# On the regularity of the optimal shapes for a class of integral functionals

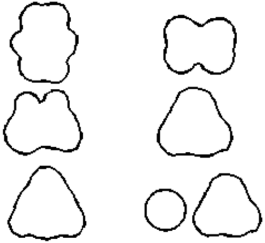
Bozhidar Velichkov
(Università di Pisa)

based on joint works with Giuseppe Buttazzo (Università di Pisa),
Francesco Paolo Maiale (Scuola Normale Superiore),
Dario Mazzoleni (Università di Pavia), Giorgio Tortone (Università di Pisa)

VAREG
Horizon 2020
European Union Funding
for Research & Innovation

# Part I

# **Shape optimization**



A shape optimization problem is a variational problem of the form

$$\min\Big\{\mathcal{F}(\Omega)\ :\ \Omega\in\mathcal{A}\Big\}.$$

**The functional** $\mathcal{F}(\Omega)$ depends on the solution of a PDE in $\Omega$, for instance

- $\mathcal{F}(\Omega) = -\displaystyle\int_\Omega u(x)\,dx$, where $u$ is the solution to

$$-\Delta u = 1 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega;$$

- $\mathcal{F}(\Omega) = \displaystyle\int_\Omega |\nabla u|^2\,dx$, where $u$ is the solution to

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \qquad \int_\Omega u^2\,dx = 1.$$

- and might also involve quantities as the perimeter or the measure of $\Omega$.

A shape optimization problem is a variational problem of the form

$$\min\Big\{\mathcal{F}(\Omega)\ :\ \Omega\in\mathcal{A}\Big\}.$$

The family $\mathcal{A}$ of **admissible sets** can be:

- **all open sets** contained in some "box" $D\subseteq\mathbb{R}^d$ or in $\mathbb{R}^d$ itself;
- **all measurable sets** contained in some "box" $D\subseteq\mathbb{R}^d$ or in $\mathbb{R}^d$ itself;
- **all sets of measure at most** $1$ contained in some "box" $D\subseteq\mathbb{R}^d$ or in $\mathbb{R}^d$ itself;
- all sets of fixed perimeter contained in some "box" $D\subseteq\mathbb{R}^d$ or in $\mathbb{R}^d$ itself;
- all convex sets contained in some "box" $D\subseteq\mathbb{R}^d$ or in $\mathbb{R}^d$ itself;
- sets satisfying exterior ball condition;
- sets with uniformly Lipschitz boundary...

A shape optimization problem is a variational problem of the form

$$\min \Big\{ \mathcal{F}(\Omega) \ : \ \Omega \in \mathcal{A} \Big\}.$$

The most studied ones are:

- **spectral functionals** $\mathcal{F}(\Omega) = F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega))$;
- **integral functionals** $\mathcal{F}(\Omega) = \displaystyle\int_\Omega j(u_\Omega, x)\, dx$ , where $u_\Omega$ is the solution of

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega\,,$$

where $f(x)$ is a given function.

- in both cases $\mathcal{A}$ is the family of all **measurable sets** with measure $\leq m$, which are contained in some "box" $D \subseteq \mathbb{R}^d$ or in $\mathbb{R}^d$.

**Minimize** $\mathcal{F}(\Omega) = F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega))$ **among all** $\Omega \subseteq D, \ |\Omega| \leq 1,$

where $\lambda_j(\Omega)$ are the eigenfunctions of the Dirichlet Laplacian on $\Omega$.

**Why they are interesting?**

- **History:** easy answer when $\mathcal{F}(\Omega) = \lambda_1(\Omega)$ and $D = \mathbb{R}^d$ (*Faber-Krahn*);
- also when $\mathcal{F}(\Omega) = \lambda_2(\Omega)$ and $D = \mathbb{R}^d$ (*Krahn-Szegö*);
- **Physical interpretation:** *Can one hear the shape of the drum ?*
- **Geometry of the domain <–> behavior of the eigenfunctions**

$$\textbf{Minimize } \mathcal{F}(\Omega) = F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \textbf{ among all } \Omega \subseteq D, \ |\Omega| \le 1,$$

where $\lambda_j(\Omega)$ are the eigenfunctions of the Dirichlet Laplacian on $\Omega$.

**Existence of optimal sets:**

- Robust existence theory when $D$ is bounded: *Buttazzo-Dal Maso* (1993).
- Existence in $\mathbb{R}^d$: *Bucur* and *Mazzoleni-Pratelli* (2011).

**Minimize** $\mathcal{F}(\Omega) = F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega))$ **among all** $\Omega \subseteq D, \ |\Omega| \leq 1,$

where $\lambda_j(\Omega)$ are the eigenfunctions of the Dirichlet Laplacian on $\Omega$.

**Regularity of the free boundary:**

- $\lambda_1(\Omega)$ and $D$ bounded: *Briançon-Lamboley* (2011), *Russ-Trey-V.* (2017), using *Alt-Caffarelli* (1981), *De Silva* (2010), *Chang Lara-Savin* (2017), *Weiss* (2000);
- $\mathcal{F}(\Omega) = \lambda_2(\Omega) + |\Omega|$ and $D$ is bounded: *Mazzoleni-Trey-V.* (2020), using *Bucur-V.* (2011), *De Philippis-Spolaor-V.* (2020) and *Russ-Trey-V.* (2017);
- $\lambda_1(\Omega) + \cdots + \lambda_k(\Omega)$ - *Mazzoleni-Terracini-V.* (2017); *Kriventsov-Lin* (2017); *Caffarelli-Shahgholian-Yeressian* (2017); *Trey* (2020); preliminary results by: *Bucur-Mazzoleni-Pratelli-V.* (2013); *David-Toro* (2013).
- $\lambda_k(\Omega)$ and more general functionals: *Kriventsov-Lin* (2018) – it is only known that the "flat" free boundaries are smooth; cusps are not excluded.

**Minimize** $\mathcal{F}(\Omega) = F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$ **among all** $\Omega \subseteq D, \ |\Omega| \leq 1,$

where $\lambda_j(\Omega)$ are the eigenfunctions of the Dirichlet Laplacian on $\Omega$.

**Main steps in proving the regularity of the free boundary:**

• Prove that the optimal sets $\Omega$ and the state functions $u_\Omega$ solve an overdetermined boundary value problem (a free bounday problem). For instance, in the case of $\lambda_1$,

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega\,, \qquad |\nabla u| = 1 \quad \text{on} \quad \partial\Omega \cap D\,, \qquad |\nabla u| \geq 1 \quad \text{on} \quad \partial\Omega \cap \partial D.$$

*Briançon-Lamboley* (2011), *Russ-Trey-V.* (2017).

---

• Prove that the if $(\Omega, u_\Omega)$ solves the free boundary problem, then $\partial\Omega$ is smooth.
*Alt-Caffarelli* (1981), *De Silva* (2010), *Chang Lara-Savin* (2017).

$$\textbf{Minimize} \quad \mathcal{F}(\Omega) = \int_\Omega j(u_\Omega, x)\,dx \ \ \textbf{among all} \ \ \Omega \subseteq D, \ \ |\Omega| \le 1,$$

where $\quad u_\Omega = 0 \ \text{ on } \ \partial\Omega, \quad \text{and} \ \ -\Delta u_\Omega = f(x) \ \text{ in } \ \Omega.$

**Why integral functionals ?** Models in Physics and Engineering.

**Existence of optimal sets:**

Existence theory for $D$ bounded and $j$ monotone: *Buttazzo-Dal Maso* (1993), and when $j$ is a small perturbation of a monotone functional: *Buttazzo-V.* (2015), while in general, there are counterexamples: *Bucur-Buttazzo* (2005).

**What about the regularity of the free boundary ?**

**Minimize** $\mathcal{F}(\Omega) = \int_\Omega -\frac{1}{2} f(x) u_\Omega \, dx + |\Omega|$ **among all sets** $\Omega \subseteq D$,

where $u_\Omega = 0$ on $\partial\Omega$, $-\Delta u_\Omega = f(x)$ in $\Omega$, **and where we assume** $f \geq 0$**.**

**This problem is equivalent to minimizing**

$$\frac{1}{2}\int_D |\nabla u|^2 \, dx - \int_D f(x) u \, dx + |\{u > 0\}| \quad \textbf{among all functions } u \in H_0^1(D).$$

The Lipschitz continuity of a minimizer $u$,

and

the regularity of the positivity set $\{u > 0\}$,

then follow from: *Alt-Caffarelli* (1981), *Weiss* (1999), *Chang-Lara-Savin* (2017), *Caffarelli-Jerison-Kenig* (2004), *Jerison-Savin (2015).*

**To prove the equivalence of the two problems, take any $v \in H_0^1(D)$. Then:**

$$\begin{aligned}
\frac{1}{2}\int_D |\nabla v|^2 - \int_D f(x)v &= \frac{1}{2}\int_D |\nabla v_+|^2 + \frac{1}{2}\int_D |\nabla v_-|^2 - \int_D f(x)\big(v_+ - v_-\big) \\
&\geq \frac{1}{2}\int_D |\nabla v_+|^2 - \int_D f(x)v_+ \\
&\geq \frac{1}{2}\int_D |\nabla w|^2 - \int_D f(x)w = -\frac{1}{2}\int_D f(x)w,
\end{aligned}$$

where $w$ is the solution to $\quad -\Delta w = f \text{ in } \{v > 0\}, \quad w \in H_0^1(\{v > 0\})$. Finally

$$\begin{aligned}
\frac{1}{2}\int_D |\nabla v|^2 - \int_D f(x)v + |\{v > 0\}| &\geq -\frac{1}{2}\int_D f(x)w + |\{w > 0\}| \\
&\geq -\frac{1}{2}\int_D f(x)u_\Omega + |\Omega| = \frac{1}{2}\int_D |\nabla u_\Omega|^2 - \int_D f(x)u_\Omega + |\{u_\Omega > 0\}|.
\end{aligned}$$

**Minimize** $\mathcal{F}(\Omega) = \int_\Omega -\frac{1}{2} f(x) u_\Omega \, dx + |\Omega|$ **among all sets** $\Omega \subseteq D$,

where $u_\Omega = 0$ on $\partial\Omega$ and $-\Delta u_\Omega = f(x)$ in $\Omega$.

**This problem is equivalent to minimizing**

$\frac{1}{2} \int_D |\nabla u|^2 \, dx - \int_D f(x) u \, dx + |\{u \neq 0\}|$ **among all functions** $u \in H^1_0(D)$.

The Lipschitz continuity of a minimizer $u$,

and

the regularity of the positivity set $\{u > 0\}$,

follow from: *Alt-Caffarelli-Friedman* (1983), *De Philippis-Spolaor-V.* (2021),

...and all the results for the "one-phase" case $f \geq 0$.

**The problem: Minimize** $\mathcal{F}(\Omega) = \int_\Omega -g(x)u_\Omega\,dx + |\Omega|$ **among all sets** $\Omega \subseteq D$**,**

**where** $u_\Omega = 0$ on $\partial\Omega$ and $-\Delta u_\Omega = f(x)$ in $\Omega$.

**Theorem.** (*Buttazzo-Maiale-Mazzoleni-Tortone-V.* and *Maiale-Tortone-V.*).

Assume that $0 \le cf(x) \le g(x) \le Cf(x)$, for some $0 < c \le C$

... and suppose that $f, g \in C^\infty(\mathbb{R}^d)$ and that $\partial D$ is $C^{1,\alpha}$.

Then, the boundary $\partial\Omega$ of any optimal set $\Omega \subseteq D$ can be decomposed as:

$$\partial\Omega = \mathrm{Reg}(\partial\Omega) \cup \mathrm{Sing}(\partial\Omega)\,,$$

where:
- $\mathrm{Reg}(\partial\Omega)$ is a $C^{1,\alpha}$ manifold;
- $\mathrm{Sing}(\partial\Omega)$ is a closed set of Hausdorff dimension at most $d-5$.

**The optimal shapes are "outwards-minimizing" or "supersolutions".** Let $B \supseteq \Omega$.

$$-\int g(x)u_\Omega + |\Omega| \le -\int g(x)u_B + |B| \quad \Rightarrow \quad \int g(x)\big(u_B - u_\Omega\big) \le |B \setminus \Omega|;$$

$$c\int f(x)\big(u_B - u_\Omega\big) \le \int g(x)\big(u_B - u_\Omega\big) \le |B \setminus \Omega|;$$

$$c\int f(x)\big(u_B - u_\Omega\big) \le |B \setminus \Omega| \quad \Rightarrow \quad -\int f(x)u_\Omega + \frac{1}{c}|\Omega| \le -\int f(x)u_B + \frac{1}{c}|B|.$$

Thus, by *Bucur-Mazzoleni-Pratelli-V.* (2013) and *Briançon-Hayouni-Pierre* (2005), $u_\Omega$ **is Lipschitz and** $\Omega$ **is open.**

**The optimal shapes are "inwards-minimizing" or "subsolutions".** Let $\omega \subseteq \Omega$.

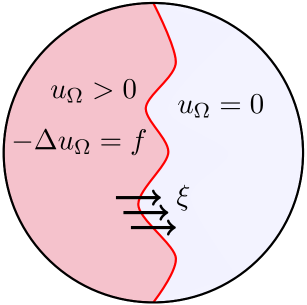
$$-\int g(x)u_\Omega + |\Omega| \le -\int g(x)u_\omega + |\omega| \quad \Rightarrow \quad |\Omega \setminus \omega| \le \int g(x)\big(u_\Omega - u_\omega\big);$$

$$|\Omega \setminus \omega| \le \int g(x)\big(u_\Omega - u_\omega\big) \le C\int f(x)\big(u_\Omega - u_\omega\big);$$

$$|\Omega \setminus \omega| \le C\int f(x)\big(u_\Omega - u_\omega\big) \quad \Rightarrow \quad -\int f(x)u_\Omega + \frac{1}{C}|\Omega| \le -\int f(x)u_\omega + \frac{1}{C}|\omega|.$$

Then, by *Bucur-V.* (2011), $u_\Omega$ is **non-degenerate**: there is $\eta > 0$ such that

$$\|u\|_{L^\infty(B_r)} \le \eta r \quad \Rightarrow \quad u \equiv 0 \ \text{ in } \ B_{r/2}.$$



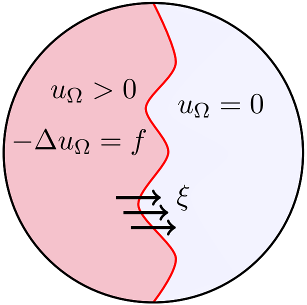

## First order variation

Let $\xi \in C_c^\infty(B_1; \mathbb{R}^d)$ be a smooth vector field.

Let $\Omega_t := (Id + t\xi)(\Omega)$.

Let $u_t \in H_0^1(\Omega_t)$ be the solution to:

$$-\Delta u_t = f(x) \quad \text{in} \quad \Omega_t, \qquad u_t = 0 \quad \text{on} \quad \partial\Omega_t$$




## First order variation

Let $\xi \in C_c^\infty(B_1; \mathbb{R}^d)$ be a smooth vector field.

Let $\Omega_t := (Id + t\xi)(\Omega)$.

Let $u_t \in H_0^1(\Omega_t)$ be the solution to:

$$-\Delta u_t = f(x) \quad \text{in} \quad \Omega_t, \qquad u_t = 0 \quad \text{on} \quad \partial\Omega_t$$

**Calculate** $\quad \delta\mathcal{F}(\Omega)[\xi] := \dfrac{d}{dt}\Big|_{t=0} \mathcal{F}(\Omega_t) = \dfrac{d}{dt}\Big|_{t=0} \displaystyle\int_{\Omega_t} j(u_{\Omega_t}, x)\,dx.$

**For simplicity:** $\quad \mathcal{F}(\Omega_t) = \displaystyle\int_{\Omega_t} j(u_{\Omega_t}, x)\,dx = -\int_{\Omega_t} g(x) u_{\Omega_t}\,dx$

Recall that $u_t \in H_0^1(\Omega_t)$ solves:

$$-\Delta u_t = f(x) \quad \text{in} \quad \Omega_t, \qquad u_t = 0 \quad \text{on} \quad \partial\Omega_t$$

Let $u'$ be the derivative of $u_t$ at $t = 0$. Then,

$$\Delta u' = 0 \quad \text{in} \quad \Omega\,, \qquad u' = -\xi \cdot \nabla u_\Omega \quad \text{on} \quad \partial\Omega.$$

Let $v_\Omega$ be the solution of

$$-\Delta v_\Omega = g(x) \quad \text{in} \quad \Omega\,, \qquad v_\Omega = 0 \quad \text{on} \quad \partial\Omega.$$

Then $$\frac{d}{dt}\Big|_{t=0} \int -g(x) u_{\Omega_t}\, dx = -\int_\Omega u' g(x)\, dx = \int_\Omega u' \Delta v_\Omega\, dx$$
$$= -\int_\Omega \nabla u' \cdot \nabla v_\Omega\, dx + \int_{\partial\Omega} u' \frac{\partial v_\Omega}{\partial n} = \int_{\partial\Omega} u' \frac{\partial v_\Omega}{\partial n}$$

$$\boxed{\delta\,\mathcal{F}(\Omega)[\xi] = \int_{\partial\Omega} u' \frac{\partial v_\Omega}{\partial n} = \int_{\partial\Omega} (-\xi \cdot \nabla u_\Omega) \frac{\partial v_\Omega}{\partial n} = \int_{\partial\Omega} (-\xi \cdot n) \frac{\partial u_\Omega}{\partial n} \frac{\partial v_\Omega}{\partial n}}$$

$$\delta \mathcal{F}(\Omega)[\xi] = \int_{\partial\Omega} u' \frac{\partial v_\Omega}{\partial n} = \int_{\partial\Omega} (-\xi \cdot \nabla u_\Omega) \frac{\partial v_\Omega}{\partial n} = \int_{\partial\Omega} (-\xi \cdot n) \frac{\partial u_\Omega}{\partial n} \frac{\partial v_\Omega}{\partial n}$$

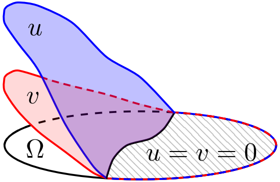
If $\Omega$ minimizes $\mathcal{F}(\Omega) = -\int_\Omega g(x) u \, dx$ among all sets of measure $|\Omega| = m$,

then $\Omega$, $u_\Omega$ and $v_\Omega$ are solutions to the system

$$\begin{cases} \Omega = \{u > 0\} = \{v > 0\} \\ -\Delta u = f(x) \quad \text{in} \quad \Omega \\ -\Delta v = g(x) \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial n} \dfrac{\partial v}{\partial n} = c \quad \text{on} \quad \partial\Omega \end{cases}$$

# Part II

# Free boundary systems



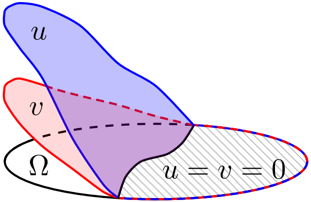
**Given:**

– *a ball* $B_1 \subseteq \mathbb{R}^d$;

– *a function* $G : \partial B_1 \to \mathbb{R}^2$,

– *an open set* $\Omega \subseteq B_1$,

– *continuous functions* $u, v : B_1 \to \mathbb{R}$,




**We say that** $(u, v, \Omega)$ **is a solution of**

**the free boundary system**

$$\begin{cases} \Omega = \{u > 0\} = \{v > 0\} \\ \Delta u = \Delta v = 0 \quad \text{in} \quad \Omega \cap B_1 \\ G(|\nabla u|, |\nabla v|) = 1 \quad \text{on} \quad \partial\Omega \cap B_1 \end{cases}$$

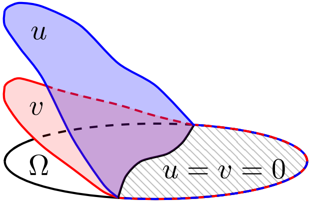
**... if**

*for every* $x_0 \in \partial\Omega \cap B_1$;

$\boxed{\textit{at which } \Omega \textit{ admits one-sided tangent ball}}$

*there are a unit vector* $\nu \in \mathbb{R}^d$;

*and constants* $\alpha > 0$ *and* $\beta > 0$




**such that:**

$$G(\alpha, \beta) = 1 \quad \text{and} \quad \begin{cases} u(x) = \alpha \left((x - x_0) \cdot \nu\right)_+ + o(|x - x_0|) \\ v(x) = \beta \left((x - x_0) \cdot \nu\right)_+ + o(|x - x_0|). \end{cases}$$

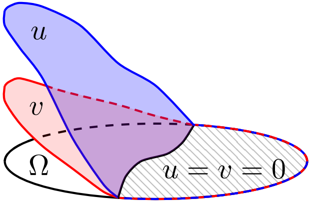
**... if**

*for every* $x_0 \in \partial\Omega \cap B_1$;

$\boxed{\textit{at which } \Omega \textit{ admits one-sided tangent ball}}$

*there are a unit vector* $\nu \in \mathbb{R}^d$;

*and constants* $\alpha > 0$ *and* $\beta > 0$



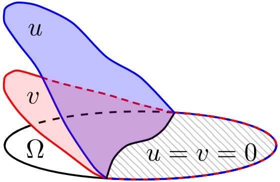

**such that:**

$$G(\alpha,\beta) = 1 \quad \text{and} \quad \begin{cases} u(x) = \alpha\,\big((x-x_0)\cdot\nu\big)_+ \; + \; o(|x-x_0|) \\ v(x) = \beta\,\big((x-x_0)\cdot\nu\big)_+ \; + \; o(|x-x_0|). \end{cases}$$

More generally, we can say that at $x_0$ there are one-homogeneous blow-ups of $u$ and $v$ of the form $u_0(x) = \alpha(x\cdot\nu)_+$ and $v_0(x) = \beta(x\cdot\nu)_+$ with $G(\alpha,\beta) = 1$.

**The only system studied:**

$$\begin{cases} \Omega = \{u^2 + v^2 > 0\} \\ \Delta u = \Delta v = 0 \quad \text{in} \quad \Omega \cap B_1 \\ |\nabla u|^2 + |\nabla v|^2 = 1 \quad \text{on} \quad \partial\Omega \cap B_1 \end{cases}$$



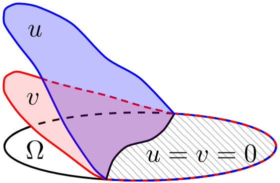

**Associated functional:** $\displaystyle\int_{B_1} |\nabla U|^2\,dx + |\{|U| > 0\}|,$ where $U = (u, v)$.

- *Caffarelli-Shahgholian-Yeressian, Mazzoleni-Terracini-V., Kriventsov-Lin* (2016), (flat free boundaries are $C^{1,\alpha}$ by assuming $u > 0$, $v > 0$ in $\Omega$);
- *Spolaor-Velichkov* (2017) - 2D, epiperimetric inequality, analysis of singularities;
- *Kriventsov-Lin* (2017) - in any dimension, no sign assumption;
- *Mazzoleni-Terracini-Velichkov* (2018), flat NTA boundaries are $C^{1,\alpha}$;
- *De Silva-Tortone* (2020), flat free boundaries are $C^{1,\alpha}$.

**Theorem (** *Maiale-Tortone-Velichkov,* 2021 **).**
Let $u, v : B_1 \to \mathbb{R}$ be non-negative continuous functions, and $\Omega \subseteq B_1$ be an open set. Suppose that:

- $(u, v, \Omega)$ is a viscosity solution to

$$\begin{cases} \Omega = \{u > 0\} = \{v > 0\} \\ -\Delta u = f \quad \text{and} \quad -\Delta v = g \quad \text{in} \quad \Omega \cap B_1 \\ |\nabla u||\nabla v| = 1 \quad \text{on} \quad \partial\Omega \cap B_1 \end{cases}$$




- $(u, v)$ is $\varepsilon$-flat for some $\varepsilon < \varepsilon_0$, that is,

$$(x \cdot \nu - \varepsilon)_+ \le u(x) \le (x \cdot \nu + \varepsilon)_+ \quad \text{and} \quad (x \cdot \nu - \varepsilon)_+ \le v(x) \le (x \cdot \nu + \varepsilon)_+ \quad \text{in} \quad B_1.$$

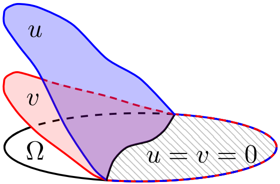
Then, $\partial\Omega$ is $C^{1,\alpha}$-regular in $B_{1/2}$.

**Aim:** Prove the following improvement of flatness theorem:

**Lemma:** Let $u, v : B_1 \to \mathbb{R}$ be non-negative continuous functions, and $\Omega \subseteq B_1$ be an open set. Suppose that:

- $(u, v, \Omega)$ is a viscosity solution to

$$\begin{cases} \Omega = \{u > 0\} = \{v > 0\} \\ -\Delta u = f \quad \text{and} \quad -\Delta v = g \quad \text{in} \quad \Omega \cap B_1 \\ |\nabla u||\nabla v| = 1 \quad \text{on} \quad \partial\Omega \cap B_1 \end{cases}$$




- $(u, v)$ is $\varepsilon$-flat for some $\varepsilon < \varepsilon_0$, that is,

$$(x \cdot \nu - \varepsilon)_+ \leq u(x) \leq (x \cdot \nu + \varepsilon)_+ \quad \text{and} \quad (x \cdot \nu - \varepsilon)_+ \leq v(x) \leq (x \cdot \nu + \varepsilon)_+ \quad \text{in} \quad B_1.$$

Then, the rescalings $u_r(x) = \frac{1}{r} u(rx)$ and $v_r(x) = \frac{1}{r} v(rx)$ are ${}^{\varepsilon}/{}_{2}$ flat in $B_1$.

**A general strategy for proving improvement-of-flatness** (*De Silva,* (2010)).

- Consider a sequence $(u_n, v_n, \Omega_n)$ of $\varepsilon_n$-flat solutions.

$$(x_d - \varepsilon_n)_+ \leq u_n(x) \leq (x_d + \varepsilon_n)_+ \quad \text{and} \quad (x_d - \varepsilon_n)_+ \leq v_n(x) \leq (x_d + \varepsilon_n)_+ \quad \text{in} \quad B_1.$$

- Prove that in any $B_r(x_0) \subseteq B_1$ the oscillation of

$$\widetilde{u}_n(x) := \frac{u_n(x) - x_d}{\varepsilon_n} \quad \text{and} \quad \widetilde{v}_n(x) := \frac{v_n(x) - x_d}{\varepsilon_n}$$

decays from $B_r(x_0)$ to $B_{r/2}(x_0)$.

- Deduce that the sequences $\widetilde{u}_n$ and $\widetilde{v}_n$ converge to some functions $\widetilde{u}_\infty$ and $\widetilde{v}_\infty$.
- Prove that $\widetilde{u}_\infty$ and $\widetilde{v}_\infty$ are (viscosity) solutions of a PDE.
- Use the uniform estimates for $\widetilde{u}_\infty$ and $\widetilde{v}_\infty$ to obtain contradiction.

**Remark.** There is a constant $c \in (0,1)$ such that:

if $h$ is a harmonic function in $B_1$, then $\operatorname{osc}\big(h; B_{1/2}\big) \leq (1-c)\operatorname{osc}(h; B_1)$.

**Proof.** Let

$$\sup_{B_1} h = M \qquad \text{and} \qquad \inf_{B_1} h = m.$$

**Case 1.** $h(0) \geq \dfrac{M+m}{2}$. Then:

$$\begin{cases} w(x) := h(x) - m \quad \text{is harmonic and nonnegative in } B_1 \\ w(0) = h(0) - m \geq \frac{M+m}{2} - m = \frac{M-m}{2} = \frac{1}{2}\operatorname{osc}(h; B_1). \end{cases}$$

Harnack $\Rightarrow \inf_{B_{1/2}} w \geq c \operatorname{osc}(h; B_1)$; on the other hand $\sup_{B_{1/2}} w \leq M - m = \operatorname{osc}(h; B_1)$...

**Lemma (Partial Harnack):** Let $u, v : B_1 \to \mathbb{R}$ be non-negative continuous functions, and $\Omega \subseteq B_1$ be an open set. Suppose that $(u, v, \Omega)$ is a solution to

$$\begin{cases} \Omega = \{u > 0\} = \{v > 0\} \\ \Delta u = \Delta v = 0 \quad \text{in} \quad \Omega \cap B_1 \\ |\nabla u||\nabla v| = 1 \quad \text{on} \quad \partial\Omega \cap B_1 \end{cases}$$

Suppose that there are constants $A$ and $B$ such that $0 \leq B - A \leq \varepsilon_0$

$$(x_d + A)_+ \leq u(x) \leq (x_d + B)_+ \quad \text{and} \quad (x_d + A)_+ \leq v(x) \leq (x_d + B)_+ \quad \text{in } B_1.$$

Then, there are $A \leq a < b \leq B$ such that $b - a \leq (1 - c)(B - A)$ and

$$(x_d + a)_+ \leq u(x) \leq (x_d + b)_+ \quad \text{and} \quad (x_d + a)_+ \leq v(x) \leq (x_d + b)_+ \quad \text{in } B_r.$$