Sharp and quantitative estimates for the p-Torsion of convex sets

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The Torsion Problem

Let $\Omega \subset \mathbb{R}^n$ an open, bounded and convex set. The torsional rigidity $T(\Omega)$ is defined as

$$T(\Omega) = \int_\Omega u(x)\, dx,$$

where u is the unique solution of the PDE problem

$$\begin{cases} -\Delta u(x) = 1 & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

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Variational characterization of Torsional Rigidity

$$T(\Omega) = \max_{\substack{\varphi \in H_0^1(\Omega) \\ \varphi \not\equiv 0}} \frac{\left(\displaystyle\int_\Omega |\varphi(x)|\, dx \right)^2}{\displaystyle\int_\Omega |\nabla \varphi(x)|^2\, dx}$$

Lower and upper estimates for the Torsion in terms of area and perimeter in the plane

Let us consider Ω an open, bounded and convex subset of $\mathbb{R}^2$ and we denote by $|\Omega|$ the volume of Ω and by $P(\Omega)$ its perimeter.

Theorem [Pólya, *J. Indian Math Soc*, (1960)]

It holds:

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} \geq \frac{1}{3}$$

and the equality sign is attained by a sequence of thinning rectangles.

Theorem [Makai, *Studies in math. analysis*, (1962)]

It holds:

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} \leq \frac{2}{3}$$

and the equality sign is attained by a sequence of thinning triangles.

Generalization to the $p-$Laplacian

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex set, $p \in (1, +\infty)$ and consider

$$-\Delta_p u := -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

We consider the following problem:

$$\begin{cases} -\Delta_p u(x) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and define the $p-$Torsional rigidity of Ω as

$$T_p(\Omega) = \max_{\substack{\varphi \in W_0^{1,p}(\Omega) \\ \varphi \not\equiv 0}} \frac{\left(\int_\Omega |\varphi(x)|\, dx\right)^{\frac{p}{p-1}}}{\left(\int_\Omega |\nabla\varphi(x)|^p\, dx\right)^{\frac{1}{p-1}}}.$$

Upper and lower estimates for the $p-$Torsional rigidity in the plane

We recall the following scaling propertiesfor every $t > 0$:

$$|t\Omega| = t^n|\Omega|, \qquad P(t\Omega) = t^{n-1}P(\Omega)$$

and

$$T_p(t\Omega) = t^{n+q}T_p(\Omega).$$

Theorem [Fragalá-Gazzola-Lamboley, *Geom. for parabolic and elliptic PDE's*, (2013)]

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and convex set. Then,

$$\frac{1}{q+1} \leq \frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} \leq \frac{2^{q+1}}{(q+2)(q+1)}, \qquad q = \frac{p}{p-1},$$

where the lower bounds holds asymptotically for a sequence of thinning rectangles, while the upper bounds for a sequence of thinning triangles.

Generalization in dimension n of the lower bound

Theorem [Della Pietra-Gavitone, *Math. Nachr.*, 2014]

Let Ω be an open, bounded and convex set of $\mathbb{R}^n$. It holds:

$$\frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} \geq \frac{1}{q+1}, \qquad q = \frac{p}{p-1}.$$

and equality holds for a sequence of thinning cylinders.

Further results

Let us define the functional

$$H_k(\Omega) = \frac{T^k(\Omega)P(\Omega)}{|\Omega|^{\alpha_k}} \qquad \alpha_k = 1 + k + \frac{2k-1}{n}, \qquad k > 0.$$

Theorem [Briani-Buttazzo-Prinari, *Applied Math. Opt.*, 2021]

Among open, bounded and convex sets $\Omega \subset \mathbb{R}^n$, $H_k(\cdot)$ is bounded if and only if $k = 1/2$. More precisely, it holds

$$\frac{1}{\sqrt{3}} \leq H_{\frac{1}{2}}(\Omega) \leq \frac{2^n n^{3n/2}}{\omega_n} \left(\frac{n}{n+2}\right)^{\frac{1}{2}},$$

and the lower bound is asymptotically achieved by a sequence of thinning cylinders.

Further results

Conjecture in dimension n, [Briani-Buttazzo-Prinari, *Applied Math. Opt.*, 2021]

Let Ω be an open, bounded and convex sets of $\mathbb{R}^n$, then

$$H_{\frac{1}{2}}(\Omega) \leq n \left(\frac{2}{(n+1)(n+2)} \right)^{\frac{1}{2}}$$

and equality is reached by sets of the form

$$\Omega_\varepsilon = \{(s,t) \,|s \in A, 0 < t < \varepsilon(1-|s|)\},$$

where A is the unit ball in $\mathbb{R}^{n-1}$.

- For $n = 2$ it coincides with the upper bound proved by Makai.
- The authors prove the conjecture in the class of convex and thin domains in $\mathbb{R}^n$.

Stability Issue

From the Polya inequality follows that along a sequence of thinning cylinders $\{\Omega_l\}_{l\in\mathbb{N}}$, we have

$$\frac{T_p(\Omega)P^q(\Omega)}{|\Omega|^{q+1}} = \mathcal{F}_p(\Omega_l) \xrightarrow{l\to 0} c_p.$$

This leads to the following stability issue: if $\mathcal{F}_p(\Omega)$ is close to c_p, can we say that Ω is close in some sense to a thin cylinder?

Some Definitions

Let us denote by w_Ω the minimal width and by $\mathrm{diam}(\Omega)$ the diameter of Ω.

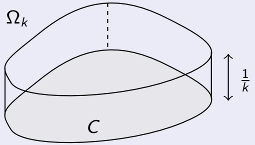
Definition

- Let Ω_k be a sequence of open, bounded and convex sets of $\mathbb{R}^n$. We say that Ω_k is a sequence of thinning domains if

$$\frac{w_{\Omega_k}}{\mathrm{diam}(\Omega_k)} \xrightarrow{k\to 0} 0.$$

- In particular, if $k > 0$ and C is an open, bounded and convex set of $\mathbb{R}^{n-1}$ with unitary $(n-1)$-dimensional measure, then, if $k \to 0$, the sequence

$$\Omega_k = C \times \left[-\frac{1}{2k}, \frac{1}{2k}\right]$$



is called a sequence of thinning cylinders. Moreover, in the case $n = 2$ the above sequence is called sequence of thinning rectangles.

A first quantitative result in the planar case

We define the following scaling invariant functional

$$\mathcal{F}_p(\Omega) = \frac{T_p(\Omega) P^q(\Omega)}{|\Omega|^{q+1}}, \qquad q = \frac{p}{p-1}.$$

Theorem 2 [Amato, Masiello, P., Sannipoli, preprint on arxiv (2021)]

Let Ω be an open, bounded and convex set of $\mathbb{R}^2$. Then,

$$\mathcal{F}_p(\Omega) - c_p \geq K(p) \frac{w_\Omega}{\mathrm{diam}(\Omega)},$$

where $K(p)$ is a positive constant that can be computed explicitly:

$$K(p) = \frac{(p-1)p}{2^{\frac{p}{p-1}} 3(3p-2)(2p-1)}.$$

Moreover, the exponent of the quantity $\frac{w_\Omega}{\mathrm{diam}(\Omega)}$ is sharp.

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Remark. We have obtained a quantitative result of this type also for $n > 2$.

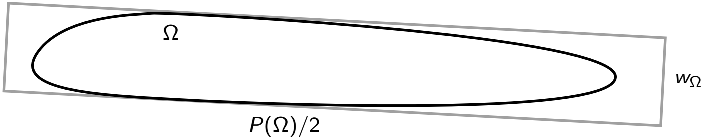
A second quantitative result in the planar case

Theorem 3 [Amato, Masiello, P., Sannipoli, preprint on arxiv (2021)]

Let Ω be an open, bounded and convex set of $\mathbb{R}^2$ and let $p = 2$. Then, there exists a positive constant M such that

$$\mathcal{F}_2(\Omega) - c_2 = \frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} - \frac{1}{3} \geq M\left(\frac{|\Omega \triangle Q|}{|\Omega|}\right)^3,$$

where $\Omega \triangle Q$ denotes the symmetric difference between Ω and a rectangle Q with sides $P(\Omega)/2$ and w_Ω, containing Ω.



Some open problems

- Sharpness of the exponent of the asymmetry in Theorem 3.
- Extend the second quantitative results contained in Theorem 3 in dimension $n > 2$.
- Prove a quantitative result of the upper bound of the torsion proved by Makay, that is, for convex sets of $\mathbb{R}^2$, it holds:

$$\frac{T(\Omega)P^2(\Omega)}{|\Omega|^3} \leq \frac{2}{3}$$

and the equality sign is attained by a sequence of thinning triangles.

Thank you for your attention!