Upper and lower bounds for some shape functionals

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We study two quantities occurring in elliptic PDEs. The first quantity is usually called torsional rigidity and is defined as

$$T(\Omega) = \int_\Omega u \, dx$$

where u is the solution of the Poisson equation

$$-\Delta u = 1 \text{ in } \Omega, \qquad u \in H_0^1(\Omega).$$

In the thermal diffusion model $T(\Omega)/|\Omega|$ is the average temperature (after a long time) of a conducting medium Ω with uniformly distributed heat sources ($f = 1$).

The second quantity is the first eigenvalue of the Dirichlet Laplacian

$$\lambda(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \ : \ u \in H_0^1(\Omega) \setminus \{0\} \right\}$$

In the thermal diffusion model, by the Fourier analysis,

$$u(t,x) = \sum_{k \geq 1} e^{-\lambda_k t} \langle u_0, u_k \rangle u_k(x),$$

so $\lambda(\Omega)$ represents the decay rate in time of the temperature when an initial temperature is given and no heat sources are present.

Under the measure constraint $|\Omega| = m$, the highest $T(\Omega)$ is given by a ball (Saint Venant inequality); similarly, the smallest $\lambda(\Omega)$ is given by a ball (Faber-Krahn inequality). We then want to study if

$$\lambda(\Omega) \sim T^{-1}(\Omega),$$

or more generally, for a suitable $q > 0$

$$\lambda(\Omega) \sim T^{-q}(\Omega),$$

where by $A(\Omega) \sim B(\Omega)$ we mean

$$0 < c_1 \leq A(\Omega)/B(\Omega) \leq c_2 < +\infty \quad \text{for all } \Omega.$$

We also aim to study the so-called Blaschke-Santaló diagram for $\lambda(\Omega)$ and $T(\Omega)$. This consists in identifying the subset $E \subset \mathbf{R}^2$

$$E = \Big\{(x, y) \;:\; x = T(\Omega),\; y = \lambda(\Omega)\Big\}$$

where Ω runs among the admissible sets. In this way, minimizing a quantity like

$$F\Big(T(\Omega), \lambda(\Omega)\Big)$$

is reduced to the optimization problem in $\mathbf{R}^2$

$$\min\Big\{F(x, y) \;:\; (x, y) \in E\Big\}.$$

The difficulty consists in the fact that characterizing the set E is hard. Here we only give some bounds by studying the inf and sup of $\lambda(\Omega)T^q(\Omega)$ when $|\Omega| = m$.

Since the two quantities scale as:

$$T(t\Omega) = t^{d+2}T(\Omega), \qquad \lambda(t\Omega) = t^{-2}\lambda(\Omega)$$

we may remove the constraint $|\Omega| = 1$ and consider the scaling free shape functional

$$F_q(\Omega) = \frac{\lambda(\Omega)T^q(\Omega)}{|\Omega|^{(dq+2q-2)/d}}$$

that we consider on various classes of admissible domains.

Research made with

- Michiel van den Berg
University of Bristol, UK

- Aldo Pratelli
Università di Pisa, Italy

generalization to the p-Laplacian made with

- Francesca Prinari
Università di Pisa, Italy

- Luca Briani
Università di Pisa, Italy

Some big names from the past:

George Pólya (1887–1985)

Gábor Szegö (1895–1985)

Endre Makai (1915–1987)

Joseph Hersch (1925–2012)

Hans F. Weinberger (1928–2017)

For the relations between $T(\Omega)$ and $\lambda(\Omega)$:

- Kohler-Jobin ZAMP 1978 (L. Brasco COCV 2014 for the nonlinear case)

- van den Berg, B., Velichkov in Birkhäuser 2015

- van den Berg, Ferone, Nitsch, Trombetti Integral Equations Operator Theory 2016

- Lucardesi, Zucco preprint

The Blaschke-Santaló diagram has been studied for other pairs of quantities:

- for $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ by D. Bucur, G.B., I. Figueiredo (SIAM J. Math. Anal. 1999);
- for $\lambda_1(\Omega)$ and $\mathrm{Per}(\Omega)$ by I. Ftouhi, J. Lamboley (on `HAL`);
- for $T(\Omega)$ and $\mathrm{cap}(\Omega)$ by M. van den Berg, G.B. (on `arxiv` and `cvgmt`);
- for $T(\Omega)$ and $\mathrm{Per}(\Omega)$ by L. Briani, G.B., F. Prinari (on `arxiv` and `cvgmt`).

We start by considering the class of all domains (with $|\Omega| = 1$). The crucial thresholds are:

• $q = 2/(d+2)$ in which the minimum of $\lambda(\Omega)T^q(\Omega)$ is reached when Ω is a ball (Kohler-Jobin 1978);
• $q = 1$ in which (Pólya inequality)

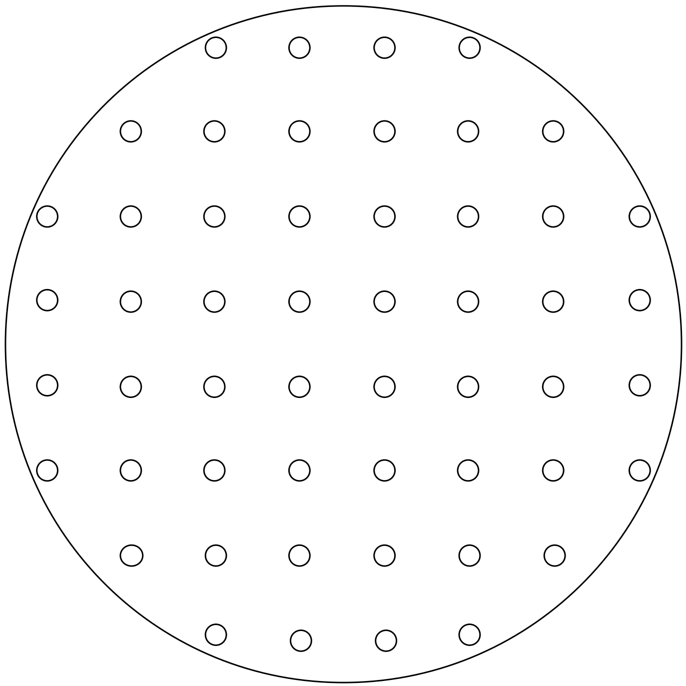
$$0 < \lambda(\Omega)T(\Omega) < 1.$$

Actually, we have $\sup \lambda(\Omega)T(\Omega) = 1$ and a maximizing sequence is made by finely perforated domains.

The finely perforated domains:

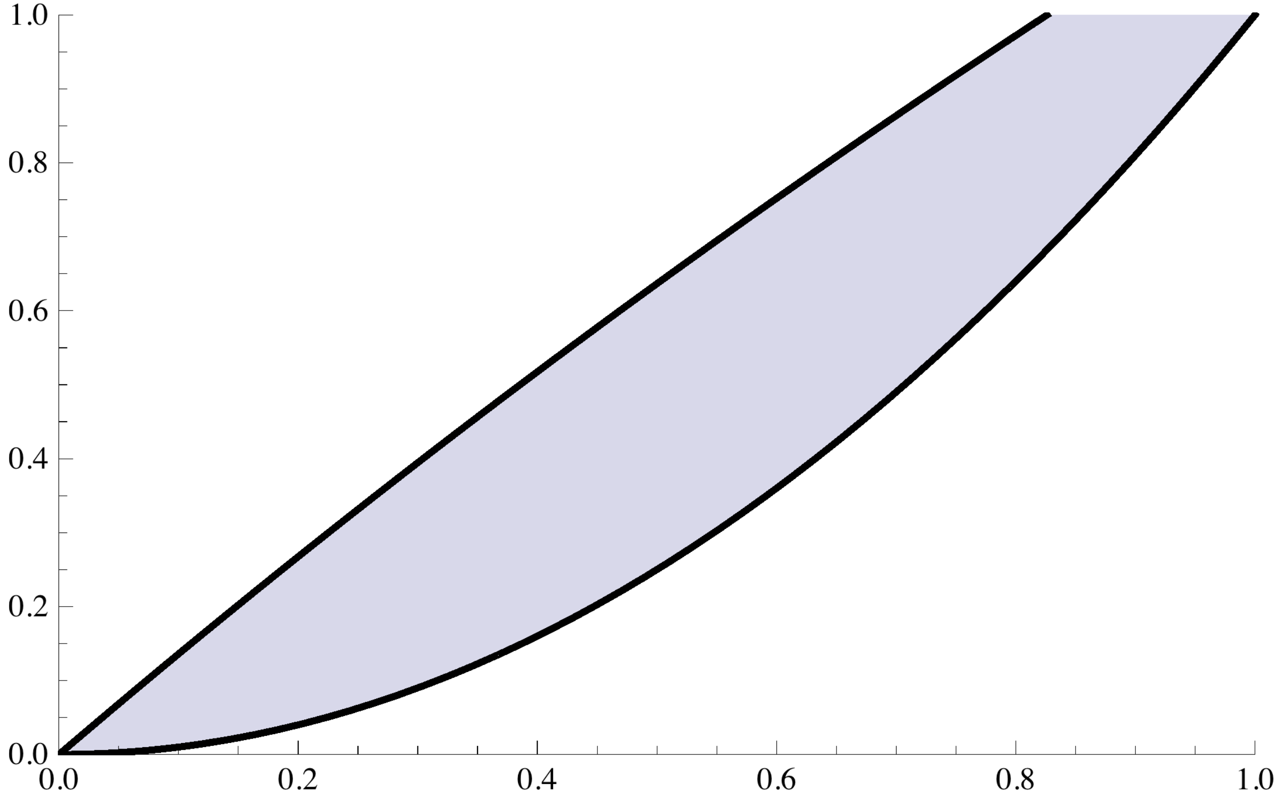
$\varepsilon =$ distance between holes $\quad r_\varepsilon =$radius of a hole

$r_\varepsilon \sim \varepsilon^{d/(d-2)}$ if $d > 2$, $\quad r_\varepsilon \sim e^{-1/\varepsilon^2}$ if $d = 2$.

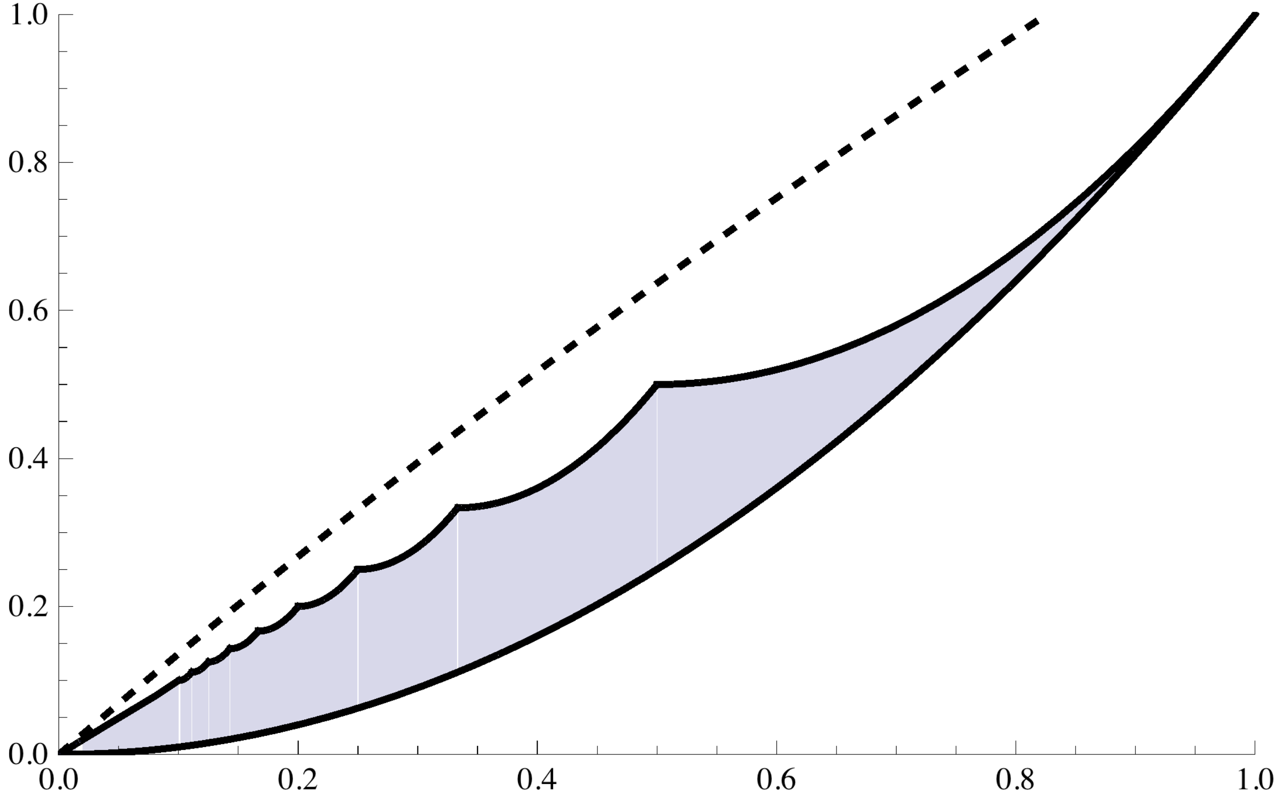


Summarizing: for **all domains** we have

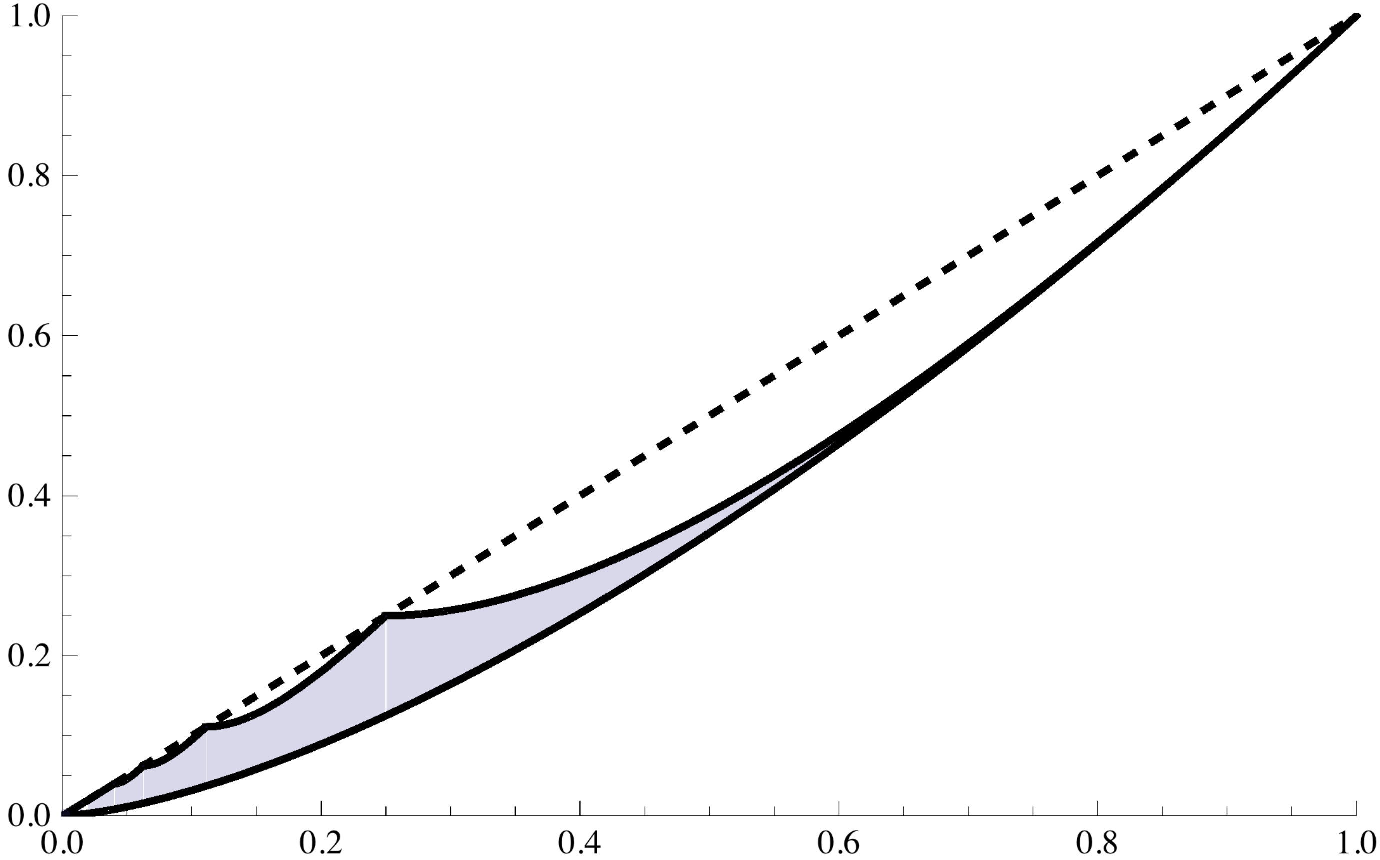
	General domains Ω	
$0 < q \leq 2/(d+2)$	$\min F_q(\Omega) = F_q(B)$	$\sup F_q(\Omega) = +\infty$
$2/(d+2) < q < 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) = +\infty$
$q = 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) = 1$
$q > 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) < +\infty$



The Blaschke-Santaló diagram with $d = 2$, for $x = \lambda(B)/\lambda(\Omega)$ and $y = T(\Omega)/T(B)$ is contained in the colored region.



In the Blaschke-Santaló diagram with $d = 2$, the colored region can be reached by domains Ω made by union of disjoint disks.



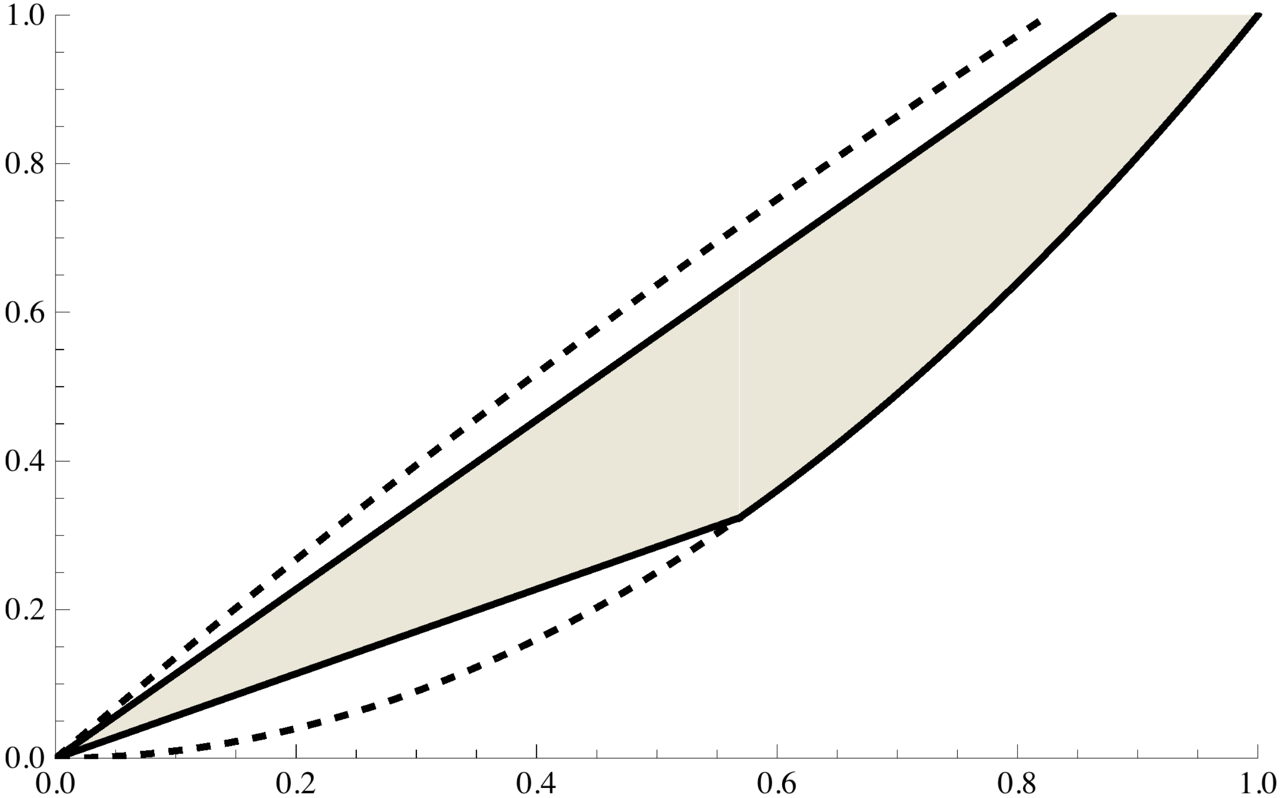
The full Blaschke-Santaló diagram in the case $d = 1$, where $x = \pi^2/\lambda(\Omega)$ and $y = 12\,T(\Omega)$.

The case Ω convex

If we consider only convex domains Ω, the Blaschke-Santaló diagram is clearly smaller. For the dimension $d = 2$ the conjecture is

$$\frac{\pi^2}{24} \le \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \le \frac{\pi^2}{12} \qquad \text{for all } \Omega$$

where the left side corresponds to Ω a thin triangle and the right side to Ω a thin rectangle.



If the Conjecture for convex domains is true, the Blaschke-Santaló diagram is contained in the colored region.

At present the only available inequalities are the ones of [BFNT2016]: for every $\Omega \subset \mathbf{R}^2$ convex

$$0.2056 \approx \frac{\pi^2}{48} \le \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \le 0.9999$$

instead of the bounds provided by the conjecture, which are

$$\begin{cases} \pi^2/24 \approx 0.4112 & \text{from below} \\ \pi^2/12 \approx 0.8225 & \text{from above.} \end{cases}$$

In dimensions $d \geq 3$ the conjecture is

$$\frac{\pi^2}{2(d+1)(d+2)} \leq \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \leq \frac{\pi^2}{12}$$

- the right side asymptotically reached by a thin slab

$$\Omega_\varepsilon = \Big\{(x', t) \ : \ 0 < t < \varepsilon\Big\}$$

  with $x' \in A_\varepsilon$, being A_ε a $d-1$ dimensional ball of measure $1/\varepsilon$

- the left side asymptotically reached by a thin cone based on A_ε above and with height $d\varepsilon$.

The convexity assumption on the admissible domains provides a strong extra compactness that allows to prove the existence of optimal domains in the cases:

$$\begin{cases} \max\Big\{\lambda(\Omega)T^q(\Omega)\ :\ \Omega \text{ convex, } |\Omega| = 1\Big\} \text{ if } q > 1 \\ \min\Big\{\lambda(\Omega)T^q(\Omega)\ :\ \Omega \text{ convex, } |\Omega| = 1\Big\} \text{ if } q < 1. \end{cases}$$

This is obtained by showing that maximizing (resp. minimizing) sequences Ω_n are not too thin, in the sense that

$$\frac{\text{inradius}(\Omega_n)}{\text{diameter}(\Omega_n)} \geq c_{d,q}\,,$$

where $c_{d,q} > 0$ depends only on d and q.

Summarizing: for **convex domains** we have

	Convex domains Ω	
$q < 1$	$\min F_q(\Omega) > 0$	$\sup F_q(\Omega) = +\infty$
$q = 1$	$\inf F_1(\Omega) = C_d^- > 0$	$\sup F_1(\Omega) = C_d^+ < 1$
$q > 1$	$\inf F_q(\Omega) = 0$	$\max F_q(\Omega) < +\infty$

The only case in which the conjecture has been proved (van den Berg-B.-Pratelli) is the case of thin domains, that is

$$\Omega_\varepsilon = \Big\{(s,t)\ :\ s \in A,\ \varepsilon h_-(s) < t < \varepsilon h_+(s)\Big\}$$

where ε is a small positive parameter and h_-, h_+ are two given functions ($h = h_+ - h_-$ is the local thickness function).

By using the asymptotics (as $\varepsilon \to 0$):

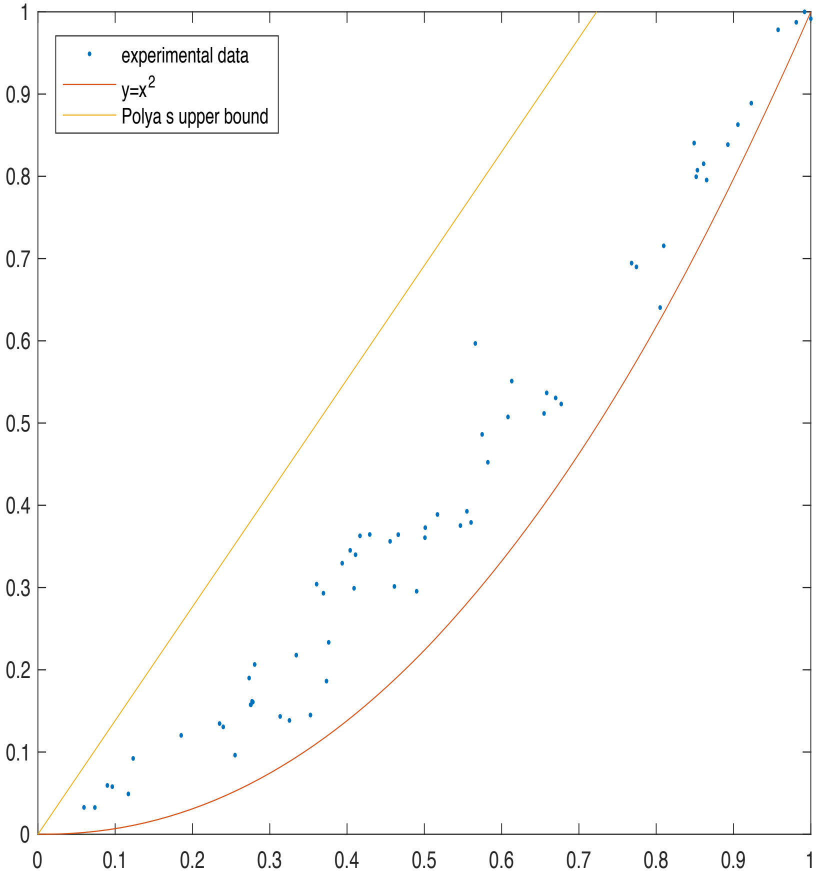
$$\lambda(\Omega_\varepsilon) \approx \frac{\varepsilon^{-2}\pi^2}{\|h\|^2_{L^\infty}} \quad \text{[Borisov-Freitas 2010]}$$

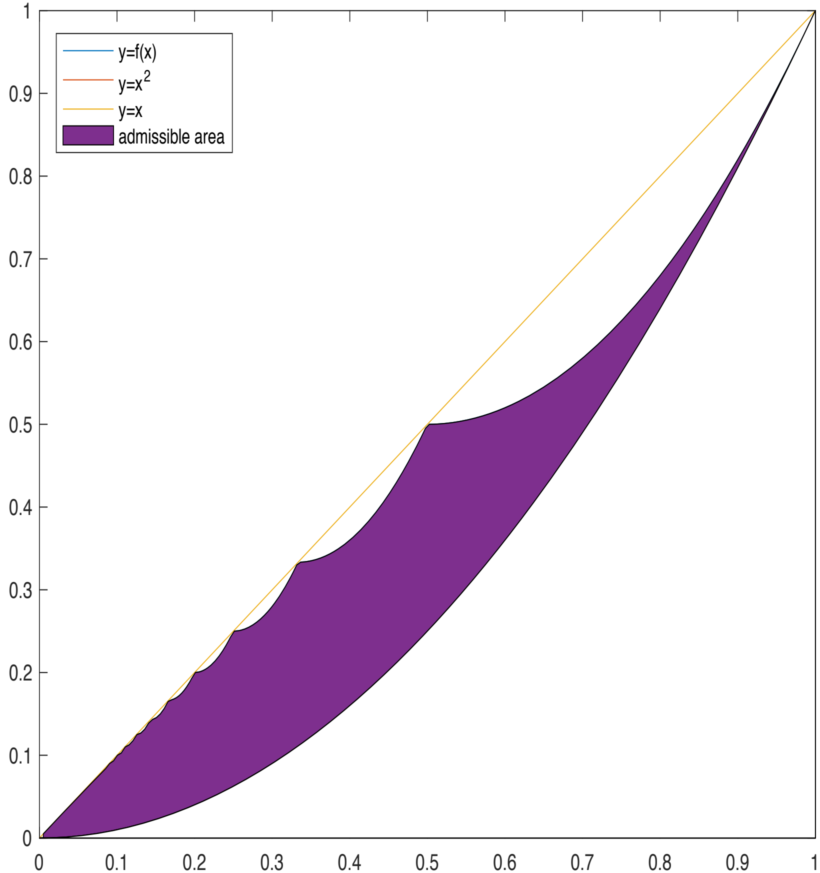
$$T(\Omega_\varepsilon) \approx \frac{\varepsilon^3}{12} \int h^3(s)\, ds \qquad |\Omega_\varepsilon| \approx \varepsilon \int h(s)\, ds.$$

the problem is reduced to the optimization of a quantity depending on h:

$$\frac{\lambda(\Omega_\varepsilon)T(\Omega_\varepsilon)}{|\Omega_\varepsilon|} \approx \frac{\pi^2}{12}\ \frac{\int h^3(s)\, ds}{\|h\|^2_{L^\infty} \int h\, ds}\ .$$

The proof then uses the convexity of Ω_ε (concavity of h) and a kind of reverse Hölder inequality.





Plot of 100 experimental domains (left), union of disks (right).

We can show that the Blaschke-Santaló diagram is the region between two graphs:

the function $y = x^{(d+2)/2}$ from below (Kohler-Jobin bound);

a suitable function $y = h(x)$ from above, where $h : [0,1] \to [0,1]$ is increasing and with

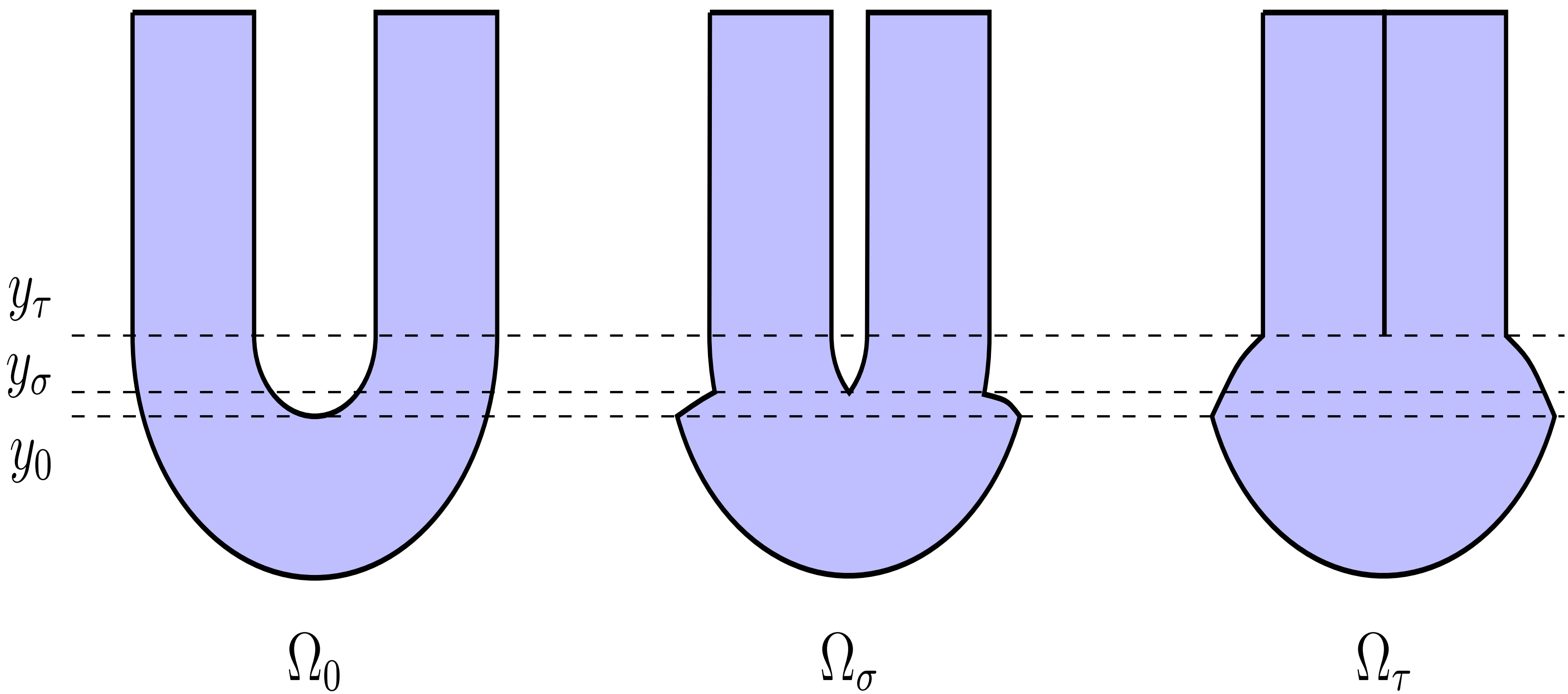
$$x^{(d+2)/2}\left(\left[x^{-d/2}\right] + \left(x^{-d/2} - \left[x^{-d/2}\right]\right)^{(d+2)/d}\right) \le h(x) \le \frac{xd(d+2)^2}{2xd + (d+2)\lambda(B)}.$$

This is obtained by using the so-called *Continuous Steiner Symmetrization*, developed by F. Brock (1995). This consists in deforming a set Ω obtaining a family Ω_t with $t \in [0,1]$, with the properties:

$$\Omega_0 = \Omega, \qquad \Omega_1 = B, \qquad |\Omega_t| = |\Omega| \ \forall t$$

$$\lambda(\Omega_t) \text{ decreasing}, \qquad T(\Omega_t) \text{ increasing}.$$

Unfortunately, the map $t \mapsto \Big(\lambda(\Omega_t), T(\Omega_t)\Big)$ is not continuous in general (only if Ω is convex) because this phenomenon may occur.

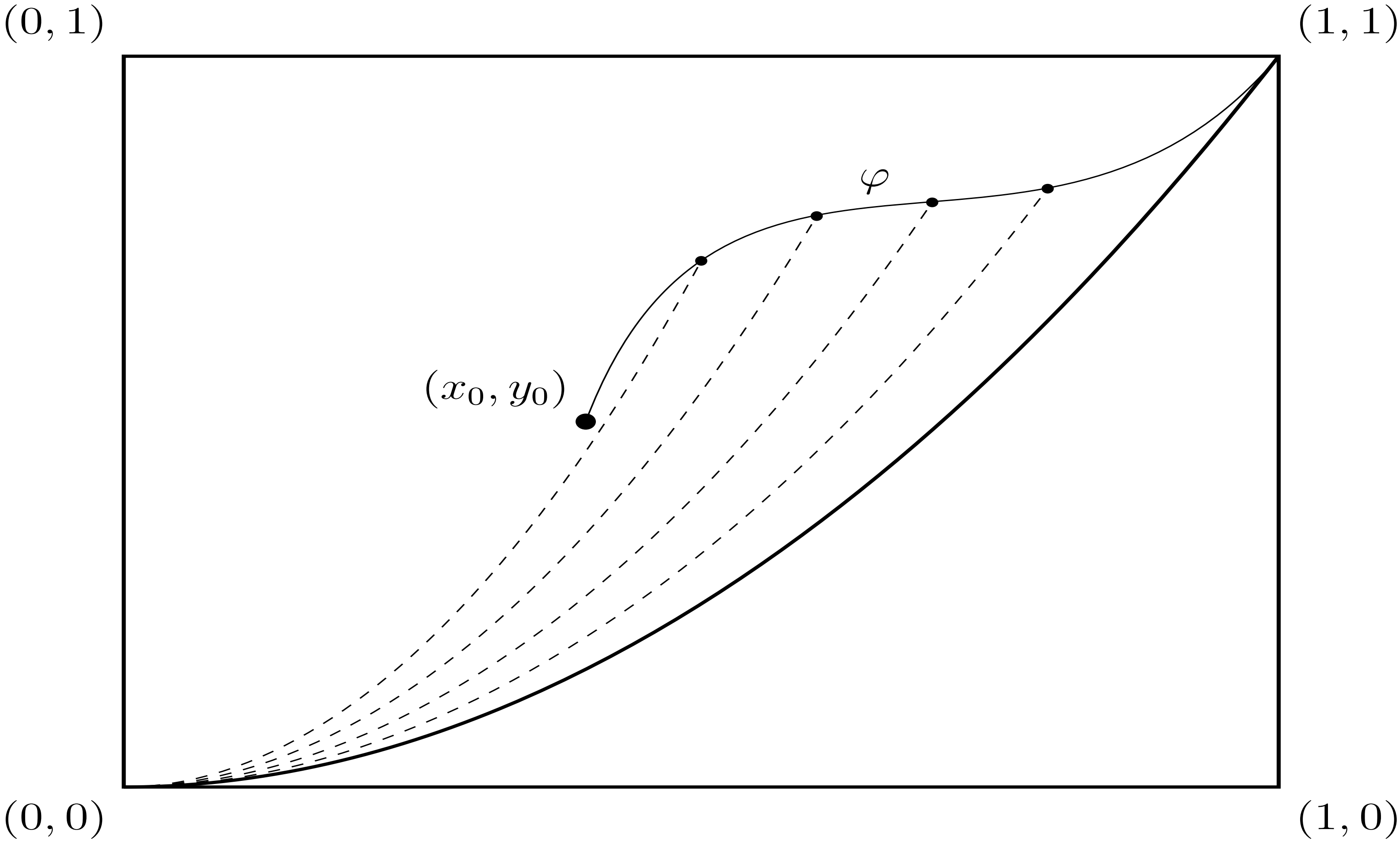


Discontinuities occur when an internal "fracture" appears.

It would be very interesting to obtain a deformation $t \mapsto \Omega_t$ really continuous, that we (A. Pratelli and I) believe possible. Nevertheless, we can show that this is true for a dense family of sets, namely for every polyhedral domain Ω.

This is enough to conclude that the Blaschke-Santaló diagram E is the region between two graphs, because we can prove that:

$$\begin{cases} E \text{ is convex horizontally} \\ E \text{ is convex vertically} \end{cases}$$



Horizontal and vertical convexity of the Blaschke-Santaló diagram.

Open questions

- characterize $\sup_{|\Omega|=1} \lambda(\Omega)T^q(\Omega)$ when $q > 1$;

- prove (or disprove) the conjecture for convex sets;

- simply connected domains or star-shaped domains? The bounds may change;

- full Blaschke-Santaló diagram;

- p-Laplacian instead of Laplacian?

- efficient experiments (random domains?).

The case $p=\infty$. We have as $p\to\infty$

$$\begin{cases}\Big(T_p(\Omega)\Big)^{1/p}\to\displaystyle\int_\Omega \mathrm{dist}_{\partial\Omega}\,dx\\ \Big(\lambda_p(\Omega)\Big)^{1/p}\to\|\,\mathrm{dist}_{\partial\Omega}\,\|_\infty^{-1}\end{cases}$$

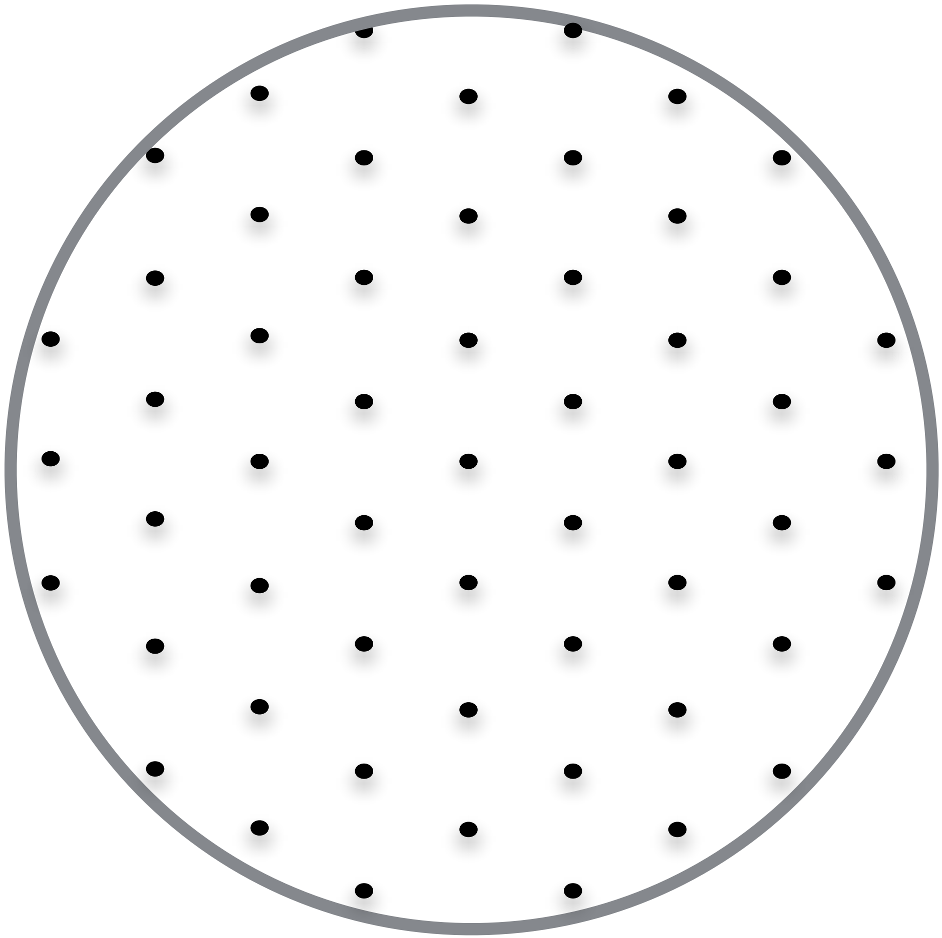
so that for the limit shape functional we have

$$\left(\frac{\lambda_p(\Omega)T_p(\Omega)}{|\Omega|^{p-1}}\right)^{1/p}\to\frac{\int_\Omega \mathrm{dist}_{\partial\Omega}\,dx}{|\Omega|\|\,\mathrm{dist}_{\partial\Omega}\,\|_\infty}=F_\infty(\Omega).$$

Problem *Is it true that (when $d=2$)*

$$\sup_\Omega F_\infty(\Omega)=\frac{1}{|E|}\int_E|x|\,dx=\frac{1}{3}+\frac{\log 3}{4}\approx 0.608$$

where E is the regular unitary exagon?



A planar domain that should asymptotically give the supremum of F_∞.