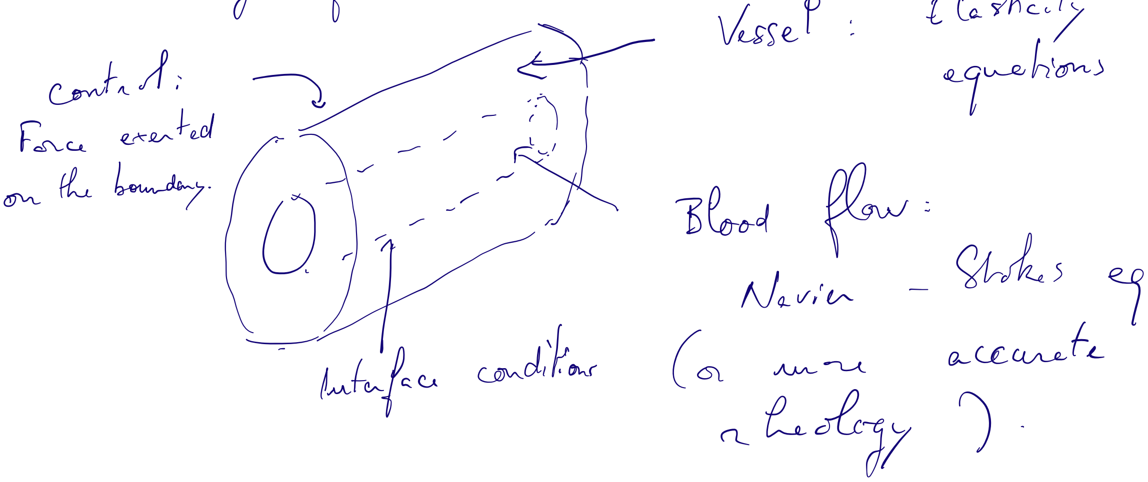
Controllability and resolvent estimates for well-posed systems.

I Motivation

1 A fluid - structure model.

Modelling of a blood flow in an elastic vessel.



Controllability? Widely open.

In fact, even well - posedness is very delicate.

Many difficulties:

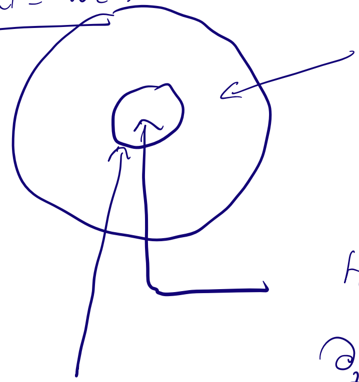
* Moving domains ⇝ Strong non-linearity
* Coupling of hyperbolic and parabolic equations

→ Regularity properties do not match.

Ref for well-posedness: [Coutand Shkoller 2005]
and more

(2) Simplified models

(a) Heat-Wave.



Wave equation
$$\partial_{tt} y - \Delta y = 0 \quad (0,T)\times(B(R_{ext})\setminus\overline{B(R_{in})})$$

Heat equation
$$\partial_t u - \Delta u = 0. \quad (0,T)\times B(R_{in})$$

Interface boundary conditions.
$$u = \partial_t y \quad \text{in } (0,T)\times S(R_{in})$$
$$\partial_\nu u = \partial_\nu y$$

Control $\quad u = v \quad$ in $(0,T)\times S(R_{ext})$.

→ What are the control properties of this model?

- In 1d. [Zhang - Zuazua 2003]: Null-controllability properties hold if $T > 2(R_{ext} - R_{in})$.
Proof by lateral propagation of the energy.

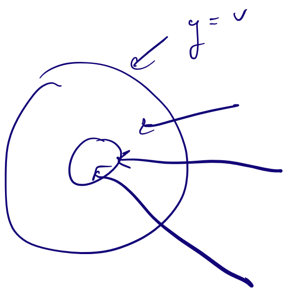
NB: Exact controllability cannot hold.

• In multi-d [Rauch Zhang Zuazua '05, Zhang Zuazua '07, Kamoun-Fatollah 2011] the stability of the model (and variants) is analyzed and is shown to depend on the geometry of the wave domain and of the heat domain.

• Still controllability of the multi-d model is open even in the simple above geometric setting.

(b) Simpler models: Waves with boundary conditions

• Replacing the heat by the Laplacian (low Reynolds, high viscosity)



$$\partial_{tt} y - \Delta y = 0.$$

$$\begin{cases} \partial_t y = v \\ \partial_\nu y = \partial_\nu v. \end{cases}$$

$$\Delta v = 0.$$

N.B. Solving $\Delta v = 0$ in $B(R_{int})$, $\partial_\nu v = g$ on $S(R_{int})$

$$v\big|_{S(R_{int})} = -(-\Delta)^{1/2} g.$$

$\Rightarrow$ the interface conditions read

$$\boxed{\partial_t y = -(-\Delta_s)^{1/2} \partial_\nu y}$$ (Here ν goes from the wave to the heat)

Rk: if $v = 0$ (no control).

$$\frac{d}{dt}\left(\frac{1}{2}\int (|\partial_t y|^2 + |\nabla y|^2)\right) - \int_{S(R_{int})} \partial_\nu y \, \partial_t y = 0.$$

$$\rightsquigarrow \frac{d}{dt}\left(\frac{1}{2}\int (|\partial_t y|^2 + |\nabla y|^2)\right) + \left\| (-\Delta_s)^{1/4} \partial_\nu y \right\|^2_{L^2(S)} = 0.$$

$\rightsquigarrow$ Damped wave equation damping term on the boundary.

Control properties? Approximate controllability holds if $T > 2(R_{ext} - R_{int})$.
(Holmgren or Spherical Harmonics + lateral propagation)

Null controllability: open pb.

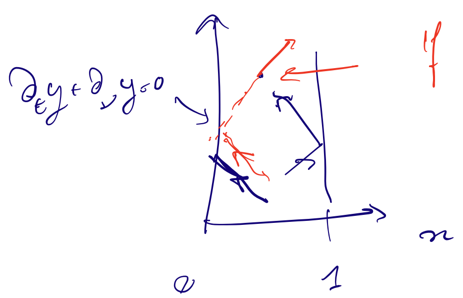
• A somewhat simpler and more classical model.

$$\begin{cases} \partial_{tt} y - \Delta y = 0 & \text{in } (0,T) \times B(R_{ext}) \setminus \overline{B(R_{int})} \\ y = v & \text{in } (0,T) \times S(R_{ext}) \\ \partial_t y + \partial_\nu y = 0 & \text{on } (0,T) \times S(R_{int}) \end{cases}$$

: "Absorbing boundary conditions". [Engquist - Majda 77]

. Here, propagation of singularities with these boundary conditions is known: One can propagate only in one direction of time.

In 1d



If this point of phase space has positive energy then the bicharacteristic before has positive energy

This is not enough to prove null-controllability

Indeed: [Bardos - Lebeau - Rauch] requires propagation of singularities + well-posedness in both directions of time (in particular in the compactness argument)

and all proofs I know use at some point that the eq can be solved backward in time !!

Not the case here.

→ Even the null-controllability of this model seems to be open.

Rk: In all the above models, due to the specific geometric setting, it is reasonable to hope for precise resolvent estimates.

(see e.g. [Bardos–Dardé & Mercado 2022]).

II Abstract setting and resolvent estimates.

Abstract setting.

$$\begin{cases} y' = Ay + Bv. \\ y(0) = y_{in}. \end{cases}$$

- A generates a C^0 semi-group on X.
- $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$

and B admissible.

$$\left(\exists t>0,\ v \longmapsto \int_0^t e^{(t-s)A} B v(s)\, ds. \right).$$
$$L^2(0,t;U) \longrightarrow X$$

All the above linearized equations fit this framework.

Q: Can we read null controllability from resolvent estimates?

Some partial answer is known.

① Necessary and sufficient condition for null / exact controllability when $\boxed{A^* = -A}$

→ Hautus test. [Miller '04, Tucsnak-Weiss book].

$\exists m, M > 0, \forall \omega \in \mathbb{R}, \forall y \in \mathcal{D}(A),$

$$\|y\|_X \leq M \|(A - i\omega) y\|_X + m \|B^* y\|_U .$$

→ Wave packets criterion. [Ramdani-Takahashi-Tenenbaum-Tucsnak]

if A has compact resolvent. $\exists$ basis of orthonormal functions $(\phi_j)_{j \in \mathbb{N}}$ with corresponding eigenvalues $(i\lambda_j)_{j \in \mathbb{N}}$

$$\exists \gamma, r > 0, \quad \forall \omega \in \mathbb{R}, \quad \forall (a_j) \in \ell^2(\mathbb{N}),$$

$$\Big\| B^* \Big(\sum_{|\lambda_j - \omega| < r} a_j \phi_j \Big) \Big\|_U \geq \gamma \Big\| \sum_{|\lambda_j - \omega| < r} a_j \phi_j \Big\|_X$$

Rk: In both cases, the time of controllability provided by these techniques is not sharp.

Can be adapted to the case $(e^{tA})_{t \geq 0}$ generates a group [Jacob Zwart '09].

② Parabolic case $\quad A^* = A < 0$.

Cf [Duyckaerts Miller JFA '12]

. Sufficient conditions for null-controllability

Thm If $\exists M \geq 0$, $\forall v \in \mathcal{D}(A)$, $\forall \omega > 0$,
$\exists \delta \in [0,1)$,

$$\|v\|^2 \leq M^2 \omega^\delta \left(\frac{1}{\omega} \|(A-\omega)v\|^2 + \|B^* v\|^2 \right).$$

Then null-controllability holds with cost $\leq C e^{C/T^\beta}$.

. A necessary condit.

Thm: If null-cont. holds in some time / any time with cost $e^{C/T}$.

Then $\exists a > 0$, $\forall v \in \mathcal{D}(A)$, $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$

$$\|v\|^2_\lambda \leq a e^{a(\mathrm{Re}\,\lambda)^{1/2}} \left(\|(A-\lambda)v\|^2_\times + \|B^* v\|^2_U \right)$$

$a e^{a(\mathrm{Re}\,\lambda)}$

Rk. Sometimes close to a necessary and sufficient condit. if adding additional spectral estimates

cf [Ammar-Khodja Benabdallah González Burgos Morancey 2019].

N.B. For parabolic equation

Lebeau-Robbiano spectral estimates

$\Rightarrow$ Null-controllability.

cf [Lebeau-Robbiano '96, Miller '06]. ....

3 Mixed hyperbolic parabolic models - More general models.

N.B. One can prove controllability results for several models of mixed hyperbolic-parabolic behavior.

[Lebeau – Zuazua '86]   thermoélasticité

[Albano – Tataru '02].   thermoelasticity

[Chaves Silva – Rosier – Zuazua]
viscoelasticity,
moving control.

[Ervedoza Guerrero '12, '16].
+ Poel
Comp. Navier – Stokes.

[Beauchard Lebeau Koenig 2020]
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Still no characterization in terms of resolvent for general models.