\ \mathcal{B}_{10} p(t,x) = 0, & (t,x) &\in \Gamma_1 \\
\mathcal{B}_{21}\partial_{\nu_\mathcal{A}} p(t,x) &= -\kappa \mathcal{B}_{21} y_d(t,x),\ \mathcal{B}_{20} p(t,x) = 0, & (t,x) &\in \Gamma_2 \\
B_{31}\partial_{\nu_\mathcal{A}} p(t,x) &= \kappa(y_0(x) - y_d(t,x)), & (t,x) &\in \Gamma_3 \\
B_{40} p(t,x) &= 0, & (t,x) &\in \Gamma_4.
\end{aligned}
$$

Transmission conditions

If we we now define $p_i := p|_{\Omega_i}, i = 1, \ldots, 4$ and use the decomposition of $\Omega = \cup_{i=1}^4 \Omega_i$ in Green identity, then after taking proper variations in ϕ, we obtain along the interfaces $\Gamma_{ij}, ij = 1, 2, 3, 4$ the transmission conditions

$$\partial_{\nu_{\mathcal{A}_1}} p_1|_{\Gamma_{12}} + \partial_{\nu_{\mathcal{A}_2}} p_2|_{\Gamma_{21}} = 0,\ p_1|_{\Gamma_{12}} = p_2|_{\Gamma_{21}}$$
$$\partial_{\nu_{\mathcal{A}_1}} p_1|_{\Gamma_{14}} + \partial_{\nu_{\mathcal{A}_4}} p_4|_{\Gamma_{14}} = 0,\ p_1|_{\Gamma_{12}} = p_2|_{\Gamma_{21}}$$
$$\partial_{\nu_{\mathcal{A}_1}} p_2|_{\Gamma_{2}} + \partial_{\nu_{\mathcal{A}_3}} p_3|_{\Gamma_{23}} = 0,\ p_2|_{\Gamma_{23}} = p_3|_{\Gamma_{32}}$$
$$\partial_{\nu_{\mathcal{A}_3}} p_3|_{\Gamma_{34}} + \partial_{\nu_{\mathcal{A}_4}} p_4|_{\Gamma_{43}} = 0,\ p_3|_{\Gamma_{34}} = p_4|_{\Gamma_{43}}$$

P.L. Lions-type domain decomposition

1. Given $p_i^n, \partial_{\nu_{\mathcal{A}_i}} p_i^n$ on Γ_{ij}, $i \neq j \in \mathcal{I}$
2. compute $p_i^{n+1}, i \in \mathcal{I}$ according to

$$-\mathrm{div}\mathcal{A}_i \nabla p_i^{n+1} + \frac{\kappa}{\nu} B_d B_d^T p_i^{n+1} = \kappa f_i \text{ in } \Omega$$

$$\mathcal{B}_{i1} \partial_{\nu_{\mathcal{A}_i}} p_i^{n+1} = 0,\ \mathcal{B}_{i0} p_i^{n+1} = 0,\ \text{on } \Gamma_{i,ext}$$

$$\partial_{\nu_{\mathcal{A}_i}} p_i^{n+1} + \beta_{ij} p_i^n = -\partial_\nu p_j^n + \beta_{ij} p_j^n := \lambda_{ij}^n,\ \ \text{on } \Gamma_{ij},\ j \in \mathcal{I}_i.$$

3. $n \to n+1$ go to ii.)

Convergence analysis

Error equations

Introduce the errors $\tilde{p}_i^n := p_i^n - p_i, n \in \mathbb{N}, i \in \mathcal{I}$. Then, $\tilde{p}_i^n$ satisfies the system:

$$-\mathrm{div}\mathcal{A}_i \nabla \tilde{p}_i^{n+1} + \frac{\kappa}{\nu}\tilde{p}_i^{n+1} = 0 \text{ in } \Omega$$

$$\mathcal{B}_{i1}\partial_{\nu_{\mathcal{A}_i}}\tilde{p}_i^{n+1} = 0,\ \mathcal{B}_{i0}\tilde{p}_i^{n+1} = 0,\ \text{ on } \Gamma_{i,ext}$$

$$\partial_{\nu_{\mathcal{A}_i}}\tilde{p}_i^{n+1} + \beta_{ij}\tilde{p}_i^n = -\partial_{\nu_{\mathcal{A}_j}}\tilde{p}_j^n + \beta_{ij}\tilde{p}_j^n := \lambda_{ij}^n, \ \text{ on } \Gamma_{ij},\ j \in \mathcal{I}_i.$$

The fixed point Ansatz: fixed point map

We introduce the space

$$\mathcal{X} := \prod_{i=1}^{N} L^2(\gamma_i)^d$$

with

$$\|X\|^2 = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}_i} \frac{1}{2\beta_{ij}} \int_{\Gamma_{ij}} |\lambda_{ij}|^2_{\mathbb{R}^d} d\gamma,$$

where $X = (\lambda_i)_{i\in\mathcal{I}}$, $\lambda_i := (\lambda_{ij})_{j\in\mathcal{I}_i}$. We now introduce the operator

$$\begin{aligned} \mathcal{T} :&\mathcal{X} \to \mathcal{X} \\ (\mathcal{X})_{ij} :&\left(-\partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j + \beta_{ij}\tilde{p}_j\right)|_{\Gamma_{ij}}, \\ (\mathcal{T}\mathcal{X})_i &= \{(\mathcal{T}\mathcal{X}_{ij}|j \in \mathcal{I}_i\} \\ \mathcal{T}X &= \{(\mathcal{T}X)_i|i \in \mathcal{I}\}. \end{aligned}$$

The iterates and their norms

Then the iteration is equivalent to the fixed point iteration

$$x^{n+1} = \mathcal{T} X^n, n = 0, 1, \ldots.$$

We compute, omitting the iteration index for a while,

$$\begin{aligned}
\|X\|_{\mathcal{X}}^2 &= \sum_{i=1}^{N} \sum_{j \in \mathcal{I}_i} \frac{1}{2\beta_{ij}} \int\limits_{\Gamma_{ij}} |\partial_{\nu_{\mathcal{A}_i}} p_i + \beta_{ij} p_i|_{\mathbb{R}^d}^2 d\gamma \\
&= \sum_{i=1}^{N} \sum_{j \in \mathcal{I}_i} \int\limits_{\Gamma_{ij}} \left(\frac{1}{2\beta_{ij}} |\partial_{\nu_{\mathcal{A}_i}} p_i|_{\mathbb{R}^d}^2 + \partial_{\nu_{\mathcal{A}_i}} p_i p_i + \frac{\beta_{ij}}{2} |p_i|_{\mathbb{R}^d}^2 \right) \\
\|\mathcal{T} X\|_{\mathcal{X}}^2 &= \sum_{i=1}^{N} \sum_{j \in \mathcal{I}_i} \frac{1}{2\beta_{ij}} \int\limits_{\Gamma_{ij}} | - \partial_{\nu_{\mathcal{A}_i}} p_j + \beta_{ij} p_j|_{\mathbb{R}^d}^2 d\gamma \\
&= \sum_{i=1}^{N} \sum_{j \in \mathcal{I}_i} \int\limits_{\Gamma_{ij}} \left(\frac{1}{2\beta_{ij}} |\partial_{\nu_{\mathcal{A}_j}} p_j|_{\mathbb{R}^d}^2 - \partial_{\nu_{\mathcal{A}_i}} p_j p_j + \frac{\beta_{ij}}{2} |p_j|_{\mathbb{R}^d}^2 \right)
\end{aligned}$$

Towards non-expansiveness

This gives

$$\|\mathcal{T}X\|_{\mathcal{X}}^2 - \|X\|_{\mathcal{X}}^2 = -2\sum_{i=1}^{N}\sum_{j\in\mathcal{I}_i}\int_{\Gamma_{ij}} \partial_{\nu_{\mathcal{A}_i}} p_i p_i d\gamma.$$

We go back to the notation of errors and use the Green's identity on the subdomain Ω_i.

$$\begin{aligned}
0 &= \int_{\Omega_i} \left(-\mathrm{div}\mathcal{A}_i\nabla\tilde{p}_i + \frac{\kappa}{\nu}\tilde{p}_i\right)\tilde{p}_i d\omega \\
&= -\int_{\partial\Gamma_{ij}} \partial_{\nu_{\mathcal{A}_i}}\tilde{p}_i(t,x)\tilde{p}_i d\gamma + \int_{\Omega_i} |\partial_t\tilde{p}_i + \Lambda\partial_x\tilde{p}_i|_{\mathbb{R}^d}^2 d\omega + \int_{\Omega_i} \frac{\kappa}{\nu}\tilde{p}_i\tilde{p}_i d\omega \\
&= -\int_{\partial\Gamma_{ij}} \partial_{\nu_{\mathcal{A}_i}}\tilde{p}_i(t,x)\tilde{p}_i d\gamma + a_i(\tilde{p}_i,\tilde{p}_i).
\end{aligned}$$

Non-expansiveness

$$\|\mathcal{T}X\|_{\mathcal{X}}^2 - \|X\|_{\mathcal{X}}^2 = -2\left\{\sum_{i=1}^{N}\int_{\Omega_i} |\partial_t \tilde{p}_i + \Lambda \partial_x \tilde{p}_i|_{\mathbb{R}^d}^2 d\omega + \int_{\Omega_i} \frac{\kappa}{\nu}\tilde{p}_i \tilde{p}_i d\omega\right\}.$$

In other words

$$\|\mathcal{T}X\|_{\mathcal{X}}^2 - \|X\|_{\mathcal{X}}^2 = -2\left\{\sum_{i=1}^{N} a_i(\tilde{p}_i, \tilde{p}_i)\right\}.$$

Under-relaxation

we introduce the following under-relaxation.

$$X^{n+1} = (1-\epsilon)\mathcal{T}X^n + \epsilon X^n, \ \epsilon \in [0,1).$$

Hence,

$$\|X^{n+1}\|_{\mathcal{X}}^2 = ((1-\epsilon)^2+\epsilon^2)\|X^n\|_{\mathcal{X}}^2 - 2(1-\epsilon)^2 \sum_{i=1}^{N} a_i(\tilde{p}_i, \tilde{p}_i) + 2\epsilon(1-\epsilon)\left(X^n, \mathcal{T}X^n\right)_{\mathcal{X}}.$$

With the definition

$$E^n := \sum_{i=1}^{N} \sum_{j\in\mathcal{I}_i} \frac{1}{2\beta_{ij}} \int_{\Gamma_{ij}} |\partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n|_{\mathbb{R}^d}^2 + \beta_{ij}|\tilde{p}_i^n|_{\mathbb{R}^d}^2 d\gamma$$

Crucial ‚energy‘-estimate

we obtain

$$\|X^n\|_{\mathcal{X}}^2 = E^n + \sum_{i=1}^{N} a_i(\tilde{p}_i, \tilde{p}_i) =: E^n + \mathcal{F}^n$$

$$\|\mathcal{T} X^n\|^2 = E^n - \mathcal{F}^n.$$

A straightforward calculation shows

$$(\mathcal{X}^n, \mathcal{T}\mathcal{X}^n) \leq E^n$$

and moreover

$$E^{n+1} \leq E^n - \sum_{i=1}^{N} \left[(1-2\epsilon) a_i(\tilde{p}_i^n, \tilde{p}_i^n) + a_i(\tilde{p}_i^{n+1}, \tilde{p}_i^{n+1}) \right].$$

Convergence: almost last step

With the coefficients $c_1(\epsilon) := 1-2\epsilon, c_{n+1}(\epsilon) = 1, c_l(\epsilon) = 2(1-\epsilon), l = 2, \dots, n,$ we obtain the crucial inequality

$$E^{n+1} + \sum_{l=1}^{n+1} c_l(\epsilon) \sum_{i=1}^{l} a_i(\tilde{p}_i^n, \tilde{p}_i^n) \leq E^1.$$

This inequality implies

$$\sum_{i=1}^{N} a_i(\tilde{p}_i^n, \tilde{p}_i^n) \to 0$$

$$E^n \leq C,$$

Convergence…..

$$\sum_{i=1}^{N}\left\{\int_{\Omega_i}|\partial_t\tilde{p}_i+\Lambda\partial_x\tilde{p}_i|^2_{\mathbb{R}^d}d\omega+\int_{\Omega_i}\frac{\kappa}{\nu}\tilde{p}_i\tilde{p}_i d\omega\right\}\to 0.$$

However, the form $a_i(\cdot,\cdot)$ is not elliptic but rather positive semi-definite. Nevertheless, we may conclude

$$\tilde{p_i}^n \to 0, \text{ strongly in } L^2(\Omega_i)^d,$$

while

$$\partial_t\tilde{p}_i+\Lambda\partial_x\tilde{p}_i \to 0, \text{ strongly in } L^2(\Omega_i)^d$$

Opial's lemma

- this does not imply $\tilde{p}_i^n \to 0$ strongly in $H^1(\Omega_i)$.
- as E^n is bounded, we can extract sub-sequences from $\partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n, \tilde{p}_i^n$ on Γ_{ij} such that $\partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n \rightharpoonup q$, $\tilde{p}_i^n|_{\Gamma_{ij}} \rightharpoonup r$, weakly in $L^2(\gamma_i)^d$.
- We may use Green's identity with a test function ϕ on Ω_i to conclude $q = r = 0$. But this holds on subsequences which may not contain to consecutive indices $n, n+1$ as required in the iteration.
- We take now advantage of the under-relaxation parameter $\epsilon \in (0,1)$.

Proposition[Opial]. Let $\mathcal{T}$ be nonexpansive with at least one fixed point. Then for each $\epsilon \in (0,1)$ the sequence $\{\mathcal{T}_\epsilon^n X\}$ is weakly convergent to a fixed point.

Here we have set $\mathcal{T}_\epsilon := \epsilon I + (1-\epsilon)\mathcal{T}$. As we have seen, our map $\mathcal{T}$ is nonexpansive and 0 is in fact the unique fixed point. Thus, we may conclude that entire sequences converge to zero and, moreover, $X^n \rightharpoonup 0$ and $\mathcal{T} X^n \rightharpoonup 0$.

Schaefer's theorem

Now, in order to prove strong convergence of $\|\mathcal{T}X^n - X^n\|_{\mathcal{X}} \to 0$, we are going to apply Schaefer's theorem. To this end we recall the definition of an asymptotic regular map $\mathcal{T}$. Let $\mathcal{C} \subset \mathcal{X}$ closed and convex and $\mathcal{T}$ nonexpansive. Then $\mathcal{T}$ is said to be *asymptotically regular* if for any $X \in \mathcal{C}$ the sequence $\{\mathcal{T}^{n+1}X - \mathcal{T}^n X\}$ tends to zero as $n \to \infty$.

Proposition[Schaefer:1957]. If $\mathcal{T}$ has at least one fixed point in $\mathcal{C}$, then the mapping $\mathcal{T}_\epsilon$ is asymptotically regular.

From this, we infer

$$\|\mathcal{T}X^n - X^n\|_{\mathcal{X}}^2 = 2E^n - 2(\mathcal{T}X^n, X^n) \to 0.$$

Strong convergence

We calculate the second term

$$(\mathcal{T}X^n, X^n) = \sum_{i=1}^{N} \sum_{j\in\mathcal{I}_i} \frac{1}{2\beta_{ij}} \int\limits_{\Gamma_{ij}} (-\partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j^n + \beta_{ij}\tilde{p}_j^n)(\partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n + \beta_{ij}\tilde{p}_i^n) d\gamma$$

$$= \sum_{i=1}^{N} \sum_{j\in\mathcal{I}_i} \frac{1}{2\beta_{ij}} \int\limits_{\Gamma_{ij}} \left\{ -\partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j^n \partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n + \beta_{ij}^2 \tilde{p}_i^n \tilde{p}_i^n + \beta_{ij} (\partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n \tilde{p}_j^n - \partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j^n \tilde{p}_i^n) \right\} d\gamma$$

$$= \sum_{i=1}^{N} \sum_{j\in\mathcal{I}_i} \frac{1}{2\beta_{ij}} \int\limits_{\Gamma_{ij}} \left\{ -\partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j^n \partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n + \beta_{ij}^2 \tilde{p}_i^n \tilde{p}_i^n \right\} d\gamma$$

Strong convergence

We obtain from this

$$2E^n - 2(\mathcal{T}X^n, X^n) = \sum_{i=1}^{N-1} \sum_{j\in\mathcal{I}_i, j>i} \int_{\Gamma_{ij}} \left(\frac{1}{\beta_{ij}} |\partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n + \partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j^n|^2 + \beta_{ij} |\tilde{p}_i^n - \tilde{p}_j^n|^2 \right) d\gamma$$

In conclusion, we obtain for all $i, j : j \in \mathcal{I}_i$

$$|\partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n + \partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j^n|_{L^2(\Gamma_{ij})} \to 0,\ n \to \infty$$

$$|\tilde{p}_i^n - \tilde{p}_j^n|_{L^2(\Gamma_{ij})} \to 0,\ n \to \infty$$

Theorem

Let $\epsilon \in [0,1)$ be given. Then the iteration defined in (**??**) converges in the following sense

1. $\epsilon = 0$

$$\begin{aligned} \partial_t \tilde{p}_i^n + \Lambda \partial_x \tilde{p}_i^n &\to 0, \text{ strongly in } L^2(\Omega_i)^d, \ n \to \infty \\ \tilde{p}_i^n &\to 0, \text{ strongly in } L^2(\Omega_i)^d, \ n \to \infty. \end{aligned}$$

   On a subsequence we have

$$\begin{aligned} \partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n &\rightharpoonup 0, \text{ weakly in } L^2(\Gamma_{ij})^d \ n \to \infty \\ \tilde{p}_i^n &\rightharpoonup 0, \text{ weakly in } L^2(\Gamma_{ij})^d \ n \to \infty \end{aligned}$$

2. $\epsilon \in (0,1)$ In addition to the first case, (i) we have

$$\begin{aligned} \partial_{\nu_{\mathcal{A}_i}} \tilde{p}_i^n + \partial_{\nu_{\mathcal{A}_j}} \tilde{p}_j^n &\to 0, \text{ strongly in } L^2(\Gamma_{ij})^d, \ n \to \infty \\ \tilde{p}_i^n - \tilde{p}_j^n &\to 0, \text{ strongly in } L^2(\Gamma_{ij})^d \ n \to \infty \end{aligned}$$

Thus, the iterates p_i^n converge to the restriction of solution $p|_{\Omega_i}$ in the Algorithm-DDM in the sense of i.) and ii.), respectively.

Theorem (a posteriori estimate)

Let the iterates $p_i^n, i = 1, 2$ be the solutions of the iteration for $\epsilon \in (0, 1)$ and let p_i solve the original optimality system. Then we have

$$\|p_1^{n+1} - p_1\|_{L^2(\Omega_1)^d} + \|p_2^n - p_2\|_{L^2(\Omega_2)^d} \leq C|p_1^{n+1} - p_2^n|_{L^2(\Gamma)^d}^{\frac{1}{2}}.$$

Proof: The proof is analogous to the one by Otto and Lube 1999 for scalar elliptic problems.

Interpretation in the original formulation

We have first from transmission condition for p

$$
\begin{aligned}
-\Lambda(\partial_t p_2 + \Lambda\partial_x p_2) + \beta_{12} p_2 &= -\Lambda(\partial_t p_1 + \Lambda\partial_x p_1) + \beta_{12} p_1 \\
\Lambda(\partial_t p_1 + \Lambda\partial_x p_1) + \beta_{21} p_1 &= \Lambda(\partial_t p_2 + \Lambda\partial_x p_2) + \beta_{21} p_2,
\end{aligned}
$$

which translates at the breakpoint $x = x_1$ (of the continuous matching) to the iteration

$$
\begin{aligned}
-\kappa\Lambda^+(y_2^+)^{n+1} + \beta_{12}(p_2^+)^{n+1} &= -\kappa\Lambda^+(y_1^+)^n + \beta_{12}(p_1^+)^n = \lambda_{21}^{+,n} \\
-\kappa\Lambda^-(y_2^-)^{n+1} + \beta_{12}(p_2^-)^{n+1} &= -\kappa\Lambda^-(y_1^-)^n + \beta_{12}(p_1^-)^n = \lambda_{21}^{-,n} \\
\kappa\Lambda^+(y_1^+)^{n+1} + \beta_{21}(p_1^+)^{n+1} &= \kappa\Lambda^+(y_2^+)^n + \beta_{21}(p_2^+)^n = \lambda_{12}^{+,n}, \\
\kappa\Lambda^-(y_1^-)^{n+1} + \beta_{21}(p_1^-)^{n+1} &= \kappa\Lambda^-(y_2^-)^n + \beta_{21}(p_2^-)^n = \lambda_{12}^{-,n}.
\end{aligned}
$$

Interpretation for the original system

We introduce $S_{ij} = I, j \in \mathbb{I}_i^t$ and $S_{ij} = \kappa\Lambda, j \in \mathcal{I}_i^x$. Moreover, we introduce the signs $\epsilon_{ij} = 1, j \in \mathcal{I}_i^x$ if j signifies the domain Ω_j right to Ω, whereas $\epsilon_{ij} = -1$ if Ω_j is left to it. Similarly, for the upper and lower neighbours of Ω_i. With this notation, the optimality system can be written as

$$
\begin{aligned}
\partial_t y_i^{n+1} + \Lambda \partial_x y_i^{n+1} &= \frac{1}{\nu} p_i^{n+1} && \text{in } \Omega_i \\
\partial_t p_i^{n+1} + \Lambda \partial_x p_i^{n+1} &= \kappa(y_i^{n+1} - y_{id}) && \text{in } \Omega_i \\
\epsilon_{ij} S_{ij} y_i^{n+1} + \beta_{ij} p_i^{n+1} &= -\epsilon_{ji} S_{ji} y_j^n + \beta_{ij} p_i^n = \lambda_{ij}^n, && \text{on } \Gamma_{ij},\ \forall j \in \mathcal{I}_i,
\end{aligned}
$$

The corresponding virtual control problem

With the virtual controls $h_{ij} \in L^2(\Gamma_{ij})^d$, $j \in \mathcal{I}_i$, we obtain the equivalent optimal control problem

$$\begin{cases} \min J(u_{di}, y_i, h_{ij}) := \dfrac{\kappa}{2} \displaystyle\int\limits_{\Omega_i} \|y_i - y_{d,i}\|^2 d\omega + \dfrac{\nu}{2} \displaystyle\int\limits_{\Omega_i} \|u_{di}\|^2 d\omega \\ \qquad\qquad + \displaystyle\sum_{j \in \mathcal{I}_i} \int\limits_{\Gamma_{ij}} \dfrac{1}{2\beta_{ij}} \left(|h_{ij}|^2 + |\lambda_{ij}^n|^2 \right) d\gamma \\ \qquad\qquad \text{subject to} \\ \partial_t y_i + \Lambda \partial_x y_i = u_i, \text{ in } \Omega_i \\ \epsilon_{ij} S_{ij} y_i = \lambda_{ij}^n + h_{ij}, \text{ on } \Gamma_{ij},\ j \in \mathcal{I}_i \end{cases}$$

Thank you for your attention!

- Reference: G. Leugering, *Space-Time-Domain Decomposition for Optimal Control Problems governed by Linear Hyperbolic Systems*

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