Event Based Control of PDEs.

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08/2022.

$$\begin{cases} \partial_{tt} z - \Delta z = u & \Omega \times \mathbb{R}_+ \\ z = 0 & \partial\Omega \times \mathbb{R}_+ \\ z(0) = z_0 \quad \partial_t z(0) = z_1 & \Omega \end{cases}$$

u is the control

$(z_0, z_1) \in H_0^1 \times L^2$ and $u \in L^2(L^2) \Rightarrow \exists ! z \in C(H_0^1) \cap C^1(L^2)$

It is well known that $u(x,t) = -\alpha \partial_t z(x,t)$ allows the exponential stabilization:

$$E(t) = \frac{1}{2} \|\partial_t z(t)\|^2_{L^2} + \frac{1}{2} \|\nabla z(t)\|^2_{L^2}$$

$$\exists\, K, \delta \ / \quad E(t) \leq K\, E(0) e^{-2\delta t} \qquad \forall t > 0.$$

Let us now assume that we would like to achieve the same kind of stability result but updating the control u only at certain instants t_k and held the control <u>constant</u> between two events. Moreover, $t_{k+1} - t_k$ is not assumed constant, but rather defined through an <u>event trigger law</u>: for instance, $\gamma > 0$

$$(*) \quad t_{k+1} = \inf \{ t > t_k, \ \|e_k(t)\|^2_{L^2} > 2\gamma E(t) \}$$

with $e_k(x,t) = \partial_t z(x,t) - \partial_t z(x, t_k)$, $\forall t \in [t_k, t_{k+1}]$

↳ déviation term.

The closed loop system is then

$$(**)\begin{cases} \partial_{tt} z - \Delta z = -\alpha \partial_t z(x, t_k) & \forall x \in \Omega,\ \forall t \in [t_k, t_{k+1}] \\ \qquad\qquad = -\alpha \partial_t z(x,t) + \alpha e_k(x,t) & \\ z|_{\partial\Omega} = 0 & \\ z(0) = z_0 \qquad \partial_t z(0) = z_1 & \Omega \end{cases}$$

Questions : Is the closed loop system (**)-(*)

1. - Well posed ?
2. - Such that Zeno behavior is avoided ?
3. - Exponentially stable ?

Définitions :

- Maximal time of existence of solutions = T^*

$$\begin{cases} T^* = +\infty & \text{if } (t_k) \text{ is a finite sequence} \\ T^* = \limsup_{k\to+\infty} t_k & \text{if not} \end{cases}$$

- Ruling out Zeno $\Leftrightarrow$ $T = +\infty$

$\Leftrightarrow$ There is no accumulation point in the $(t_k)_{k\geq 0}$ sequence

Reference:

Koudohode - Baudouin - Tarbouriech [Automatica]
hal 03273260.

Proofs: ① By an iterative process, from $[0, t_1)$ to $[t_1, t_2)$ ... until T^*, existence and regularity of solutions to (**) is rather classical.

But we need it with the regularity $H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega)$ for the proof concerning Zeno.

② We first need a specific lemma:

- Lemma L: $\exists c>0 \;/\; \forall t\in[0,T^*)$

$$E(0)e^{-2ct} \leq E(t) \leq E(0)e^{2ct}.$$

→ proof by easy energy estimations.

- No Zeno: Study $\varphi(t) = \dfrac{\|e_k(t)\|^2_{L^2}}{2\gamma E(t)} \geq 0$

$\varphi(t_k)$

φ jumps from $\lim_{t\to t_{k+1}} \varphi(t) = 1$ to $\varphi(t_{k+1}) = 0$ when a triggering event occurs. The time evolution of φ is the key point to rule out Zeno.

We estimate $\partial_t\varphi(t)$ and prove, using the H^2 regularity of $z(t)$ that $\dot\varphi(t) \leq A + \dfrac{B}{\sqrt{E(t)}}$.

With lemma L $\Rightarrow$ $\dot\varphi(t) \leq A + \dfrac{Be^{cT^*}}{\sqrt{E(0)}}$

and $\int_{t_k}^{t_{k+1}}$ brings a contradiction if $T^* = +\infty$.

③ An appropriate Lyapunov functional allows to prove the exponential stability

$$V(t) = E(t) + \frac{\alpha\varepsilon}{2}\int_\Omega |z(t)|^2 dx + \varepsilon\int_\Omega z(t)\partial_t z(t)\,dx$$

→ in the continuous control case $u(x,t) = -\alpha\partial_t z(x,t)$

→ in the event-triggered control case as well, using that $\forall t \in [t_k, t_{k+1}[$, $\|e_k(t)\|^2_{L^2} \leq 2\gamma E(t)$.

Other Even Triggered Control situations for PDE already under study / known :

- Schrodinger Equation (PhD. F. Koudohode)

$$i\partial_t z + \Delta z = -i\,\alpha(x)\, z(x, t_k).$$

with $\alpha > 0$ only on $\omega \subset\subset \Omega$.

- Abstract System (Baudouin/Ervedoza)

$$\begin{cases} \dot{z} = Az + Bu \\ u = KCz(t_k) \end{cases}$$

$A : \mathcal{D}(A) \subset H \to H$. C^0-semi gpe...

Hyp: . $A^* = -A$

. $B \in \mathcal{L}(U, H)$, $C \in \mathcal{L}(H, U)$

. $A + BKC$ inf. gen. of an exp. stable C_0-semigpe..

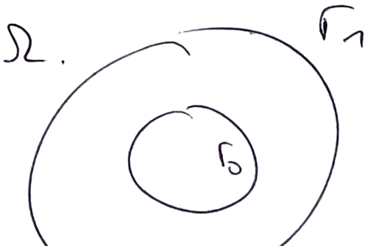
- Boundary Control of the waves. (Baudouin, Marx, Valein, Tarbouriech)

$$\begin{cases} \partial_{tt} z - \Delta z = 0 \\ z|_{\Gamma_0} = 0 \\ \partial_\nu z|_{\Gamma_1} = u \\ z(0) = z_0 \quad \partial_t z(0) = z_1 \end{cases}$$

$\Gamma_1 = \{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\}$

$\Gamma_0 \cap \Gamma_1 = \emptyset$

$\Gamma_0 \cup \Gamma_1 = \partial\Omega$.



$$u(x,t) = -(x - x_0)\cdot\nu(x)\, \partial_t z(t_k, x)$$

* Event-triggered law 1

$$t_{k+1} = \inf \{ t \geq t_k, \ \|\partial_t z(t) - \partial_t z(t_k)\|^2_{L^2(\Gamma_1)} \geq 2\gamma E(t) + \nu_0 \}$$

$\Rightarrow$ Practical Stability towards an attractor

$$\mathcal{A} = \{ v \in H^1_{\Gamma_0}(\Omega) \ / \ \partial_t v \in L^2, \ E_v(t) \leq r \}$$

where $r = r(\nu_0, \gamma, \dots)$

* E-T law 2

$$t_{k+1} = \inf \{ \text{————} \ \|.\|_{L^2(\Gamma_1)} \geq 2\gamma E(t) + \varepsilon_0 e^{-2\theta t} \}$$

$\Rightarrow$ Exponential Convergence

choose $\varepsilon_0 \in]0, E(0)]$ then $\exists \, k, \ \delta < \theta$ so that

$$E(t) \leq k E(0) e^{-2\delta t} \quad \forall t > 0.$$

Both laws rule out Zeno, but for that, need the extra term ν_0 / ε_0 because we don't have the lemma L in that case.

Open questions:

- Take continuous feedback controls of some of your favorite PDE and trigger its update with some well designed law. Prove then you have
  $\rightarrow$ well posedness $\rightarrow$ no Zeno $\rightarrow$ Exponential Stability.

- Check if this kind of Event triggered law work also for other approaches than Lyapunov functional approach. It works with observability techniques also. What more?